Generalized g–Fractional vector Representation Formula and integral Inequalities for Banach space valued functions

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Abstract

Here we give a very general fractional Bochner integral representation formula for Banach space valued functions. We derive generalized left and righ fractional Opial type inequalities, fractional Ostrowski type inequalities and fractional Grüss type inequalities. All these inequalities are very general having in their background Bochner type integrals.

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1 Background

We need

Definition 1 *([2]) Let* $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ *a Banach space,* $g \in C^1([a,b])$ *and increasing,* $f \in C([a,b], X)$, $\nu > 0$.

We define the left Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I_{a+;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (g(x) - g(z))^{\nu-1} g'(z) f(z) \, dz, \tag{1}$$

$\forall\, x \in [a,b]$, *where* Γ *is the gamma function.*

The last integral is of Bochner type. Since $f \in C([a,b], X)$, *then* $f \in L_\infty([a,b], X)$. *By [2] we get that* $I_{a+;g}^{\nu} f \in C([a,b], X)$. *Above we set* $I_{a+;g}^{0} f := f$ *and see that* $\left(I_{a+;g}^{\nu} f\right)(a) = 0$.

When g is the identity function id, we get that $I^{\nu}_{a+;id} = I^{\nu}_{a+}$, the ordinary left Riemann-Liouville fractional integral

$$\left(I^{\nu}_{a+}f\right)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\,dt, \tag{2}$$

$\forall\ x \in [a,b]$, $\left(I^{\nu}_{a+}f\right)(a) = 0$.

We need

Theorem 2 *([2]) Let $\mu, \nu > 0$ and $f \in C([a,b], X)$. Then*

$$I^{\mu}_{a+;g} I^{\nu}_{a+;g} f = I^{\mu+\nu}_{a+;g} f = I^{\nu}_{a+;g} I^{\mu}_{a+;g} f. \tag{3}$$

We need

Definition 3 *([2]) Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$.*

We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I^{\nu}_{b-;g}f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (g(z) - g(x))^{\nu-1} g'(z) f(z)\,dz, \tag{4}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_{\infty}([a,b], X)$. By [2] we get that $I^{\nu}_{b-;g} f \in C([a,b], X)$. Above we set $I^{0}_{b-;g} f := f$ and see that $\left(I^{\nu}_{b-;g} f\right)(b) = 0$.

When g is the identity function id, we get that $I^{\nu}_{b-;id} = I^{\nu}_{b-}$, the ordinary right Riemann-Liouville fractional integral

$$\left(I^{\nu}_{b-}f\right)(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)\,dt, \tag{5}$$

$\forall\ x \in [a,b]$, with $\left(I^{\nu}_{b-}f\right)(b) = 0$.

We need

Theorem 4 *([2]) Let $\mu, \nu > 0$ and $f \in C([a,b], X)$. Then*

$$I^{\mu}_{b-;g} I^{\nu}_{b-;g} f = I^{\mu+\nu}_{b-;g} f = I^{\nu}_{b-;g} I^{\mu}_{b-;g} f. \tag{6}$$

We will use

Definition 5 *([2]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha}f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t))^{n-\alpha-1} g'(t) \left(f\circ g^{-1}\right)^{(n)}(g(t))\,dt, \tag{7}$$

$\forall\ x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{a+;g}^{\alpha}f\right) \in C([a,b],X)$.

We see that

$$\left(I_{a+;g}^{n-\alpha}\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha}f\right)(x), \quad \forall\ x\in[a,b]. \tag{8}$$

We set

$$D_{a+;g}^{n}f(x) := \left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(x) \in C([a,b],X), \quad n\in\mathbb{N}, \tag{9}$$

$$D_{a+;g}^{0}f(x) = f(x), \quad \forall\ x\in[a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha}f = D_{a+;id}^{\alpha}f = D_{*a}^{\alpha}f, \tag{10}$$

the usual left X-valued Caputo fractional derivative, see [3].

