# Convergence Results in Fuzzy Metric Spaces and Applications

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**Abstract**

In order to acknowledge Singh et al.[6] original concept of Complex Valued Fuzzy Metric Spaces, we investigated and illustrated a number of Fixed Point Results in Complex Valued Fuzzy b-Metric Spaces in this study. Our findings represent important extensions and expansions of some findings in the current framework. The conclusions are supported with creative examples and applications, which help to clarify the established theory.

**Key words** Complex Valued Fuzzy Metric Spaces(CVFMS), Complex Valued Fuzzy b-Metric Spaces(CVFbMS), Fixed Point Theorem(FPTh).

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# 1 Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [12] in a noteworthy article in year 1965. After that many authors have contributed in the concept of fuzzy sets and fuzzy metric spaces in different ways.
Kramosil and Michalek [4] initiated the conviction of a fuzzy metric space (FMS) by generalizing the conviction of the probabilistic metric space to the fuzzy circumstance. Grabiec [3] enlisted the fuzzy version of Banach contraction principle (BCP) introduced by Kramosil and Michalek. George and Veeramani [2] generalized the conviction of FMS due to Kramosil and Michalek [4] and determin the Hausdorff topology of fuzzy metric spaces. This proved a landmark in fixed point theory (FPT) of FMS and subsequently many of articles appeared for FPTh in this spaces.
Ramot et al.[5] investigate the innovative concept of complex fuzzy sets by extended fuzzy sets to complex fuzzy sets. According to Ramot et al.[5], the complex fuzzy set is depicted by a membership function, whose range is not delimited to $[0, 1]$ but expanded to the unit circle in the complex plane. Membership in a complex fuzzy set remains "as fuzzy" as membership in a traditional fuzzy set.

**Definition 1.1.** *[5]* A ***Complex Fuzzy Set*** S, defined on the discourse Universe $\mathcal{U}$, is described by a membership function $\mu_s(\breve{a})$ that allots every element $\breve{a} \in \mathcal{U}$, a complex valued grade of membership in S. The values $\mu_s(\breve{a})$ lie within the unit circle in the complex plane, and are thus of the form

$$\mu_s(\breve{a}) = r_s(\breve{a}).e^{iw_s(\breve{a})}, \quad (i = \sqrt{-1}),$$

where $r_s(\breve{a})$ and $w_s(\breve{a})$ both real-valued, with $r_s(\breve{a}) \in [0, 1]$.
The complex fuzzy set $S$, may be represented as the set of ordered pairs, given by

$$S = \{(\breve{a}, \mu_s(\breve{a}))|\breve{a} \in \mathcal{U}\}.$$

Ramot et al.[5] Given that complex fuzzy sets are extensions of ordinary fuzzy sets, it is feasible to express every ordinary fuzzy set in terms of a complex fuzzy set. On the other hand,Bakhtin [7] first proposed the idea of b-metric space by extended the triangle inequality of metric space from its weaker form and this conviction was elaborately use by Czerwik [8]. The concept of quasi b-metric space was proposed by Shah et al.[10] and slightly modified by Nadaban [11].

Azam et al.[1] introduced the conviction of complex valued metric spaces (CVMS). According to them two complex number can be compare by the following way by endowing partial order relation $' \precsim'$ on Complex Numbers . Let $\mathcal{C}$ be the set of complex numbers and $z_1, z_2 \in \mathcal{C}$ then

$$z_1 \precsim z_2$$

if and only if

$$\mathbb{R}(z_1) \leq \mathbb{R}(z_2) \ \ and \ \ \mathbb{I}(z_1) \leq \mathbb{I}(z_2).$$

It follows that $z_1 \precsim z_2$, if one the following conditions are satisfied:
*(C-1)* $\mathbb{R}(z_1) = \mathbb{R}(z_2)$ and $\mathbb{I}(z_1) = \mathbb{I}(z_2)$;
*(C-2)* $\mathbb{R}(z_2) > \mathbb{R}(z_1)$ and $\mathbb{I}(z_2) = \mathbb{I}(z_1)$;
*(C-3)* $\mathbb{R}(z_1) = \mathbb{R}(z_2)$ and $\mathbb{I}(z_2) > \mathbb{I}(z_1)$;
*(C-4)* $\mathbb{R}(z_2) > \mathbb{R}(z_1)$ and $\mathbb{I}(z_2) > \mathbb{I}(z_1)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C-2), (C-3) and (C-4) is satisfied while $z_1 \prec z_2$ if only (C-4) is satisfied . Observe that

$$0 \preceq z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Recently,acknowledging the innovative conviction of complex valued fuzzy set, Singh et al.[6] proposed the concept of CVFMS and several allied topological aspects for CVFMS

**Definition 1.2.** *[6]* A binary operation $* : r_s(cos\vartheta + isin\vartheta) \times r_s(cos\vartheta + isin\vartheta) \rightarrow r_s(cos\vartheta + isin\vartheta)$, wherein $r_s \in [0, 1]$ and a fix $\vartheta \in [0, \frac{\pi}{2}]$, is known as complex valued continuous triangular norm i.e. $t$-norm if the following conditions hold :
*(i)* $*$ is Associative and Commutative,
*(ii)* $*$ is Continuous,
*(iii)* $x * (cos\vartheta + isin\vartheta) = x, \forall x \in r_s(cos\vartheta + isin\vartheta)$, where $r_s \in [0, 1]$ and $\vartheta \in [0, \frac{\pi}{2}]$,
*(iv)* $x * y \preceq p * q$ whenever $x \preceq p$ and $y \preceq q$, for all $x, y, p, q \in r_s(cos\vartheta + isin\vartheta)$ where $r_s \in [0, 1]$ and a fix $\vartheta \in [0, \frac{\pi}{2}]$.

**Example 1.1.** *[6]* $x * y = \min(x, y)$.

**Example 1.2.** *[6]* $x * y = max(x + y - (cos\vartheta + isin\vartheta), 0)$, *for a fix* $\theta \in [0, \frac{\pi}{2}]$.

