Contraction in Partially Ordered Fuzzy Metric Space

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ABSTRACT

The aim of authors in this paper is to establish the results for generalized weakly contraction mappings in partially ordered fuzzy metric spaces. Our results generalized [1, Theorem 2.1] from metric to fuzzy metric spaces. To support and validate the results, illustrative examples are provided.

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [5] in 1965 to mathematically simulate real-life situations characterized by ambiguity and uncertainty caused by non-probabilistic factors.

Definition 1.1. A continuous t-norm is a binary operation T on $[0,1]$ satisfying the following conditions:

i. T is a commutative and associative;

ii. $T(a, 1) = a$ for all $a \in [0,1]$;

iii. $T(a, b) = T(c, d)$ whenever $a = c$ and $b = d$, $(a, b, c, d \in [0,1])$;

iv. The mapping $T: [0,1] \times [0,1] \to [0,1]$ is continuous.

The following are examples of continuous t-norm:

i. $T_M(a,\ b) = =\ \min\{a, b\}$;

ii. $T_P(a,\ b) = =\ ab$.

Kramosil and Michalek [4] introduced the concept of fuzzy metric space, the formal definition is as follows:

Definition 1.2. A fuzzy metric space is a triple $(X, M,*)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that the following axioms holds:

i. $M(x, y, 0) = 0\ \forall\ x, y \in X$;

ii. $M(x, y, t) = 1$ iff $x = y\ \forall\ t > 0$;

iii. $M(x, y, t) = M(y, x, t)\ \forall\ x, y \in X, t > 0$;

iv. $M(x, y, \cdot): [0, \infty) \to [0,1]$ is left continuous $\forall\, x, y \in X$;

v. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)\ \forall\, x, y, z \in X$ and $s, t > 0$.

We shall refer to these spaces as KM-fuzzy metric spaces. This concept was further modified by George and Veeramani [2] as follows:

Definition 1.3. A fuzzy metric space is a triple $(X, M,*)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ such that the following axioms holds:

i. $M(x, y, t) > 0\ \forall\, x, y \in X, t > 0$;

ii. $M(x, y, t) = 1$ iff $x = y\ \forall\, t > 0$;

iii. $M(x, y, t) = M(y, x, t)\ \forall\, x, y \in X, t > 0$;

iv. $M(x, y, \cdot): (0, \infty) \to (0,1]$ is continuous $\forall x, y \in X$;

v. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)\ \forall\, x, y, z \in X$ and $s, t > 0$.

Notice that condition (v) in Definition (1.3) is a fuzzy version of triangular inequality. The value $M(x, y, t)$ can be thought of as degree of nearness between x and y with respect to t and from condition (2) we can relate the value 0 and 1 of a fuzzy metric to the notions of ∞ and 0 of classical metric, respectively.

Example 1.4. Consider the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$ is the usual Euclidean distance on the real line. Now, let us define the fuzzy set $M(x, y, t)$ as $M(x, y, t) = \frac{t}{t+|x-y|}$ for $t > 0$. Now, let the maximum norm $*$ be defined as $a * b = \max\{a, b\}$. Then, the triplet $(\mathbb{R}, M,*)$ forms a fuzzy metric space.

Definition 1.5. (f non-decreasing mapping [3]) Let $(X, \preceq)$ be partial order set and f is self mapping on X. We say f is non-decreasing if $x, y \in X,\ x \preceq y$ implies $F(x) \preceq F(y)$.

2 MAIN THEOREMS

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set and (X, M, T) be a complete fuzzy metric space. Assume there is a non-decreasing function $\psi: [0, \infty) \to [0, \infty)$ with

$$\lim_{n\to\infty} \psi^n(t) = 0, \text{ for each } t > 0$$

and also consider that $f: X \to X$ is a non-decreasing mapping such that

i) the following contraction holds

$$\left(\frac{1}{M(f(x), f(y), t)} - 1\right) \leq \psi\left(\frac{1}{M(x, y, t)} - 1\right), \text{ for all } x \succeq y \tag{1}$$

ii) and f is continuous.

If there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has a fixed point.