We will use

Definition 6 *([2]) Let $\alpha > 0$, $\lceil\alpha\rceil = n$, $\lceil\cdot\rceil$ the ceiling of the number. Let $f \in C^n([a,b],X)$, where $[a,b]\subset\mathbb{R}$, and $(X,\|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^n([g(a),g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t)-g(x))^{n-\alpha-1} g'(t) \left(f\circ g^{-1}\right)^{(n)}(g(t))\,dt, \tag{11}$$

$\forall\ x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{b-;g}^{\alpha}f\right) \in C([a,b],X)$.

We see that

$$I_{b-;g}^{n-\alpha}\left((-1)^n\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x), \quad a\le x\le b. \tag{12}$$

We set

$$D_{b-;g}^{n}f(x) := (-1)^n\left(\left(f\circ g^{-1}\right)^{n}\circ g\right)(x) \in C([a,b],X), \quad n\in\mathbb{N}, \tag{13}$$

$$D_{b-;g}^{0}f(x) := f(x), \quad \forall\ x\in[a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha}f(x) = D_{b-;id}^{\alpha}f(x) = D_{b-}^{\alpha}f, \tag{14}$$

the usual right X-valued Caputo fractional derivative, see [3].

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We make

Remark 7 *All as in Definition 5. We have (by Theorem 2.5, p. 7, [5])*

$$\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(g(x)-g(t))^{n-\alpha-1}g'(t)\left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\|dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(a)}^{g(x)}(g(x)-g(t))^{n-\alpha-1}dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(x)-g(a))^{n-\alpha}. \tag{15}$$

That is

$$\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(x)-g(a))^{n-\alpha}, \tag{16}$$

$\forall\, x\in[a,b]$.

If $\alpha\notin\mathbb{N}$, then $\left(D_{a+;g}^{\alpha}f\right)(a)=0$.

Similarly, by Definition 6 we derive

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_{x}^{b}(g(t)-g(x))^{n-\alpha-1}g'(t)\left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\|dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(x)}^{g(b)}(g(t)-g(x))^{n-\alpha-1}dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(b)-g(x))^{n-\alpha}. \tag{17}$$

That is

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)}(g(b)-g(x))^{n-\alpha}, \tag{18}$$

$\forall\, x\in[a,b]$.

If $\alpha\notin\mathbb{N}$, then $\left(D_{b-;g}^{\alpha}f\right)(b)=0$.

Notation 8 *We denote by*

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha}\quad (n\ \text{times}),\ n\in\mathbb{N}, \tag{19}$$

$$I_{a+;g}^{n\alpha} := I_{a+;g}^{\alpha}I_{a+;g}^{\alpha}...I_{a+;g}^{\alpha}, \tag{20}$$

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha}D_{b-;g}^{\alpha}...D_{b-;g}^{\alpha}, \tag{21}$$

and

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha}I_{b-;g}^{\alpha}...I_{b-;g}^{\alpha}, \tag{22}$$

(n times), $n\in\mathbb{N}$.

We are motivated by the following generalized fractional Ostrowski type inequality:

Theorem 9 *([2]) Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$, and $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space. Let $x_0 \in [a,b]$ be fixed. Assume that $F_k^{x_0} := D_{x_0-;g}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a, x_0], X)$ and $\left(D_{x_0-;g}^{i\alpha} f\right)(x_0) = 0$, $i = 2, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0, b], X)$ and $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 2, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha} (b - x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha} (x_0 - a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}. \quad (23)$$

In this work we will present several generalized fractional Bochner integral inequalities.

We mention the following g-left generalized X-valued Taylor's formula:

Theorem 10 *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\,dt =$$

$$f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \quad (24)$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\,dz, \quad \forall\, x \in [a,b].$$

We mention the following g-right generalized X-valued Taylor's formula:

Theorem 11 *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) +$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt =$$
$$f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+ \tag{25}$$
$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,\quad \forall\ x\in[a,b].$$

For the Bochner integral excellent resources are [4], [6], [7] and [1], pp. 422-428.