The value of $\vartheta$ is considered to be fixed in the interval $[0, \frac{\pi}{2}]$ with the assumption that the complex fuzzy set $S = \{(x, \mu_s(x))|x \in \mathcal{U}\}$ interacts with other complex fuzzy sets through the partial order relation due to Azam et al.[1].
Complex valued fuzzy metric spaces defined as follows.

**Definition 1.3.** *[6]* if $\mathcal{X}$ is an arbitrary non empty set, $*$ is a complex valued continuous t-norm and $\mathcal{M} : \mathcal{X} \times \mathcal{X} \times (0,\infty) \to r_s(cos\vartheta + isin\vartheta)$ is a complex valued fuzzy set , then the triplet $(\mathcal{X}, \mathcal{M}, *)$ is called complex valued fuzzy meric space space , where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:

*($\mathcal{CF}-1$)* $\mathcal{M}(\breve{a}, \breve{b}, t) \succ 0$,
*($\mathcal{CF}-2$)* $\mathcal{M}(\breve{a}, \breve{b}, t) = cos\vartheta + isin\vartheta$ for all $t > 0 \Leftrightarrow \breve{a} = \breve{b}$,
*($\mathcal{CF}-3$)* $\mathcal{M}(\breve{a}, \breve{b}, t) = \mathcal{M}(\breve{b}, \breve{a}, t)$,
*($\mathcal{CF}-4$)* $\mathcal{M}(\breve{a}, \breve{b}, t) * \mathcal{M}(\breve{b}, \breve{c}, s) \preceq \mathcal{M}(\breve{a}, \breve{c}, (t+s)$,
*($\mathcal{CF}-5$)* $\mathcal{M}(\breve{a}, \breve{b}, .) : (0,\infty) \to r_s(cos\vartheta + isin\vartheta)$

is continuous, for all $\breve{a}, \breve{b}, \breve{c} \in \mathcal{X}$, $s, t > 0$, $r_s \in [0,1]$ and $\vartheta \in [0, \frac{\pi}{2}]$ .

Also $(\mathcal{M}, *)$ is called a complex valued fuzzy metric.

**Remark 1.1.** *It is obvious that at $\vartheta = 0$ , CVFMS becomes real valued FMS.*

**Example 1.3.** *[6]*Let $\mathcal{X} = \mathcal{R}$. We define $\breve{a} * \breve{b} = min\{\breve{a}, \breve{b}\}, \forall \breve{a}, \breve{b} \in r_s(cos\vartheta + isin\vartheta)$, where $r_s \in [0,1]$ and $\vartheta \in [0, \frac{\pi}{2}]$. Furthermore for all $\breve{a}, \breve{b} \in X$ and $t \in (0,\infty)$, we define

$$\mathcal{M}(\breve{a}, \breve{b}, t) = (cos\vartheta + isin\vartheta)e^{-\frac{|\breve{a}-\breve{b}|}{t}}.$$

Then $(\mathcal{X}, \mathcal{M}, *)$ is a CVFMS.

In the fixed point theory (see [14],[15]), it is of interest to investigate the classes of $t$-norms $*$ and sequences $\{\breve{a}_n\}$ from the interval $r_s(cos\vartheta + isin\vartheta)$, where $r_s \in [0,1]$ such that $\lim_{n\to\infty} \breve{a}_n = (cos\vartheta + isin\vartheta)$ and

$$\lim_{n\to\infty} *_{i=n}^{\infty} \breve{a}_i = \lim_{n\to\infty} *_{i=n}^{\infty} \breve{a}_{n+i} = (cos\vartheta + isin\vartheta). \tag{1.1}$$

**Definition 1.4.** Let $\mathcal{X}$ is an arbitrary non empty set, let $\kappa \geq 1$ be a given real number and $*$ is a complex valued continuous t-norm and $\mathcal{M} : \mathcal{X} \times \mathcal{X} \times (0,\infty) \to r_s e^{i\vartheta}$ is a complex valued fuzzy set, where $r_s \in [0,1]$ and $\vartheta \in [0, \frac{\pi}{2}]$, Then $(\mathcal{X}, \mathcal{M}, *, \|)$ is called a CVFbMS if it holding the following conditions:

*(CFb-1)* $\mathcal{M}(\breve{a}, \breve{b}, t) \succ 0$,
*(CFb-2)* $\mathcal{M}(\breve{a}, \breve{b}, t) = (cos\vartheta + isin\vartheta)$ for all $t > 0 \Leftrightarrow \breve{a} = \breve{b}$,
*(CFb-3)* $\mathcal{M}(\breve{a}, \breve{b}, t) = \mathcal{M}(\breve{b}, \breve{a}, t)$,
*(CFb-4)* $\mathcal{M}(\breve{a}, \breve{b}, t) * \mathcal{M}(\breve{b}, \breve{c}, s)) \preceq \mathcal{M}(\breve{a}, \breve{c}, \kappa(t+s))$,
*(CFb-5)* $\mathcal{M}(\breve{a}, \breve{b}, .) : (0, \infty) \to \breve{r}_s(cos\vartheta + isin\vartheta)$

is continuous,for all $\breve{a}, \breve{b}, \breve{c} \in \mathcal{X}$, $s, t > 0$, $r_s \in [0,1]$ and $\vartheta \in [0, \frac{\pi}{2}]$ .
Also $(\mathcal{M}, *, k)$ is called a CVFbMS.