Proof. For $t > 0$, it should be noted first that $\psi(t) < t$. This may be seen if we assume that there is a non-decreasing function ψ and that there exists an integer $t_0 > 0$ such that $t_0 \leq \psi(t_0)$.

$$t_0 \leq \psi(t_0) \leq \psi^2(t_0) \leq \cdots \leq \psi^n(t_0).$$

This implies

$$t_0 \leq \psi^n(t_0) \text{ for each } n = 1,2, \dots$$

But, it is assumed that

$$\lim_{n\to\infty} \psi^n(t_0) = 0.$$

This implies

$$t_0 \leq 0$$

which is a contradiction. Therefore, $\psi(t) < t$ for $t > 0$ and also $\psi(0) = 0$. If x_0 is a fixed point, i.e., $f(x_0) = x_0$, then we are done. So, let us suppose that $f(x_0) \neq x_0$. Since $x_0 \preceq f(x_0)$ and f is non-decreasing mapping, we have

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq \cdots$$

Now, since $x_0 \preceq f(x_0)$, putting $x = f(x_0)$ and $y = x_0$ in inequality (1), we have

$$\left(\frac{1}{M(f^2(x_0), f(x_0), t)} - 1\right) \leq \psi\left(\frac{1}{M(f(x_0), x_0, t)} - 1\right)$$

and also, since $f(x_0) \preceq f^2(x_0)$, we have

$$\left(\frac{1}{M(f^3(x_0), f^2(x_0), t)} - 1\right) \leq \psi\left(\frac{1}{M(f^2(x_0), f(x_0), t)} - 1\right)$$
$$\leq \psi^2\left(\frac{1}{M(f(x_0), x_0, t)} - 1\right).$$

By using principle of Mathematical Induction,

$$\left(\frac{1}{M(f^{n+1}(x_0), f^n(x_0), t)} - 1\right) \leq \psi^n\left(\frac{1}{M(f(x_0), x_0, t)} - 1\right). \tag{2}$$

But it is assumed that

$$\lim_{n\to\infty} \psi^n(t) = 0 \text{ for all } t > 0$$

So, as (2) shows, the inequality is

$$\lim_{n\to\infty} \left(\frac{1}{M(f^{n+1}(x_0), f^n(x_0), t)} - 1\right) \leq 0.$$

This implies $$\lim_{n\to\infty} M(f^{n+1}(x_0), f^n(x_0), t) \geq 1.$$

It is well-known that $M(x,y,t)$ lies in the interval [0,1], therefore

$$\lim_{n\to\infty} M(f^{n+1}(x_0), f^n(x_0), t) = 1. \tag{3}$$

Now, since $f^n(x_0) \preceq f^{n+1}(x_0) \preceq f^{n+2}(x_0)$, so using the triangle inequality of fuzzy metric space, we have

$$M(f^{n+2}(x_0), f^n(x_0), t+s) \geq T\big(M(f^{n+2}(x_0), f^{n+1}(x_0), t), M(f^{n+1}(x_0), f^n(x_0), s)\big)$$

where, suppose that T is a product $t-$ norm, then

$$M(f^{n+2}(x_0), f^n(x_0), t+s) \geq \big(M(f^{n+2}(x_0), f^{n+1}(x_0), t)\big) * M(f^{n+1}(x_0), f^n(x_0), s) \tag{4}$$

Now, using inequality (2) in right hand side of inequality (4)

$$\begin{aligned}\left(\frac{1}{M(f^{n+2}(x_0), f^{n+1}(x_0), t)} - 1\right) &\leq \psi\left(\frac{1}{M(f^{n+1}(x_0), f^n(x_0), t)} - 1\right)\\ &\leq \psi^{n+1}\left(\frac{1}{M(f(x_0), x_0, t)} - 1\right)\end{aligned}$$

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+2}(x_0), f^{n+1}(x_0), t)} - 1\right) \leq \lim_{n\to\infty}\psi^{n+1}\left(\frac{1}{M\big(f(x_0)\big), x_0, t)} - 1\right).$$

This implies

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^{n+1}(x_0), t) = 1. \tag{5}$$

Using equations (3) and (5) in inequality (4), we get

$$\begin{aligned}\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t+s) &\geq 1 * 1\\ &= 1\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t_1) = 1, \text{ where } t_1 = t+s. \tag{6}$$

Now, since $f^n(x_0) \preceq f^{n+2}(x_0)$, so by the same arguments used to obtained inequality (4) one has

$$M(f^{n+3}(x_0), f^n(x_0), t+s) \geq M\left(f^{n+3}(x_0), f^{n+1}(x_0), \frac{t}{2}\right) * M\left(f^{n+1}(x_0), f^n(x_0), \frac{s}{2}\right) \tag{7}$$

Now,

$$M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) * M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{s}{2}\right) \tag{8}$$