2 Main Results

We give the following representation formula:

Theorem 12 *All as in Theorem 10. Then*

$$f(y)=\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)}\int_a^b (g(x)-g(y))^k\,dx+R_1(y), \tag{26}$$

for any $y\in[a,b]$, where

$$R_1(y)=-\frac{1}{\Gamma(\alpha)(b-a)}$$
$$\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right.$$
$$\left.\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right]. \tag{27}$$

here χ_A stands for the characteristic function set A, where A is an arbitrary set.

One may write also that

$$R_1(y)=-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right. \tag{28}$$
$$\left.+\int_y^b\left(\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right],$$

for any $y\in[a,b]$.

Putting things together, one has

$$f(y)=\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)}\int_a^b (g(x)-g(y))^k\,dx$$

$$-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt\right)dx\right.$$

$$\left.\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt\right)dx\right]. \qquad (29)$$

In particular, one has

$$f(y)-\frac{1}{b-a}\int_a^b f(x)\,dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)}\int_a^b (g(x)-g(y))^k\,dx = R_1(y), \qquad (30)$$

for any $y \in [a,b]$.

Proof. Here $x, y \in [a,b]$. We keep y as fixed.
By Theorem 10 we get:

$$f(x) = f(y)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k + \qquad (31)$$

$$\frac{1}{\Gamma(\alpha)}\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt, \quad \text{for any } x \geq y.$$

By Theorem 11 we get:

$$f(x) = f(y)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k + \qquad (32)$$

$$\frac{1}{\Gamma(\alpha)}\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt, \quad \text{for any } x \leq y.$$

By (31), (32) we notice that

$$\int_a^b f(x)\,dx = \int_a^y f(x)\,dx + \int_y^b f(x)\,dx = \qquad (33)$$

$$\int_a^y f(y)\,dx + \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}\int_a^y (g(x)-g(y))^k\,dx +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^y\left(\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt\right)dx +$$

$$\int_y^b f(y)\,dx + \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}\int_y^b (g(x)-g(y))^k\,dx +$$

$$\frac{1}{\Gamma(\alpha)}\int_y^b\left(\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt\right)dx.$$

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Hence it holds

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(y) + \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx+$$
$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx+\right.$$
$$\left.\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right]. \tag{34}$$

Therefore we obtain

$$f(y) = \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx-$$
$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx+\right.$$
$$\left.\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right]. \tag{35}$$

Hence the remainder

$$R_1(y) := -\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right.$$
$$\left.+\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right] =$$
$$-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right.$$
$$\left.+\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right]. \tag{36}$$

The theorem is proved. ■

Next we present a left fractional Opial type inequality:

Theorem 13 *All as in Theorem 10. Additionally assume that* $\alpha \geq 1$, $g \in C^1([a,b])$, *and* $\left(f\circ g^{-1}\right)^{(k)}(g(a)) = 0$, *for* $k = 0,1,...,n-1$. *Let* $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. *Then*

$$\int_a^x \|f(w)\|\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\| g'(w)\,dw \leq \frac{1}{\Gamma(\alpha)\,2^{\frac{1}{q}}}\cdot \tag{37}$$

$$\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x (g'(w))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\|^q dw\right)^{\frac{2}{q}},$$

$\forall\ x \in [a,b]$.

Proof. By Theorem 10, we have that

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{a+;g}^{\alpha}f\right)(t)\,dt, \quad \forall\, x\in[a,b]. \tag{38}$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\|\leq\frac{1}{\Gamma(\alpha)}\left(\int_a^x (g(x)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \tag{39}$$

Call

$$z(x):=\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt, \tag{40}$$

$z(a)=0$.