**Remark 1.2.** *The class of CVFbMS is larger than the class of CVFMS , since a CVFbMS is a CVFMS when $\kappa = 1$.*

**Example 1.4.** *[17] Let $\mathcal{M}(\breve{a}, \breve{b}, t) = e^{i\vartheta} e^{\frac{-|\breve{a}-\breve{b}|^p}{t}}$, where $p > 1$ is a real number. Then $\mathcal{M}$ is a complex valued fuzzy b-metric with $\kappa = 2^{p-1}$.*

**Example 1.5.** Let $\mathcal{X}$ be the set of real numbers.We define $\breve{a} * \breve{b} = \breve{a}\breve{b}, \forall \breve{a}, \breve{b} \in r_s e^{i\vartheta}$, where $r_s \in [0,1]$ and $\vartheta \in [0, \frac{\pi}{2}]$. Furthermore for all $\breve{a}, \breve{b} \in \mathcal{X}$ and $t \in (0, \infty)$, we define

$$\mathcal{M}(\breve{a}, \breve{b}, t) = e^{i\vartheta} e^{\frac{-d(\breve{a}, \breve{b})}{t}}.$$

Then it is not difficult to prove that $(\mathcal{X}, \mathcal{M}, *)$ is a CVFbMS. Obviously condition (CFb-1),(CFb-2),(CFb-3),(CFb-5) of Definition 1.4 are satisfied. For each $\breve{a}, \breve{b}, \breve{c} \in \mathcal{X}$, we obtain

$$\begin{aligned}
\mathcal{M}(\breve{a}, \breve{b}, \kappa(t+s)) &= (cos\vartheta + isin\vartheta) e^{\frac{-d(\breve{a}, \breve{b})}{\kappa(t+s)}} \\
&\succeq e^{i\vartheta} e^{-\kappa \frac{d(\breve{a}, \breve{c}) + d(\breve{c}, \breve{b})}{\kappa(t+s)}} \\
&= e^{i\vartheta} e^{-\kappa \frac{d(\breve{a}, \breve{c})}{\kappa(t+s)}} . e^{i\vartheta} e^{-\kappa \frac{d(\breve{c}, \breve{b})}{\kappa(t+s)}} \\
&\succeq e^{i\vartheta} e^{\frac{-d(\breve{a}, \breve{b})}{t}} . e^{i\vartheta} e^{\frac{-d(\breve{c}, \breve{b})}{s}} \\
&= \mathcal{M}(\breve{a}, \breve{c}, t) * \mathcal{M}(\breve{c}, \breve{b}, s).
\end{aligned}$$

Here condition (CFb-4) of Definition 1.4 is satisfied, hence $(\mathcal{X}, \mathcal{M}, *)$ is a CVFbMS.

**Definition 1.5.** *[17] A function $f : \mathcal{R} \to \mathcal{R}$ is called k-non decreasing if $x > ky$ implies $f(x) \geq f(y)$ for all $x, y \in \mathcal{R}$.*

A point $\breve{a} \in \mathcal{X}$ is said to be an interior point of set $\mathcal{H} \subset \mathcal{X}$, whenever there exists $\breve{r} \in C, 0 \prec \breve{r} \prec (cos\vartheta + isin\vartheta)$ such that

$$\mathcal{B}(\breve{a}, \breve{r}, t) = \{\breve{y} \in \mathcal{X} : \mathcal{M}(\breve{a}, \breve{b}, t) \succ (cos\vartheta + isin\vartheta) - r\} \subset \mathcal{H},$$

where $\vartheta \in [0, \frac{\pi}{2}]$.
The subset $\mathcal{H}$ of $\mathcal{X}$ is said to be **open** if each element of $\mathcal{H}$ is an interior point of $\mathcal{H}$.
**Topology on $\mathcal{X}$:** Assume that $(\mathcal{X}, \mathcal{M}, *)$ be a CVFbMS. Let $\mu$ be the set of all $\mathcal{H} \subset X$ : $with$ $\breve{a} \in \mathcal{H}$ if and only if there exists $t > 0$ and $\breve{r} \in C, 0 \prec \breve{r} \prec (cos\vartheta + isin\vartheta)$, $\vartheta \in [0, \frac{\pi}{2}]$ such that $\mathcal{B}(\breve{a}, \breve{r}, t) \subset \mathcal{H}$.
Then $\mu$ is a topology on $\mathcal{X}$.

**Definition 1.6.** Let $(\mathcal{X}, \mathcal{M}, *)$ be a CVFbMS and $\mu$ be the topology induced by complex valued fuzzy metric. Then for a sequence $\{\breve{a}_n\} \in \mathcal{X}$ converges to $\breve{x}$ if and only if $\mathcal{M}(\breve{a}_n, \breve{a}, t) \to (cos\vartheta + isin\vartheta)$ as $n \to \infty$ or $|\mathcal{M}(\breve{a}_n, \breve{a}, t)| \to 1$ as $n \to \infty$ for each $t > 0$.

**Definition 1.7. Cauchy sequence:** A sequence $\breve{a}_n$ in a CVFbMS $(\mathcal{X}, \mathcal{M}, *)$ is a Cauchy sequence if and only if

$$\lim_{n\to\infty} \mathcal{M}(\breve{a}_{n+p}, \breve{a}_n, t) = (cos\vartheta + isin\vartheta), \quad p > 0, t > 0$$

or

$$\lim_{n\to\infty} |\mathcal{M}(\breve{a}_{n+p}, \breve{a}_n, t)| = 1, \quad p > 0, t > 0.$$

**Definition 1.8.** A CVFbMS in which every Cauchy sequence is convergent, is called complex valued complete fuzzy b-metric space.

**Definition 1.9. Bounded Set:** Let $(\mathcal{X}, \mathcal{M}, *)$ be a CVFbMS . A subset $'\mathcal{P}'$ of $X$ is said to be $F_c$ bounded if and only if there exist $t > 0$ and $\breve{r} \in C$, $0 \prec r \prec (cos\vartheta + isin\vartheta)$ such that

$$\mathcal{M}(\breve{a}, \breve{b}, t) \succ (cos\vartheta + isin\vartheta) - r, \quad for\ all\ \ \breve{a}, \breve{b} \in \mathcal{P}.$$

## 2 Elementary topological concepts:

**Definition 2.1.** *Let $\urcorner : \mathcal{R} \to \mathcal{R}$ be a function. Then $\urcorner$ is called b-non decreasing, if $\breve{x} > \breve{b}y$ this implies $\urcorner\breve{x} \geq \urcorner\breve{y}$ for each $\breve{x}, \breve{y} \in R$.*