Using inequality (2) in right hand side of inequality (8), we have

$$\left(\frac{1}{M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right)} - 1\right) \leq \psi\left(\frac{1}{M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{t}{2}\right)} - 1\right)$$

$$\leq \psi^{n+2}\left(\frac{1}{M\left(f^{n}(x_0), x_0, \frac{t}{2}\right)} - 1\right)$$

$$\lim_{n\to\infty} M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) = 1. \quad (9)$$

From (5), (8) and (9), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq 1 * 1$$

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) = 1. \quad (10)$$

Using (3) and (10) in (7), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n}(x_0), t+s) = 1. \quad (11)$$

By using principle of Mathematical Induction, we get

$$\lim_{n\to\infty} M(f^{n+k}(x_0), f^{n}(x_0), t+s) = 1 \text{ for } k \geq 1,$$

This implies that $< f^n(x_0) >$ is a Cauchy sequence in X and since (X, M, T) is a complete fuzzy metric space, so there exists $x \in X$ such that

$$\lim_{n\to\infty} M(f^n(x_0), x, t) = 1 \; \forall \; t > 0$$

This implies that $f^n(x_0) = x$. As, it is assumed that f is continuous, so clearly $x = f(x)$. Hence x is fixed point for the mapping f.

Remark 2.2. One can notice that f non-decreasing mapping and can be replaced by f non-increasing in Theorem 2.5, provided $x_0 \preceq f(x_0)$ is replaced by $f(x_0) \preceq x_0$.

Remark 2.3. One can also notice that if $\psi: [0, \infty) \to [0, \infty)$ is continuous mapping with $\psi(t) < t,\ t > 0$, then for all $t > 0$,

$$\lim_{n\to\infty} \psi^n(t) = 0.$$

For fixed $t > 0$, if $\gamma_n = \psi^n(t)$, then $\gamma_n = \psi(\gamma_{n-1}) \leq \gamma_{n-1}$, this implies that γ_n is non-decreasing.

Example 2.4. Let $X = \mathbb{R}$ be a partial order set under the usual order $\leq$ and (X, M, T) be fuzzy metric space. Consider $M(x, y, t) = e^{\frac{-|x-y|}{t}}$ and $\psi(t) = \frac{t}{2}$ is non-decreasing function. Suppose $f(x) = \frac{x+1}{2}$. Clearly these functions satisfy all conditions of Theorem 2.1 and f has a unique fixed point 1.

Theorem 2.5. Let $(X,\preceq)$ be a partially ordered set and (X,M,T) be a complete fuzzy metric space. Assume there is a non-decreasing function $\psi:[0,\infty)\to[0,\infty)$ with

$$\lim_{n\to\infty}\psi^n(t)=0,\quad \text{for each } t>0$$

and also consider that $f: X\to X$ is a non-decreasing mapping such that

i) for all $x\succeq y$, the following contraction holds:

$$\left(\frac{1}{M(f(x),f(y),t)}-1\right) \le \psi\left(\max\left\{\left(\frac{1}{M(x,y,t)}-1\right),\left(\frac{1}{M(x,f(x),t)}-1\right),\left(\frac{1}{M(y,f(y),t)}-1\right),\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right)+\left(\frac{1}{M(y,f(x),t)}-1\right)\right]\right\}\right) \quad (12)$$

ii) and f is continuous.

If there an exists $x_0\in X$ such that $x_0\preceq f(x_0)$, then f has a fixed point.

Proof. For $t>0$, it should be noted first that $\psi(t)<t$. This may be seen if we assume that there is a non-decreasing function ψ and that there exists a positive integer t_0 such that $t_0\le\psi(t_0)$.

$$t_0\le\psi(t_0)\le\psi^2(t_0)\le\cdots\le\psi^n(t_0)$$

This implies

$$t_0\le\psi^n(t_0)\ \text{ for each } n=1,2,\ldots$$

But it is assumed that

$$\lim_{n\to\infty}\psi^n(t_0)=0$$

This implies

$$t_0\le 0$$

which is a contradiction. Therefore $\psi(t)<t$ for $t>0$ and also $\psi(0)=0$.