Thus

$$z'(x)=(g'(x))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\|^q\geq 0, \tag{41}$$

and

$$(z'(x))^{\frac{1}{q}}=g'(x)\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\|\geq 0, \quad \forall\, x\in[a,b]. \tag{42}$$

Consequently, we get

$$\|f(w)\|\, g'(w)\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\|\leq \tag{43}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}(z(w)\,z'(w))^{\frac{1}{q}}, \quad \forall\, w\in[a,b].$$

Then

$$\int_a^x \|f(w)\|\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\| g'(w)\,dw\leq \tag{44}$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}(z(w)\,z'(w))^{\frac{1}{q}}dw\leq$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x z(w)\,z'(w)\,dw\right)^{\frac{1}{q}}= \tag{45}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\cdot$$

$$\left(\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt\right)^{\frac{2}{q}}\cdot 2^{-\frac{1}{q}}. \tag{46}$$

The theorem is proved. ■

We also give a right fractional Opial type inequality:

Theorem 14 *All as in Theorem 11. Additionally assume that* $\alpha \geq 1$*,* $g \in C^1([a,b])$*, and* $\left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$*,* $k = 0, 1, ..., n-1$*. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| g'(w)\, dw \leq \frac{1}{2^{\frac{1}{q}}\Gamma(\alpha)} \cdot \tag{47}$$

$$\left(\int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)}\, dt\right) dw\right)^{\frac{1}{p}} \left(\int_x^b (g'(w))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\|^q dw\right)^{\frac{2}{q}},$$

all $a \leq x \leq b$*.*

Proof. By Theorem 11, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \quad \text{all } a \leq x \leq b. \tag{48}$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (g(t) - g(x))^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} \left(\int_x^b (g'(t))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \tag{49}$$

Call

$$z(x) := \int_x^b (g'(t))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(t)\right\|^q dt, \tag{50}$$

$z(b) = 0$.

Hence

$$z'(x) = -(g'(x))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\|^q \leq 0, \tag{51}$$

and

$$-z'(x) = (g'(x))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\|^q \geq 0, \tag{52}$$

and

$$(-z'(x))^{\frac{1}{q}} = g'(x) \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\| \geq 0, \quad \forall\, x \in [a,b]. \tag{53}$$

Consequently, we get

$$\|f(w)\| g'(w) \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} (z(w)(-z'(w)))^{\frac{1}{q}}, \quad \forall\, w \in [a,b]. \tag{54}$$

Then

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| g'(w)\, dw \leq \tag{55}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} (-z(w) z'(w))^{\frac{1}{q}}\, dw \leq$$

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$$\frac{1}{\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(-\int_x^b z(w)\,z'(w)\,dw\right)^{\frac{1}{q}} = \tag{56}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}} =$$

$$\frac{1}{2^{\frac{1}{q}}\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\cdot$$

$$\left(\int_x^b (g'(t))^q\left\|\left(D^{\alpha}_{b-;g}f\right)(t)\right\|^q dt\right)^{\frac{2}{q}}. \tag{57}$$

The theorem is proved. ■

Two extreme fractional Opial type inequalities follow (case $p=1$, $q=\infty$).

Theorem 15 *All as in Theorem 10. Assume that* $\left(f\circ g^{-1}\right)^{(k)}(g(a)) = 0$, $k=0,1,...,n-1$. *Then*

$$\int_a^x \|f(w)\|\left\|D^{\alpha}_{a+;g}f(w)\right\|dw \le \frac{\left\|D^{\alpha}_{a+;g}f\right\|^2_{\infty}}{\Gamma(\alpha+1)}\left(\int_a^x (g(w)-g(a))^{\alpha}\,dw\right), \tag{58}$$

all $a\le x\le b$.