**Lemma 2.1.** *Let $(\mathcal{X}, \mathcal{M}, *)$be a CVFbMS such that*

$$\lim_{t\to\infty} \mathcal{M}(\breve{a}, \breve{b}, t) = cos\vartheta + isin\vartheta$$

*, for all $\breve{a}, \breve{b} \in \mathcal{X}$ ,if*

$$\mathcal{M}(\breve{a}, \breve{b}, t) \preceq \mathcal{M}(\breve{a}, \breve{b}, kt)$$

*for all $\breve{a}, \breve{b} \in \mathcal{X}$,$0 \prec k \prec 1$, $t \in (0, \infty)$, then $\breve{a} = \breve{b}$.*

*Proof.* Suppose $\exists$, $k \in (0,1)$ such that

$$\mathcal{M}(\breve{a}, \breve{b}, t) \preceq \mathcal{M}(\breve{a}, \breve{b}, kt),$$

$\forall \breve{a}, \breve{b} \in$X,

t$\in (0, \infty)$ so that

$$\mathcal{M}(\breve{a}, \breve{b}, \frac{t}{k}) \preceq \mathcal{M}(, \breve{a}, \breve{b}, t)$$

repeated application gives

$$\mathcal{M}(\breve{a}, \breve{a}\frac{t}{k^n}) \preceq \mathcal{M}(\breve{a}, \breve{a}\frac{t}{k^{n-1}}) \preceq \mathcal{M}(\breve{a}, \breve{b}, \frac{t}{k^{n-2}}) \preceq \cdots \preceq \mathcal{M}(\breve{a}, \breve{b}\frac{t}{k}) \preceq \mathcal{M}(\breve{a}, \breve{b}, t)$$

for some positive integer n,on making $n \to \infty$, reduces to

$$\mathcal{M}(\breve{a}, \breve{b}, t) \succeq \cos\theta + isin\theta$$

This implies $\mathcal{M}(\breve{a}, \breve{b}, t) = cos\vartheta + isin\vartheta$. Thus we have $\breve{a} = \breve{b}$. □

**Lemma 2.2.** *Assume that $\breve{a}_n$ be a sequence in a CVFMS $(X, M, *)$ with*

$$\lim_{t\to\infty} M(\breve{a}, \breve{b}, t) = cos\vartheta + isin\vartheta,$$

*for all $\breve{a}, \breve{b} \in X$. If $\exists$ a number $k \in (0,1)$ such that*

$$M(\breve{a}_{n+1}, \breve{a}_{n+2}, kt) \succeq M(\breve{a}_n, \breve{a}_{n+1}, t), \forall\ t \succ 0 \text{ and } n = 1, 2, 3, ...$$

*then $\{\breve{a}_n\}$ is a Cauchy sequence.*

*Proof.* For $n = 0$ we have $\mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t}{k}\Big) \preceq \mathcal{M}(\breve{a}_1, \breve{a}_2, t)$ ,$\forall\ t \succ 0$ and $k \in (0,1)$ By induction one sets

$$\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t}{k^{n+1}}\Big) \preceq \mathcal{M}(\breve{a}_{n+1}, \breve{a}_{n+2}, t), \forall n$$

thus for any positive integer p and using (CFb-4) we have

$$\mathcal{M}(\breve{a}_n, \breve{a}_{n+p}, t) \succeq \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t}{p}\Big) * ...(p - times) * \mathcal{M}\Big(\breve{a}_{n+p-1}, \breve{a}_{n+p}, \frac{t}{p}\Big)$$

$$\succeq \mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t}{pk^n}\Big) * ...(p-times) * \mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t}{pk^{n+p-1}}\Big)$$

Which on letting $n \to \infty$ reduces to

$$\lim_{n\to\infty} \mathcal{M}(\breve{a}_n, \breve{a}_{n+p}, t) \succeq (cos\vartheta + isin\vartheta)*(cos\vartheta + isin\vartheta)*(cos\vartheta + isin\vartheta)*...*(cos\vartheta + isin\vartheta)$$

Since $k \prec 1$ and $\lim_{n\to\infty} \mathcal{M}(\breve{a}, \breve{b}, t) = (cos\vartheta + isin\vartheta)$

$$\lim_{n\to\infty} \mathcal{M}(\breve{a}_n, \breve{a}_{n+p}, t) \succeq (cos\vartheta + isin\vartheta)$$

This necessitates that $\{\breve{a}_n\}$ is Cauchy sequence in $\mathcal{X}$.

□

# 3 Main Result

In this section, utilizing the concept of CVFbMS we proved some fixed point results.

**Theorem 3.1.** *Let $(\mathcal{X}, \mathcal{M}, *)$ be a CVFbMS $\daleth : \mathcal{X} \to \mathcal{X}$. Let there exists $\lambda \in (0, \frac{1}{k})$ such that*

$$\mathcal{M}(\daleth\breve{a}, \daleth\breve{b}, t) \succsim \mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \quad a, b \in \mathcal{X} \quad t > 0 \tag{3.1}$$

*there exist $\breve{a}_0 \in \mathcal{X}$ and $v \in (0,1)$ such that*

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathcal{M}\Big(\breve{a}_0, \daleth\breve{a}_0, \frac{t}{v^i}\Big) = cos\vartheta + isin\vartheta \quad t > 0. \tag{3.2}$$

*Then $\daleth$ posses an unique fixed point in $\mathcal{X}$.*

*Proof.* Let $\breve{a}_0 \in \mathcal{X}$ and $\daleth\breve{a}_n = \breve{a}_{n+1}$, n is an element of N. If we consider $\breve{a}_n = \breve{a}$ and $\breve{a}_{n-1} = \breve{b}$ in (3.1), then we get

$$\mathcal{M}(\breve{a}_n, \breve{a}_{n+1}, t) \succsim \mathcal{M}\Big(\breve{a}_{n-1}, \breve{a}_n, \frac{t}{\lambda}\Big), \quad n \in N \quad t > 0 \tag{3.3}$$

Rest is to prove that sequence $\{\breve{a}_n\}$ is a Cauchy sequence.
Let $\eth \in (\lambda k, 1)$. Then the series $\eth_{i=1}^{\infty}\eth^i$ is tend to a finite point and there exists $n_0 \in N$ such that $\eth_{i=1}^{\infty}\eth^i < (cos\vartheta + isin\vartheta)$ for every $n > n_0$.