If x_0 is a fixed point, i.e., $f(x_0)=x_0$, then we are done. So, let us suppose that $f(x_0)\ne x_0$. Since $x_0\preceq f(x_0)$ and f is non-decreasing mapping, we have

$$x_0\preceq f(x_0)\preceq f^2(x_0)\preceq\cdots\preceq f^n(x_0)\preceq\cdots$$

Case 1: If

$$\max\left\{\left(\frac{1}{M(x,y,t)}-1\right),\left(\frac{1}{M(x,f(x),t)}-1\right),\left(\frac{1}{M(y,f(y),t)}-1\right),\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right)+\left(\frac{1}{M(y,f(x),t)}-1\right)\right]\right\} = \left(\frac{1}{M(x,y,t)}-1\right) \quad (13)$$

Then $$\left(\frac{1}{M(f(x),f(y),t)}-1\right)\leq\psi\left(\frac{1}{M(x,y,t)}-1\right).$$

Now, since $x_0 \preceq f(x_0)$, putting $x = f(x_0)$ and $y = x_0$ in inequality (13) we have

$$\left(\frac{1}{M(f^2(x_0),f(x_0),t)}-1\right)\leq\psi\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$

and also, since $f(x_0) \preceq f^2(x_0)$, we have

$$\left(\frac{1}{M(f^3(x_0),f^2(x_0),t)}-1\right)\leq\psi\left(\frac{1}{M(f^2(x_0),f(x_0),t)}-1\right)$$
$$\leq\psi^2\left(\frac{1}{M(f(x_0),x_0,t)}-1\right).$$

By using principle of Mathematical Induction,

$$\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\leq\psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right). \quad (14)$$

But it is assumed that

$$\lim_{n\to\infty}\psi^n(t)=0 \text{ for all } t>0.$$

So, as (14) is reduces to

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\leq 0.$$

This implies $$\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)\geq 1.$$

It is well-known that $M(x,y,t)$ lies in the interval [0,1], therefore

$$\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)=1 \quad (15)$$

Now, since $f^n(x_0) \preceq f^{n+1}(x_0) \preceq f^{n+2}(x_0)$, so using the triangle inequality of fuzzy metric space, we have

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\geq T\big(M(f^{n+2}(x_0),f^{n+1}(x_0),t),M(f^{n+1}(x_0),f^n(x_0),s)\big)$$

where, suppose that T is a product t - norm, then

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\geq\big(M(f^{n+2}(x_0),f^{n+1}(x_0),t)\big)*M(f^{n+1}(x_0),f^n(x_0),s) \quad (16)$$

Now, using inequality (14) in right hand side of inequality (16), we have

$$\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\leq\psi\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)$$
$$\leq\psi^{n+1}\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$

This implies $$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\leq\lim_{n\to\infty}\psi^{n+1}\left(\frac{1}{M(f(x_0)),x_0,t)}-1\right)$$

Thus

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^{n+1}(x_0), t) = 1. \tag{17}$$

Using equations (15) and (17) in inequality (16), we get

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^{n}(x_0), t+s) \geq 1 * 1$$
$$= 1$$

Therefore,

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^{n}(x_0), t_1) = 1 \text{ where } t_1 = t + s \tag{18}$$

Now, since $f^n(x_0) \preceq f^{n+2}(x_0)$, so by the same arguments used to obtained inequality (16) one has

$$M(f^{n+3}(x_0), f^{n}(x_0), t+s) \geq M\left(f^{n+3}(x_0), f^{n+1}(x_0), \frac{t}{2}\right) * M\left(f^{n+1}(x_0), f^{n}(x_0), \frac{s}{2}\right). \tag{19}$$

Now,

$$M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) * M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{s}{2}\right) \tag{20}$$

Using inequality (14) in right hand side of inequality (20), one obtains

$$\left(\frac{1}{M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right)} - 1\right) \leq \psi\left(\frac{1}{M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{t}{2}\right)} - 1\right)$$
$$\leq \psi^{n+2}\left(\frac{1}{M\left(f^{n}(x_0), x_0, \frac{t}{2}\right)} - 1\right)$$

$$\left(\frac{1}{M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right)} - 1\right) \leq \psi\left(\frac{1}{M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{t}{2}\right)} - 1\right)$$
$$\leq \psi^{n+2}\left(\frac{1}{M\left(f^{n}(x_0), x_0, \frac{t}{2}\right)} - 1\right)$$

$$\lim_{n\to\infty} M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) = 1. \tag{21}$$

From (17), (20) and (21), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq 1 * 1$$
$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) = 1. \tag{22}$$

Using (15) and (22) in (19), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n}(x_0), t+s) = 1. \tag{23}$$

By using principle of Mathematical Induction, we get

$$\lim_{n\to\infty} M(f^{n+k}(x_0), f^n(x_0), t+s) = 1 \text{ for } k \geq 1$$

This implies that $< f^n(x_0) >$ is a Cauchy sequence in X and since (X, M, T) is a complete fuzzy metric space, so there exists $x \in X$ such that

$$\lim_{n\to\infty} M(f^n(x_0), x, t) = 1 \; \forall \; t > 0.$$

This implies that $f^n(x_0) = x$. As, it is assumed that f is continuous, so clearly $x = f(x)$. Hence x is fixed point for the mapping f.