Proof. For any $w\in[a,b]$, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_a^w (g(w)-g(t))^{\alpha-1}g'(t)\left(D^{\alpha}_{a+;g}f\right)(t)\,dt, \tag{59}$$

and

$$\|f(x)\| \le \frac{1}{\Gamma(\alpha)}\left(\int_a^w (g(w)-g(t))^{\alpha-1}g'(t)\,dt\right)\left\|D^{\alpha}_{a+;g}f\right\|_{\infty}$$

$$= \frac{\left\|D^{\alpha}_{a+;g}f\right\|_{\infty}}{\Gamma(\alpha+1)}(g(w)-g(a))^{\alpha}. \tag{60}$$

Hence we obtain

$$\|f(w)\|\left\|D^{\alpha}_{a+;g}f(w)\right\| \le \frac{\left\|D^{\alpha}_{a+;g}f\right\|^2_{\infty}}{\Gamma(\alpha+1)}(g(w)-g(a))^{\alpha}. \tag{61}$$

Integrating (61) over $[a,x]$ we derive (58). ■

Theorem 16 *All as in Theorem 11. Assume that* $\left(f\circ g^{-1}\right)^{(k)}(g(b)) = 0$, $k = 0,1,...,n-1$. *Then*

$$\int_x^b \|f(w)\|\left\|D^{\alpha}_{b-;g}f(w)\right\|dw \le \frac{\left\|D^{\alpha}_{b-;g}f\right\|^2_{\infty}}{\Gamma(\alpha+1)}\left(\int_x^b (g(b)-g(w))^{\alpha}\,dw\right), \tag{62}$$

all $a\le x\le b$.

Proof. For any $w \in [a,b]$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_w^b (g(t) - g(w))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \tag{63}$$

and

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_w^b (g(t) - g(w))^{\alpha-1} g'(t)\, dt \right) \left\|D_{b-;g}^{\alpha} f\right\|_{\infty}$$

$$= \frac{\left\|D_{b-;g}^{\alpha} f\right\|_{\infty}}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \tag{64}$$

Hence we obtain

$$\|f(w)\| \left\|D_{b-;g}^{\alpha} f(w)\right\| \leq \frac{\left\|D_{b-;g}^{\alpha} f\right\|_{\infty}^2}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \tag{65}$$

Integrating (65) over $[x,b]$ we derive (62). ■

Next we present three fractional Ostrowski type inequalities:

Theorem 17 *All as in Theorem 10. Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x)\, dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(y))}{k!\,(b-a)} \int_a^b (g(x) - g(y))^k\, dx \right\| \leq \tag{66}$$

$$\frac{1}{\Gamma(\alpha+1)(b-a)} \cdot$$

$$\left[(g(y) - g(a))^{\alpha} (y-a) \left\|D_{y-;g}^{\alpha} f\right\|_{\infty} + (g(b) - g(y))^{\alpha} (b-y) \left\|D_{y+;g}^{\alpha} f\right\|_{\infty} \right],$$

$\forall\, y \in [a,b]$.

Proof. Define

$$\left(D_{y+;g}^{\alpha} f\right)(t) = 0, \ \text{ for } t < y, \quad \text{and} \quad \left(D_{y-;g}^{\alpha} f\right)(t) = 0, \ \text{ for } t > y. \tag{67}$$

Notice for $0 < \alpha \notin \mathbb{N}$ by Remark 7 we have

$$\left(D_{a+;g}^{\alpha} f\right)(a) = 0. \tag{68}$$

Similarly it holds $(0 < \alpha \notin \mathbb{N})$ by Remark 7 that

$$\left(D_{b-;g}^{\alpha} f\right)(b) = 0. \tag{69}$$

Thus

$$\left(D_{y+;g}^{\alpha} f\right)(y) = 0, \quad \left(D_{y-;g}^{\alpha} f\right)(y) = 0, \tag{70}$$

$0 < \alpha \notin \mathbb{N}$, any $y \in [a,b]$.