Let $n > m > n_0$, since $\mathcal{M}$ is a $k$-non decreasing by (CFb-4) for every $t > 0$, we have

$$\begin{aligned}
\mathcal{M}(\breve{a}_n, \breve{a}_{n+m}, t) &\succsim \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+m}, \frac{t\eth_{i=n}^{n+m-1}\eth^i}{k}\Big) \\
&\succsim \Big(\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t\eth^n}{k^2}\Big) * \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_{n+m}, \frac{t\eth_{i=n+1}^{n+m-1}\eth^i}{k^2}\Big)\Big) \\
&\succsim \Big(\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t\eth^n}{k^2}\Big) * \Big(\mathcal{M}\Big(a_{n+1}, a_{n+2}, \frac{t\eth^{n+1}}{k^3}\Big) \\
&\quad * \cdots * \mathcal{M}\Big(\breve{a}_{n+m-1}, \breve{a}_{n+m}, \frac{t\eth^{n+m-1}}{k^m}\Big), \ldots\Big)\Big)
\end{aligned}$$

From (3.3) it follows that

$$\mathcal{M}(\breve{a}_n, \breve{a}_{n+1}, t) \succsim \mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t}{\lambda^n}\Big),\ n \in N,\ t > 0.$$

Since $n > m$ and $k > 1$, we have

$$\begin{aligned}
\mathcal{M}(\breve{a}_n, \breve{a}_{n+m}, t) &\succsim \Big(\mathcal{M}\Big(a_0, a_1, \frac{t\eth^n}{k^2\lambda^n}\Big) * \Big(\mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t\eth^{n+1}}{k^3\lambda^{n+1}}\Big), \\
&\quad * \cdots * \mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t\eth^{n+m-1}}{k^{m+1}\lambda^{n+m-1}}\Big), \ldots\Big)\Big) \\
&\succsim *_{i=n}^{n+m-1}\mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t\eth^i}{k^{i-n+2}\lambda^i}\Big) \\
&\succsim *_{i=n}^{n+m-1}\mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t\eth^i}{k^i\lambda^i}\Big) \\
&\succsim *_{i=n}^{\infty}\mathcal{M}\Big(\breve{a}_0, \breve{a}_1, \frac{t}{v^i}\Big).
\end{aligned}$$

Where $v = \frac{k\lambda}{\eth}$, since $v = (0,1)$ by lemma 2.2 we concluded that $\{\breve{a}_n\}$ is a Cauchy sequence. Since$(\mathcal{X}, \mathcal{M}, *)$ is complete, hence $\exists\ \breve{a} \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \breve{a}_n = \breve{a} \textit{ and } \lim_{n\to\infty} \mathcal{M}(\daleth\breve{a}, \breve{a}_n, t) = cos\vartheta + isin\vartheta \ \ t > 0. \tag{3.4}$$

(3.1) and (CFb-4) are used to show that $\breve{a}$ is a fixed point for $\daleth$:

$$\begin{aligned}
\mathcal{M}(\daleth\breve{a}, \breve{a}, t) &\succsim \Big(\mathcal{M}\Big(\daleth\breve{a}, \breve{a}_n, \frac{t}{2k}\Big) * \mathcal{M}\Big(\breve{a}_n, \breve{a}, \frac{t}{2k}\Big)\Big) \\
&\succsim \Big(\mathcal{M}\Big(\breve{a}, \breve{a}_{n-1}, \frac{t}{2k\lambda}\Big) * \mathcal{M}\Big(\breve{a}_n, a, \frac{t}{2k}\lambda\Big)\Big)
\end{aligned}$$

for all $t > 0$. By 3.4 as $n \to \infty$, we get

$$\mathcal{M}(\urcorner\breve{a}, \breve{a}, t) \succsim (cos\vartheta + isin\vartheta) * (cos\vartheta + isin\vartheta)) = (cos\vartheta + isin\vartheta).$$

Suppose that $\breve{a}$ and $\breve{b}$ are fixed point of $\urcorner$. By (3.1) we have

$$\mathcal{M}(\breve{a}, \breve{b}, t) = \mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) \succsim \mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big),\ t > 0$$

and **??** implies that

$$\mathcal{M}(\breve{a}, \breve{b}, t) \succsim \mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda^n}\Big)\ \ n \in N,\ \ t > 0$$

Now

$$\mathcal{M}(\breve{a}, \breve{b}, t) \succsim \lim_{n\to\infty} \mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda^n}\Big) = (cos\vartheta + isin\vartheta),\ \ t > 0$$

and by (CFb-2) it follows that $\breve{a} = \breve{b}$. □

**Example 3.1.** *Let $\mathcal{X} = [0, 1]$. By Example 4, for $p = 2$, it follows that $(\mathcal{X}, \mathcal{M}, *)$ is a CVFbMS with $k = 2$ and complex valued fuzzy b-metric*

$$\mathcal{M}(\urcorner\breve{a}, \breve{b}, t) = (cos\vartheta + isin\vartheta)e^{\frac{-(\breve{a}-\breve{b})^2}{t}}\ \breve{a}, \breve{b} \in \mathcal{X},\ t > 0.$$

*Let $\urcorner\breve{a} = \mu\breve{a}, \mu < \frac{1}{\sqrt{3}}, \breve{a} \in X$. Then*

$$\begin{aligned}\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) &= (cos\vartheta + isin\vartheta)e^{\frac{-\mu^2(\breve{a}-\breve{b})^2}{t}}\\ &\succsim (cos\vartheta + isin\vartheta)e^{\frac{-\lambda(\breve{a}-\breve{b})^2}{t}}\\ &= \mathcal{M}\Big(\breve{a}, \urcorner\breve{a}, \frac{t}{\lambda}\Big),\ \breve{a}, \breve{b} \in \mathcal{X},\ t > 0,\end{aligned}$$