Case 2: If

$$\max\left\{\left(\frac{1}{M(x,y,t)}-1\right),\left(\frac{1}{M(x,f(x),t)}-1\right),\left(\frac{1}{M(y,f(y),t)}-1\right),\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right) + \left(\frac{1}{M(y,f(x),t)}-1\right)\right]\right\} = \left(\frac{1}{M(x,f(x),t)}-1\right) \quad (24)$$

Then $$\left(\frac{1}{M(f(x),f(y),t)}-1\right) \leq \psi\left(\left(\frac{1}{M(x,f(x),t)}-1\right)\right).$$

Putting $x = f(x_0)$ and $y = x_0$ in inequality (24) we have

$$\left(\frac{1}{M(f^2(x_0),f(x_0),t)}-1\right) \leq \psi\left(\frac{1}{M(f(x_0),f^2(x_0),t)}-1\right)$$

and also, since $f(x_0) \leq f^2(x_0)$, we have

$$\left(\frac{1}{M(f^3(x_0),f^2(x_0),t)}-1\right) \leq \psi\left(\frac{1}{M(f^2(x_0),f(x_0),t)}-1\right)$$
$$\leq \psi^2\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$

By using principle of Mathematical Induction,

$$\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right) \leq \psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$

$$\operatorname*{Lim}_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right) \leq \lim_{n\to\infty}\psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right) \quad (25)$$

But it is assumed that

$$\lim_{n\to\infty}\psi^n(t) = 0 \text{ for all } t > 0.$$

Therefore, the inequality (25) turns out to be

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right) \leq 0.$$

This implies $$\lim_{n\to\infty} M(f^{n+1}(x_0), f^n(x_0), t) \geq 1.$$

It is well-known that $M(x, y, t)$ lies in the interval [0,1], therefore

$$\lim_{n\to\infty} M(f^{n+1}(x_0), f^n(x_0), t) = 1 \tag{26}$$

Now, since $f^n(x_0) \preceq f^{n+1}(x_0) \preceq f^{n+2}(x_0)$, so using the triangle inequality of fuzzy metric space, we have

$$M(f^{n+2}(x_0), f^n(x_0), t+s) \geq T\big(M(f^{n+2}(x_0), f^{n+1}(x_0), t), M(f^{n+1}(x_0), f^n(x_0), s)\big)$$

where, suppose that T is a product t -norm, then

$$M(f^{n+2}(x_0), f^n(x_0), t+s) \geq \big(M(f^{n+2}(x_0), f^{n+1}(x_0), t)\big) * M(f^{n+1}(x_0), f^n(x_0), s) \tag{27}$$

Now, using inequality (25) in right hand side of inequality (27), we have

$$\begin{aligned}\left(\frac{1}{M(f^{n+2}(x_0), f^{n+1}(x_0), t)} - 1\right) &\leq \psi\left(\frac{1}{M(f^{n+1}(x_0), f^n(x_0), t)} - 1\right) \\ &\leq \psi^{n+1}\left(\frac{1}{M(f(x_0), x_0, t)} - 1\right)\end{aligned}$$

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+2}(x_0), f^{n+1}(x_0), t)} - 1\right) \leq \lim_{n\to\infty}\psi^{n+1}\left(\frac{1}{M\big(f(x_0)\big), x_0, t)} - 1\right).$$

This implies

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^{n+1}(x_0), t) = 1 \tag{28}$$

Using equations (26) and (28) in inequality (27), we get

$$\begin{aligned}\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t+s) &\geq 1 * 1 \\ &= 1.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t_1) = 1, \text{ where } t_1 = t+s. \tag{29}$$

Now, since $f^n(x_0) \preceq f^{n+2}(x_0)$, so by the same arguments used to obtained inequality (27) one has

$$M(f^{n+3}(x_0), f^n(x_0), t+s) \geq M\left(f^{n+3}(x_0), f^{n+1}(x_0), \frac{t}{2}\right) * M\left(f^{n+1}(x_0), f^n(x_0), \frac{s}{2}\right) \tag{30}$$

Now,

$$M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) * M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{s}{2}\right) \tag{31}$$