We observe that

$$\|R_1(y)\| \overset{(28)}{\leq} \frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \left(\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\,dt \right) dx \right) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} \right. \tag{71}$$

$$\left. + \left(\int_y^b \left(\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\,dt \right) dx \right) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right] =$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \frac{(g(y)-g(x))^{\alpha}}{\alpha} dx \right) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} \right.$$

$$\left. + \left(\int_y^b \frac{(g(x)-g(y))^{\alpha}}{\alpha} dx \right) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right] \leq$$

$$\frac{1}{\Gamma(\alpha+1)(b-a)} \left[(g(y)-g(a))^{\alpha} (y-a) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} + \right.$$

$$\left. (g(b)-g(y))^{\alpha} (b-y) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right]. \tag{72}$$

We have proved that

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha+1)(b-a)} \left[(g(y)-g(a))^{\alpha} (y-a) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} \right. \tag{73}$$

$$\left. + (g(b)-g(y))^{\alpha} (b-y) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right],$$

any $y \in [a,b]$.

We have established the theorem. ■

Theorem 18 *All as in Theorem 10. Here we take $\alpha \geq 1$. Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x)\,dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(y))}{k!\,(b-a)} \int_a^b (g(x)-g(y))^k\,dx \right\| \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1} \right.$$

$$\left. + \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1} \right], \tag{74}$$

$\forall\ y \in [a,b]$.

Proof. We can rewrite

$$R_1(y) = -\frac{1}{\Gamma(\alpha)(b-a)} \left[\int_a^y \left(\int_{g(x)}^{g(y)} (z-g(x))^{\alpha-1} \left(\left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\,dz \right) dx \right. \tag{75}$$

$$\left. + \int_y^b \left(\int_{g(y)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right)(z)\,dz \right) dx \right].$$

We assumed $\alpha \geq 1$, then

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \cdot$$

$$\left[\int_a^y \left(\int_{g(x)}^{g(y)} (z-g(x))^{\alpha-1} \left\|\left(\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx \right. \tag{76}$$

$$\left. + \int_y^b \left(\int_{g(y)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \left(\int_{g(x)}^{g(y)} \left\|\left(\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right)(g(y)-g(a))^{\alpha-1}\right.$$

$$\left. + \left(\int_y^b \left(\int_{g(y)}^{g(x)} \left\|\left(\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right)(g(b)-g(y))^{\alpha-1}\right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1}\right. \tag{77}$$

$$\left. + \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1}\right].$$

So when $\alpha \geq 1$, we obtained

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \left[\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1}\right.$$

$$\left. + \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1}\right]. \tag{78}$$

Clearly here g^{-1} is continuous, thus $\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1} \in C([g(a), g(y)], X)$, and $\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1} \in C([g(y), g(b)], X)$. Therefore

$$\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]}, \quad \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} < \infty. \tag{79}$$

The proof of the theorem now is complete. ■

Theorem 19 *All as in Theorem 10. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. Then*

$$\left\|f(y) - \frac{1}{b-a} \int_a^b f(x)\,dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)} \int_a^b (g(x)-g(y))^k\,dx\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \tag{80}$$

$$\left[(g(y)-g(a))^{\alpha-1+\frac{1}{p}} (y-a) \left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{q,[g(a),g(y)]}\right.$$

$$\left. + (g(b)-g(y))^{\alpha-1+\frac{1}{p}} (b-y) \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{q,[g(y),g(b)]}\right],$$

$\forall\, y \in [a,b]$.

Proof. Here we use (75).

We get that

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \left[\int_a^y \left(\int_{g(x)}^{g(y)} (z-g(x))^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \cdot \right.$$

$$\left(\int_{g(x)}^{g(y)} \left\| \left(\left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx + \int_y^b \left(\int_{g(y)}^{g(x)} (g(x)-z)^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \cdot$$

$$\left. \left(\int_{g(y)}^{g(x)} \left\| \left(\left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \frac{(g(y)-g(x))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} dx \right) \left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]} \right.$$

$$\left. + \left(\int_a^y \frac{(g(x)-g(y))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} dx \right) \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} \right]. \tag{81}$$

(here it is $\alpha - 1 + \frac{1}{p} > 0$)

Hence it holds

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \tag{82}$$

$$\left[(g(y)-g(a))^{\alpha-1+\frac{1}{p}} (y-a) \left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]} + \right.$$

$$\left. (g(b)-g(y))^{\alpha-1+\frac{1}{p}} (b-y) \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} \right].$$