*for $\frac{1}{k} > \lambda > \mu^2$. So, equation (3.1) of Theorem 1 is fulfilled, and $f$ has a unique fixed point in $\mathcal{X}$.*

**Theorem 3.2.** *Let $(\mathcal{X}, \mathcal{M}, \mathcal{T})$ is a complex valued complete fuzzy b-metric space, and let $\urcorner : \mathcal{X} \to \mathcal{X}$. Suppose that there exists $\lambda \in (0, \frac{1}{k})$ such that*

$$\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) \succsim \min\Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\urcorner\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\urcorner\breve{b}, \breve{b}, \frac{t}{\lambda}\Big)\Big\} \tag{3.5}$$

*for all $\breve{a}, \breve{b} \in X,\ t > 0$, and there exist $\breve{a}_0 \in X$ and $v \in (0, 1)$ such that*

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathcal{M}\Big(\breve{a}_0, \urcorner\breve{a}_0, \frac{t}{v^i}\Big) = e^{i\vartheta} \tag{3.6}$$

*for all $t > 0$. Then $\urcorner$ has a unique fixed point in$\mathcal{X}$.*

*Proof.* Let $\breve{a}_0 \in N$ and $\breve{a}_{n+1} = f\breve{a}_n, n \in N$. By (3.5) with $\breve{a}_n = \breve{a}$ and $\breve{a}_{n-1} = \breve{b}$, using (CFb4) and the assumption that $T = T_{min}$ for every $n \in N$ and every $t > 0$, we have

$$\begin{aligned}
\mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) &\succsim \min\Big\{\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big)\Big\} \\
&\succsim \min\Big\{\mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_n, \frac{t}{\lambda}\Big)\Big\} \\
&\succsim \min\Big\{\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{k\lambda}\Big)\Big\}.
\end{aligned}$$

If $\mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) \succsim \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{\lambda}\Big), n \in N,\ t > 0$ then by Lemma (**??**) it follows that $\breve{a}_n = \breve{a}_{n+1},\ n \in N$. So

$$\mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) \succsim \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big), n \in N,\ t > 0$$

and by Lemma 2.2 we have that $\{\breve{a}_n\}$ is a Cauchy sequence. Hence there exists $\breve{a} \in X$ such that

$\lim\limits_{n\to\infty} \breve{a}_n = \breve{a}$ *and* $\lim\limits_{n\to\infty} \mathcal{M}(\breve{a}, \breve{a}_n, t) = e^{i\vartheta},\ t > 0(3.7)$

Let us prove that $\breve{a}$ is a fixed point for $\daleth$. Let $\eth_1 \in (\lambda k, 1)$ and $\eth_2 = 1 - \eth_1$. By (3.5) we have

$$\begin{aligned}
\mathcal{M}(\daleth\breve{a}, \breve{a}, t) &\succsim \Big(\mathcal{M}\Big(\daleth\breve{a}, \daleth\breve{a}_n, \frac{t\eth_1}{k}\Big) * \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}, \frac{t\eth_2}{k}\Big)\Big) \\
&\succsim \Big(\min\Big\{\mathcal{M}\Big(\breve{a}, \breve{a}_n, \frac{t\eth_1}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t\eth_1}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t\eth_1}{k\lambda}\Big)\Big\} * \\
&\quad \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}, \frac{t\eth_2}{k}\Big)\Big).
\end{aligned}$$

Taking $n \to \infty$ and using (3), we get

$$\begin{aligned}
\mathcal{M}(\daleth\breve{a}, \breve{a}, t) &\succsim \Big(\min cos\vartheta + isin\vartheta), \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t\sigma_1}{k\lambda}\Big), (cos\vartheta + isin\vartheta)\Big\} * (cos\vartheta + isin\vartheta)\Big) \\
&= \Big(\mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t\sigma_1}{k\lambda}\Big) * (cos\vartheta + isin\vartheta)\Big) = \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t}{v}\Big),\ t > 0,
\end{aligned}$$

where $\frac{k\lambda}{\eth_1} \in (0, 1)$. So

$$\mathcal{M}(\daleth\breve{a}, \breve{a}, t) \succsim \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t}{v}\Big),\ t > 0,$$

and by Lemma 2.1 it follows that $\urcorner\breve{a} = \breve{a}$.
Suppose that $\breve{a}$ and $\breve{b}$ are fixed points for $\urcorner$, that is, $\urcorner\breve{a} = \breve{a}$ and $\urcorner\breve{b} = \breve{b}$. By condition (3.5) we get

$$\begin{aligned} \mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) &\succsim \Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}, \urcorner\breve{a}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}, \urcorner\breve{b}, \frac{t}{\lambda}\Big)\Big\}, \\ &= \Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), (cos\vartheta + isin\vartheta), (cos\vartheta + isin\vartheta)\Big\} \\ &= \mathcal{M}\Big(\breve{a}, \breve{b}), \frac{t}{\lambda}\Big) = \mathcal{M}\Big(\urcorner\breve{a}, \urcorner\breve{b}, \frac{t}{\lambda}\Big). \end{aligned}$$

for $t > 0$, and by Lemma 2.1 it follows that $\urcorner\breve{a} = \urcorner\breve{b}$, that is, $\breve{a} = \breve{b}$.