Using inequality (25) in right hand side of inequality (31), one obtains

$$\left(\frac{1}{M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right)} - 1\right) \leq \psi\left(\frac{1}{M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{t}{2}\right)} - 1\right)$$

$$\leq \psi^{n+2}\left(\frac{1}{M\left(f^{n}(x_0), x_0, \frac{t}{2}\right)} - 1\right)$$

$$\lim_{n\to\infty} M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) = 1. \tag{32}$$

From (28), (31) and (32), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq 1 * 1.$$

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) = 1\,. \tag{33}$$

Using (26) and (33) in (30), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n}(x_0), t+s) = 1 \tag{34}$$

By using principle of Mathematical Induction, we get

$$\lim_{n\to\infty} M(f^{n+k}(x_0), f^{n}(x_0), t+s) = 1 \text{ for } k \geq 1.$$

This implies that $< f^n(x_0) >$ is a Cauchy sequence in X and since (X, M, T) is a complete fuzzy metric space, so there exists $x \in X$ such that

$$\lim_{n\to\infty} M(f^{n}(x_0), x, t) = 1 \;\forall\; t > 0$$

This implies that $f^n(x_0) = x$. As, it is assumed that f is continuous, so clearly $x = f(x)$. Hence x is fixed point for the mapping f.

Case 3: If

$$\max\left\{\left(\frac{1}{M(x,y,t)} - 1\right), \left(\frac{1}{M(x,f(x),t)} - 1\right), \left(\frac{1}{M(y,f(y),t)} - 1\right), \frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)} - 1\right) + \left(\frac{1}{M(y,f(x),t)} - 1\right)\right]\right\} = \left(\frac{1}{M(y,f(y),t)} - 1\right) \tag{35}$$

then $\left(\frac{1}{M(f(x),f(y),t)} - 1\right) \leq \psi\left(\frac{1}{M(y,f(y),t)} - 1\right)$

Putting $x = f(x_0)$ and $y = x_0$ in inequality (35), we have

$$\left(\frac{1}{M(f^2(x_0), f(x_0), t)} - 1\right) \leq \psi\left(\frac{1}{M(x_0), f(x_0), t)} - 1\right)$$

and also, since $f(x_0) \preceq f^2(x_0)$, we have

$$\left(\frac{1}{M(f^3(x_0),f^2(x_0),t)}-1\right)\le \psi\left(\frac{1}{M(f(x_0),f^2(x_0),t)}-1\right)$$
$$\le \psi^2\left(\frac{1}{M(x_0),f(x_0),t)}-1\right).$$

By using principle of Mathematical Induction,

$$\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\le \psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$
$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\le \lim_{n\to\infty}\psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right) \tag{36}$$

But it is assumed that

$$\lim_{n\to\infty}\psi^n(t)=0 \text{ for all } t>0.$$

Therefore, the inequality (36) turns out to be

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\le 0.$$

This implies $\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)\ge 1.$

It is well-known that $M(x,y,t)$ lies in the interval [0,1], therefore

$$\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)=1. \tag{37}$$

Now, since $f^n(x_0)\le f^{n+1}(x_0)\le f^{n+2}(x_0)$, so using the triangle inequality of fuzzy metric space, we have

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\ge T\big(M(f^{n+2}(x_0),f^{n+1}(x_0),t),M(f^{n+1}(x_0),f^n(x_0),s)\big)$$

where, suppose that T is a product t- norm, then

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\ge \big(M(f^{n+2}(x_0),f^{n+1}(x_0),t)\big)*M(f^{n+1}(x_0),f^n(x_0),s). \tag{38}$$

Now, using inequality (36) in right hand side of inequality (38)

$$\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\le \psi\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)$$
$$\le \psi^{n+1}\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$
$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\le \lim_{n\to\infty}\psi^{n+1}\left(\frac{1}{M\big(f(x_0)\big),x_0,t)}-1\right).$$

This implies

$$\lim_{n\to\infty}M(f^{n+2}(x_0),f^{n+1}(x_0),t)=1. \tag{39}$$

Using equations (37) and (39) in inequality (38), we get

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t+s) \geq 1 * 1$$
$$= 1.$$

Therefore,

$$\lim_{n\to\infty} M(f^{n+2}(x_0), f^n(x_0), t_1) = 1, \text{ where } t_1 = t+s. \tag{40}$$

Now, since $f^n(x_0) \preceq f^{n+2}(x_0)$, so by the same arguments used to obtained inequality (38) one has

$$M(f^{n+3}(x_0), f^n(x_0), t+s) \geq M\left(f^{n+3}(x_0), f^{n+1}(x_0), \frac{t}{2}\right) * M\left(f^{n+1}(x_0), f^n(x_0), \frac{s}{2}\right). \tag{41}$$