Clearly here

$$\left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]}, \quad \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} < \infty.$$

We have proved the theorem. ■

Next we give some fractional Grüss type inequalities:

Theorem 20 *Let f, h as in Theorem 10. Here $R_1(y)$ will be renamed as $R_1(f, y)$, so we can consider $R_1(h, y)$. Then*

1)

$$\Delta_n(f,h) := \frac{1}{b-a} \int_a^b f(x) h(x) dx - \frac{\left(\int_a^b f(x) dx \right) \left(\int_a^b h(x) dx \right)}{(b-a)^2} +$$

$$\frac{1}{2(b-a)^2} \sum_{k=1}^{n-1} \frac{1}{k!} \left[\int_a^b \left(\int_a^b \left(h(y) \left(f \circ g^{-1} \right)^{(k)} (g(y)) + \right. \right. \right.$$

$$f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\Big)(g(x)-g(y))^{k}dx\Big)dy\Big]=$$

$$\frac{1}{2(b-a)}\left[\int_{a}^{b}\left(h(y)R_{1}(f,y)+f(y)R_{1}(h,y)\right)dy\right]=:K_{n}(f,h),\tag{83}$$

2) it holds

$$\|\Delta_{n}(f,h)\|\leq\frac{(g(b)-g(a))^{\alpha}}{2\Gamma(\alpha+1)}\left[\|h\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|D_{y-;g}^{\alpha}f\right\|_{\infty}+\left\|D_{y+;g}^{\alpha}f\right\|_{\infty}\right)\right)\right.$$

$$\left.+\|f\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|D_{y-;g}^{\alpha}h\right\|_{\infty}+\left\|D_{y+;g}^{\alpha}h\right\|_{\infty}\right)\right)\right],\tag{84}$$

3) if $\alpha\geq1$, we get:

$$\|\Delta_{n}(f,h)\|\leq\frac{1}{2\Gamma(\alpha)(b-a)}(g(b)-g(a))^{\alpha-1}\cdot\tag{85}$$

$$\left\{\|h\|_{1}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\|f\|_{1}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)\right\},$$

4) if $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, we get:

$$\|\Delta_{n}(f,h)\|\leq\frac{(g(b)-g(a))^{\alpha-1+\frac{1}{p}}}{2\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot\tag{86}$$

$$\left\{\|h\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\|f\|_{\infty}\left(\sup_{y\in[a,b]}\left(\left\|\left(D_{y-;g}^{\alpha}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D_{y+;g}^{\alpha}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)\right\}.$$

All right hand sides of (84)-(86) are finite.

Proof. By Theorem 10 we have

$$h(y)f(y)=\frac{h(y)}{b-a}\int_{a}^{b}f(x)dx-$$

$$\sum_{k=1}^{n-1}\frac{h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)}\int_{a}^{b}(g(x)-g(y))^{k}dx+h(y)R_{1}(f,y),\tag{87}$$

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and

$$f(y)h(y)=\frac{f(y)}{b-a}\int_a^b h(x)\,dx-$$

$$\sum_{k=1}^{n-1}\frac{f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx+f(y)R_1(h,y), \tag{88}$$

$\forall\ y\in[a,b]$.

Then integrating (87) we find

$$\int_a^b h(y)f(y)\,dy=\frac{\left(\int_a^b h(y)\,dy\right)}{b-a}\left(\int_a^b f(x)\,dx\right)-$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\int_a^b\int_a^b h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))\,(g(x)-g(y))^k\,dxdy$$

$$+\int_a^b h(y)R_1(f,y)\,dy, \tag{89}$$

and integrating (88) we obtain

$$\int_a^b f(y)h(y)\,dy=\frac{\left(\int_a^b f(y)\,dy\right)\left(\int_a^b h(x)\,dx\right)}{b-a}-$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\int_a^b\int_a^b f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\,(g(x)-g(y))^k\,dxdy$$