**Theorem 3.3.** *Let $(\mathcal{X}, \mathcal{M}, \mathcal{T})$ is a complex valued complete fuzzy b-metric space, and let $\urcorner : \mathcal{X} \to \mathcal{X}$. Suppose that there exists $\lambda \in (0, \frac{1}{k})$ such that*

$$\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) \succsim \min\Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\urcorner\breve{a}, \urcorner b, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\urcorner\breve{a}, \breve{b}, \frac{t}{\lambda}\Big)\Big\} \quad (3.8)$$

*for all $\breve{a}, \breve{b} \in X,\ t > 0$, and there exist $\breve{a}_0 \in X$ and $v \in (0, 1)$ such that*

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathcal{M}\Big(\breve{a}_0, \urcorner\breve{a}_0, \frac{t}{v^i}\Big) = e^{i\vartheta} \quad (3.9)$$

*for all $t > 0$. Then $\urcorner$ has a unique fixed point in $X$.*

*Proof.* Let $\breve{a}_0 \in N$ and $\breve{a}_{n+1} = \urcorner\breve{a}_n, n \in N$. By (3.8) with $\breve{a}_n = \breve{a}$ and $\breve{a}_{n-1} = \breve{b}$, for every $n \in N$ and every $t > 0$, we have

$$\begin{aligned} \mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) &\succsim \min\Big\{\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big)\Big\} \\ &\succsim \min\Big\{\mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}_n, \frac{t}{\lambda}\Big)\Big\}. \end{aligned}$$

If $\mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) \succsim \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}, \frac{t}{\lambda}\Big), n \in N,\ t > 0$ then by Lemma 2.2 it follows that $\breve{a}_n = \breve{a}_{n+1},\ n \in N$. So

$$\mathcal{M}(\breve{a}_{n+1}, \breve{a}_n, t) \succsim \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n-1}, \frac{t}{\lambda}\Big), n \in N,\ t > 0$$

and by Lemma 2.2 we have that $\{\breve{a}_n\}$ is a Cauchy sequence. Hence there exists $\breve{a} \in X$ such that

$$\lim_{n\to\infty} \breve{a}_n = \breve{a} \ \ and \ \ \lim_{n\to\infty} \mathcal{M}(\breve{a}, \breve{a}_n, t) = (cos\vartheta + isin\vartheta),\ t > o \quad (3.10)$$

Let us prove that $\breve{a}$ is a fixed point for $\daleth$. Let $\eth_1 \in (\lambda k, 1)$ and $\eth_2 = 1 - \eth_1$. By (3.8) we have

$$\begin{aligned} \mathcal{M}(\daleth\breve{a}, \breve{a}, t) &\succsim \Big(\mathcal{M}\Big(\daleth\breve{a}, \daleth\breve{a}_n, \frac{t\eth_1}{k}\Big) * \mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}, \frac{t\eth_2}{k}\Big)\Big) \\ &\succsim \Big(\min\Big\{\mathcal{M}\Big(\breve{a}, \breve{a}_n, \frac{t\eth_1}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}, \daleth a, \frac{t\eth_1}{k\lambda}\Big), \mathcal{M}\Big(\breve{a}_n, \breve{a}_{n+1}, \frac{t\eth_1}{k\lambda}\Big)\Big\}, \\ &\quad *\mathcal{M}\Big(\breve{a}_{n+1}, \breve{a}, \frac{t\eth_2}{k}\Big)\Big). \end{aligned}$$

Taking $n \to \infty$ and using 3.10, we get

$$\begin{aligned} \mathcal{M}(\daleth\breve{a}, \breve{a}, t) &\succsim \Big(\min\Big\{(cos\vartheta + isin\vartheta), \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t\sigma_1}{k\lambda}\Big), (cos\vartheta + isin\vartheta)\Big\} * (cos\vartheta + isin\vartheta)\Big) \\ &= \Big(\mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t\sigma_1}{k\lambda}\Big) * (cos\vartheta + isin\vartheta)\Big) = \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t}{v}\Big),\ t > 0, \end{aligned}$$

where $\frac{k\lambda}{\sigma_1} \in (0, 1)$. So

$$\mathcal{M}(\daleth\breve{a}, \breve{a}, t) \succsim \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t}{v}\Big),\ t > 0,$$

and by Lemma 2.1 it follows that $\daleth\breve{a} = \breve{a}$.
Suppose that $\breve{a}$ and $\breve{b}$ are fixed points for $\daleth$, that is, $\daleth\breve{a} = \breve{a}$ and $\daleth\breve{b} = \breve{b}$. By condition 3.5 we get

$$\begin{aligned} \mathcal{M}(\daleth\breve{a}, \daleth\breve{b}, t) &\succsim min\Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{a}, \daleth\breve{a}, \frac{t}{\lambda}\Big), \mathcal{M}\Big(\breve{b}, \daleth\breve{b}, \frac{t}{\lambda}\Big)\Big\}, \\ &= min\Big\{\mathcal{M}\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big), (cos\vartheta + isin\vartheta), (cos\vartheta + isin\vartheta)\Big\} \\ &= M\Big(\breve{a}, \breve{b}, \frac{t}{\lambda}\Big) = \mathcal{M}\Big(\daleth\breve{a}, \daleth\breve{b}, \frac{t}{\lambda}\Big). \end{aligned}$$

for $t > 0$, and by Lemma 2.1 it follows that $\daleth\breve{a} = \daleth\breve{b}$, that is, $\breve{a} = \breve{b}$. □

**Example 3.2.** *Let* $\mathcal{X} = (0, 2), \mathcal{M}\ (\breve{a}, \breve{b}, t) = (cos\vartheta + isin\vartheta)e^{\frac{-|\breve{a} - \breve{b}y|^2}{t}}, * = *_p$, *Then* $(\mathcal{X}, \mathcal{M}, *)$ *is a complex valued complete b-fuzzy metric space with* $k = 2$. *Let*

$$\daleth(\breve{a}) = \begin{cases} 2 - \breve{a}, & \text{if } \breve{a} \in (0, 1), \\ (cos\vartheta + isin\vartheta), & \text{if } \breve{a} \in [1, 2), \vartheta \in [0, \frac{\pi}{2}] \end{cases}$$

*Case 1. If* $\breve{a}, \breve{b} \in [1,2)$*, then* $\mathcal{M}(f\breve{a}, f\breve{b}, t) = cos\vartheta + isin\vartheta,\ t > 0,$ *and conditions (3.8) are trivially fulfilled.*
*Case 2. If* $\breve{a} \in [1,2)$ *and* $\breve{b} \in (0,1)$*, then, for* $\lambda \in \left(\frac{1}{4}, \frac{1}{2}\right)$*, we have*

$$\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) = (cos\vartheta + isin\vartheta)e^{\frac{-(1-\breve{a})^2}{t}} \succsim (cos\vartheta + isin\vartheta)e^{\frac{-4\lambda(1-\breve{b})^2}{t}} = \mathcal{M}\left(\urcorner\breve{a}, \breve{b}, \frac{t}{\lambda}\right),\ t > 0.$$