Now, $M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) * M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{s}{2}\right)$. (42)

Using inequality (36) in right hand side of inequality (42)

$$\left(\frac{1}{M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right)} - 1\right) \leq \psi\left(\frac{1}{M\left(f^{n+2}(x_0), f^{n+1}(x_0), \frac{t}{2}\right)} - 1\right)$$
$$\leq \psi^{n+2}\left(\frac{1}{M\left(f^n(x_0), x_0, \frac{t}{2}\right)} - 1\right)$$

$$\lim_{n\to\infty} M\left(f^{n+3}(x_0), f^{n+2}(x_0), \frac{t}{2}\right) = 1. \tag{43}$$

From (39), (42) and (43), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) \geq 1 * 1$$
$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^{n+1}(x_0), t) = 1 \tag{44}$$

Using (37) and (44) in (41), we get

$$\lim_{n\to\infty} M(f^{n+3}(x_0), f^n(x_0), t+s) = 1. \tag{45}$$

By using principle of Mathematical Induction, we get

$$\lim_{n\to\infty} M(f^{n+k}(x_0), f^n(x_0), t+s) = 1 \text{ for } k \geq 1.$$

This implies that $< f^n(x_0) >$ is a Cauchy sequence in X and since (X, M, T) is a complete fuzzy metric space, so there exists $x \in X$ such that

$$\lim_{n\to\infty} M(f^n(x_0), x, t) = 1 \ \forall\, t > 0$$

This implies that $f^n(x_0) = x$. As, it is assumed that f is continuous, so clearly $x = f(x)$. Hence x is fixed point for the mapping f.

Case 4: If

$$\max\left\{\left(\frac{1}{M(x,y,t)}-1\right),\left(\frac{1}{M(x,f(x),t)}-1\right),\left(\frac{1}{M(y,f(y),t)}-1\right),\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right)\right.\right.$$
$$\left.\left.+\left(\frac{1}{M(y,f(x),t)}-1\right)\right]\right\}$$
$$=\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right)+\left(\frac{1}{M(y,f(x),t)}-1\right)\right], \tag{46}$$

then $\left(\frac{1}{M(f(x),f(y),t)}-1\right)\leq\psi\left(\frac{1}{2}\left[\left(\frac{1}{M(x,f(y),t)}-1\right)+\left(\frac{1}{M(y,f(x),t)}-1\right)\right]\right)$

Putting $x=f(x_0)$ and $y=x_0$ in inequality (46), we have

$$\left(\frac{1}{M(f^2(x_0),f(x_0),t)}-1\right)\leq\psi\left(\frac{1}{2}\left[\left(\frac{1}{M(f(x_0),f(x_0),t)}-1\right)+\left(\frac{1}{M(x_0,f^2(x_0),t)}-1\right)\right]\right)$$
$$=\psi\left(\frac{1}{2}\left(\frac{1}{M(x_0,f^2(x_0),t)}-1\right)\right)\leq\psi^2\left(\frac{1}{M(x_0),f(x_0),t)}-1\right).$$

By using principle of Mathematical Induction,

$$\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\leq\psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$
$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\leq\lim_{n\to\infty}\psi^n\left(\frac{1}{M(f(x_0),x_0,t)}-1\right) \tag{47}$$

But it is assumed that

$$\lim_{n\to\infty}\psi^n(t)=0 \text{ for all } t>0.$$

Therefore, the inequality (47) turns out to be

$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+1}(x_0),f^n(x_0),t)}-1\right)\leq 0.$$

This implies $\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)\geq 1.$

It is well-known that $M(x,y,t)$ lies in the interval [0,1], therefore

$$\lim_{n\to\infty}M(f^{n+1}(x_0),f^n(x_0),t)=1 \tag{48}$$

Now, since $f^n(x_0)\preceq f^{n+1}(x_0)\preceq f^{n+2}(x_0)$, so using the triangle inequality of fuzzy metric space, we have

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\geq T\big(M(f^{n+2}(x_0),f^{n+1}(x_0),t),M(f^{n+1}(x_0),f^n(x_0),s)\big)$$

where, suppose that T is a product t-norm, then

$$M(f^{n+2}(x_0),f^n(x_0),t+s)\geq\big(M(f^{n+2}(x_0),f^{n+1}(x_0),t)\big)*M(f^{n+1}(x_0),f^n(x_0),s). \tag{49}$$