$$+\int_a^b f(y)R_1(h,y)\,dy. \tag{90}$$

Adding the last two equalities (89) and (90), we get:

$$2\int_a^b f(x)h(x)\,dx=\frac{2\left(\int_a^b f(x)\,dx\right)\left(\int_a^b h(x)\,dx\right)}{b-a}-$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\left[\left[\int_a^b\int_a^b h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))+f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\right]\cdot\right.$$

$$\left.(g(x)-g(y))^k\,dxdy\right]+\int_a^b\left(h(y)R_1(f,y)+f(y)R_1(h,y)\right)dy. \tag{91}$$

Divide the last (91) by $2(b-a)$ to obtain (83).

Then, we upper bound $K_n(f,h)$ using Theorems 17, 18, 19, to obtain (84)-(86), respectively.

We use also that a norm is a continuous function. The theorem is proved. ■

We make

Remark 21 *(in support of the proof of Theorem 20) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$. We have*

$$\left(D_{y+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{y}^{x} (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{92}$$

$\forall\ x \in [y, b]$, and

$$\left(D_{y-;g}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{y} (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{93}$$

$\forall\ x \in [a, y]$, both are Bochner type integrals.

By change of variables for Bochner integrals, see [6], Lemma B. 4.10 and [7], p. 158, we get:

$$\left(D_{y+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{g(y)}^{g(x)} (g(x) - z)^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz =$$

$$\left(D_{g(y)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \forall\ x \in [y, b], \tag{94}$$

and

$$\left(D_{y-;g}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{g(x)}^{g(y)} (z - g(x))^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz =$$

$$\left(D_{g(y)-}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \forall\ x \in [a, y]. \tag{95}$$

Here $D_{g(y)+}^{\alpha}, D_{g(y)-}^{\alpha}$ are the left and right X-valued Caputo fractional differentiation operators.

Fix $w : w \geq x_0 \geq y_0$; $w, x_0, y_0 \in [a, b]$, then $g(w) \geq g(x_0) \geq g(y_0)$. Hence

$$\left\| \left(D_{y_0+;g}^{\alpha} f\right)(w) - \left(D_{x_0+;g}^{\alpha} f\right)(w) \right\| =$$

$$\left\| \left(D_{g(y_0)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(w)) - \left(D_{g(x_0)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(w)) \right\| =$$

$$\frac{1}{\Gamma(n-\alpha)} \left\| \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz \right\| \leq \tag{96}$$

$$\frac{1}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} \left\| \left(f \circ g^{-1}\right)^{(n)} (z) \right\| dz \leq$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{\infty,[g(a),g(b)]}}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} dz =$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{\infty,[g(a),g(b)]}}{\Gamma(n-\alpha+1)} \left[(g(y_0) - z)^{n-\alpha} - (g(x_0) - z)^{n-\alpha} \right] \to 0,$$

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as $y_0 \to x_0$, *then* $g(y_0) \to g(x_0)$, *proving continuity of* $\left(D^{\alpha}_{g(y)+}\left(f \circ g^{-1}\right)\right)(g(x))$ *with respect to* $g(y)$, *and of course continuity of* $\left(D^{\alpha}_{y+;g} f\right)(x)$ *in* $y \in [a,b]$.

Similarly, it is proved that $\left(D^{\alpha}_{y-;g} f\right)(x)$ *is continuous in* $y \in [a,b]$, *the proof is omitted.*

Remark 22 *Some examples for* g *follow:*

$$\begin{gathered} g(x) = e^x, \;\; x \in [a,b] \subset \mathbb{R}, \\ g(x) = \sin x, \\ g(x) = \tan x, \\ \text{where } x \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right], \text{ where } \varepsilon > 0 \text{ small.} \end{gathered}$$

Indeed, the above examples of g *are strictly increasing and contimuous functions.*

One can apply all of our results here for the above specific choices of g. *We choose to omit this job.*

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