*Case 3. As in the previous case, for* $\lambda \in \left(\frac{1}{4}, \frac{1}{2}\right)$ *we have*

$$\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) \succsim \mathcal{M}\left(\urcorner\breve{a}, \urcorner\breve{b}, \frac{t}{\lambda}\right),\ \breve{a} \in (0,1)\ \breve{b} \in [1,2),\ t > 0.$$

*If* $\breve{a} \in (0,1)$ *then, for*$\lambda \in \left(\frac{1}{4}, \frac{1}{2}\right)$,

$$\begin{aligned}
\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) &= (cos\vartheta + isin\vartheta)e^{\frac{-(\breve{a}-\breve{b})^2}{t}} \\
&\succsim (cos\vartheta + isin\vartheta)e^{\frac{-(1-\breve{b})^2}{t}} \\
&\succsim (cos\vartheta + isin\vartheta)e^{\frac{-4\lambda(1-\breve{b})^2}{t}} \\
&= \mathcal{M}\left(\urcorner\breve{b}, \breve{b}, \frac{t}{\lambda}\right),\ \breve{a} > \breve{b}.
\end{aligned}$$

*and*

$$\mathcal{M}(\urcorner\breve{a}, \urcorner\breve{b}, t) \succsim \mathcal{M}\left(\urcorner\breve{a}, \breve{a}, \frac{t}{\lambda}\right),\ \breve{a} < \breve{b},\ t > 0$$

*So conditions (3.5) are satisfied for all* $\breve{a}, \breve{b} \in \mathcal{X},\ t > 0$*, and by Lemma 2.1 it follows that* $\breve{a} = 1$ *is a unique fixed point for* $\urcorner$.

# 4 Application

In this part, we apply a fixed point theorem to guarantee that a Fredholm integral problem has only one unique solution.[18]

$$x(t) = \int_a^b F(t,s)h(x(s))ds + g(t), \tag{4.1}$$

for all $t \in \mathcal{I} = [a,b]$, where $F \in \mathcal{C}(\mathcal{I}X\mathcal{I}, \mathcal{R})$ and $g, h \in \mathcal{C}(\mathcal{I}, \mathcal{R})$. Our next result ensures the existence of a unique solution to the Eq.4.1.

**Theorem 4.1.** *Suppose the following statements are true*
*(p1) There exists* $k \in (0, \frac{1}{\sqrt{2}})$*such that for all* $\breve{x}, \breve{y} \in \mathcal{C}(\mathcal{I}, \mathcal{R})$

$$|h(\breve{x}) - h(\breve{y})| \preceq \sqrt{k}|\breve{x} - \breve{y}|$$

*(p2)*

$$\sup_{t\in I} \int_a^b \digamma^2(t,s)ds = \lambda \prec \frac{1}{\sqrt{2}}.$$

*Then, Eq (4.1) has a unique solution.*

*Proof.* let $\mathcal{X} = \mathcal{C}(\mathcal{I}, \mathcal{R})$.For all $\Game, \mho \in \mathcal{X}$,take

$$d(\Game, \mho) = \sup_{t\in I} |\Game(t) - \mho(t)|^2.$$

Choose $a * c = ac$ for all $c \in r_s(cos\vartheta + isin\vartheta)$ where $r_s \in [0, 1]$ and $\vartheta \in [0, \frac{\pi}{2}]$ and $\mathcal{M}(\Game, \mho, t) = (cos\vartheta + isin\vartheta)\frac{t}{t+d(\Game,\mho)}$ for $\Game, \mho \in \mathcal{X}$and $t \succ 0$. Obviously, $(\mathcal{X}, \mathcal{M}, *, \underline{=}2)$ is a complete fuzzy b-metric space. Define

$$f : \mathcal{X} \to \mathcal{X}$$

by

$$f\Game(t) = \int_a^b \digamma(t,s)h(\Game(s))ds + g(t),$$

for all $\Game, \mho \in \mathcal{X}$, we have

$$\begin{aligned}
f\Game(t) - f\mho(t) &= \int_a^b \digamma(t,s)(h(\Game(s)) - h(\mho(s)))ds \\
&\preceq \sqrt{k} \int_a^b \digamma(t,s)|\Game(s) - \mho(s)|ds \\
&\preceq \sqrt{k}\Big[\int_a^b \digamma^2(t,s)ds\Big]^{\frac{1}{2}}\Big[\int_a^b ((|\Game(s) - \mho(s)|)^2 ds\Big]^{\frac{1}{2}} \\
&\preceq \sqrt{k}\sqrt{\lambda}(d(\Game, \mho))^{\frac{1}{2}}
\end{aligned}$$

we deduce for all $\Game, \mho \in X$,

$$d(f\Game, f\mho) \succeq k\lambda d(\Game, \mho)$$

This inequality is equivalent to

$$\mathcal{M}(f\partial, f\mho, t) \succeq \mathcal{M}(\partial, \mho, \frac{t}{\lambda}).$$

Hence all hypotheses of Theorem 3.1 are full-filled and the solution of equation 4.1 is unique.

# 5 Conclusion

In this paper, we researched and demonstrated various Fixed Point Results in Complex Valued Fuzzy b-Metric Spaces, acknowledging the novel notion of Complex Valued Fuzzy Metric Spaces. Our results are significant extension generalizations of some results in the existing theory.To support our new finding, in the Application part the solution of a Fedholm Integral equation is given. For further research the question may arrise that can we use this result in other important field of non-linear analysis ?. Like to solve non liner diffrential and Fractional Integral equations.

**Conflict of Interest:-** The authors declare that they have no conflict of interest.

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