Now, using inequality (47) in right hand side of inequality (49)

$$\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\leq\psi\left(\frac{1}{M(f^{n+1}(x_0),f^{n}(x_0),t)}-1\right)$$
$$\leq\psi^{n+1}\left(\frac{1}{M(f(x_0),x_0,t)}-1\right)$$
$$\lim_{n\to\infty}\left(\frac{1}{M(f^{n+2}(x_0),f^{n+1}(x_0),t)}-1\right)\leq\lim_{n\to\infty}\psi^{n+1}\left(\frac{1}{M\big(f(x_0)\big),x_0,t)}-1\right).$$

This implies

$$\lim_{n\to\infty}M(f^{n+2}(x_0),f^{n+1}(x_0),t)=1. \tag{50}$$

Using equations (48) and (50) in inequality (49), we get

$$\lim_{n\to\infty}M(f^{n+2}(x_0),f^{n}(x_0),t+s)\geq 1*1$$
$$=1.$$

Therefore,

$$\lim_{n\to\infty}M(f^{n+2}(x_0),f^{n}(x_0),t_1)=1,\ \text{ where } t_1=t+s. \tag{51}$$

Now, since $f^n(x_0)\leq f^{n+2}(x_0)$, so by the same arguments used to obtained inequality (49) one has

$$M(f^{n+3}(x_0),f^{n}(x_0),t+s)\geq M\left(f^{n+3}(x_0),f^{n+1}(x_0),\frac{t}{2}\right)*M\left(f^{n+1}(x_0),f^{n}(x_0),\frac{s}{2}\right) \tag{52}$$

Now, $M(f^{n+3}(x_0),f^{n+1}(x_0),t)\geq M\left(f^{n+3}(x_0),f^{n+2}(x_0),\frac{t}{2}\right)*M\left(f^{n+2}(x_0),f^{n+1}(x_0),\frac{s}{2}\right)$ (53)

Using inequality (47) in right hand side of inequality (53)

$$\left(\frac{1}{M\left(f^{n+3}(x_0),f^{n+2}(x_0),\frac{t}{2}\right)}-1\right)\leq\psi\left(\frac{1}{M\left(f^{n+2}(x_0),f^{n+1}(x_0),\frac{t}{2}\right)}-1\right)$$
$$\leq\psi^{n+2}\left(\frac{1}{M\left(f^{n}(x_0),x_0,\frac{t}{2}\right)}-1\right)$$
$$\lim_{n\to\infty}M\left(f^{n+3}(x_0),f^{n+2}(x_0),\frac{t}{2}\right)=1. \tag{54}$$

From (50), (53) and (58), we get

$$\lim_{n\to\infty}M(f^{n+3}(x_0),f^{n+1}(x_0),t)\geq 1*1$$
$$\lim_{n\to\infty}M(f^{n+3}(x_0),f^{n+1}(x_0),t)=1. \tag{55}$$

Using (48) and (55) in (52), we get

$$\lim_{n\to\infty}M(f^{n+3}(x_0),f^{n}(x_0),t+s)=1. \tag{56}$$

By using principle of Mathematical Induction, we get

$$\lim_{n\to\infty} M(f^{n+k}(x_0), f^n(x_0), t+s) = 1 \text{ for } k \geq 1$$

This implies that $< f^n(x_0) >$ is a Cauchy sequence in X and since (X, M, T) is a complete fuzzy metric space, so there exists $x \in X$ such that

$$\lim_{n\to\infty} M(f^n(x_0), x, t) = 1 \; \forall \; t > 0$$

This implies that $f^n(x_0) = x$. As, it is assumed that f is continuous, so clearly $x = f(x)$. Hence x is fixed point for the mapping f.

Example 2.6. Let $X = [0,1]$ be a partial order set under the usual order $\leq$ and (X, M, T) be complete fuzzy metric space. Consider $M(x, y, t) = e^{-t|x-y|}$ and $\psi(t) = \frac{t}{2}$ is non-decreasing function. Suppose $f(x) = \frac{x}{2}$. Consequently, these functions satisfy all conditions of Theorem 2.2 and f has a unique fixed point 0.

Conflicts of interest: Authors do not have conflicts of interest.

Acknowledgement: The authors express their gratitude to anonymous referees for careful reading of the manuscript and for providing valuable suggestions. The first author is thankful to the Council of Scientific & Industrial Research (CSIR) for providing the research grant vide sanctioned letter number 09/0862(16414)/2023 - EMR - I

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