On Compound Exponential New XLindley Distribution: Properties, Simulation, Fuzzy reliability and Application in Decennial Census of Population

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Abstract

This study presents an innovative lifetime distribution formed by integrating the exponential and the new XLindley distributions, termed the compound exponential new XLindley distribution (CENXLD). The resultant distribution displays decreasing hazard rate properties, thereby rendering it applicable to a multitude of contexts in reliability and survival analysis. The investigation delineates various numerical characteristics associated with the moment, as well as the maximum likelihood, least squares, and weighted least squares methodologies. It elucidates the techniques for parameter estimation pertinent to the newly proposed distribution and introduces a simulation to illustrate its efficacy. Furthermore, a section on fuzzy reliability alongside simulation is included. In addition, this new model is compared against established one-parameter distributions utilizing real-world data from the decennial population census.

Keywords: New XLindley distribution, one-parameter distribution, exponential distribution, simulation.

1. Introduction

Several parametric distributions, such as the exponential, gamma, and Weibull distributions, are frequently employed in statistical literature for the analysis of lifetime data. Among these, the Lindley distribution has garnered significant interest owing to its proficiency in modeling lifetime data, rendering it useful in diverse domains. Initially presented by Lindley in 1958, the one-parameter Lindley distribution (LD) has demonstrated its utility as an effective instrument in this area. The probability density function (pdf) of the LD, corresponding to a random variable X, is defined as follows:

$$f(x,\theta) = \begin{cases} \dfrac{\theta^2(1+x)e^{-\theta x}}{1+\theta}, & x,\theta > 0 \\ 0 & otherwise \end{cases}$$

Gui et al. (2014) analyzed the Lindley distribution and clarified its interpretation as a combination of exponential and gamma distributions. Numerous scholars have investigated the potential for expanding this distribution by amalgamating it with other models. For instance, Arslan et al. (2017) explored its relationship with the Weibull distribution and suggested the Generalized Lindley distribution as a feasible alternative. Likewise, Asgharzadesh et al. (2014) presented the Poisson-Lindley distribution, which demonstrates various forms for the failure rate function. Zeghdoudi and Nedjar (2015) offered a distinct one-parameter Lindley distribution, referred to as GamL, which proves to be particularly advantageous in modeling lifetime data, survival analysis, and actuarial science. Its probability density function is articulated as follows:

$$f(x;\theta,\beta) = \frac{\theta^2((\beta+\beta\theta-\theta)x+1)e^{-\theta x}}{\beta(1+\theta)}; \quad x > 0, \theta > 0, \beta > 0.$$

Nedjar and Zeghdoudi (2016) investigated the diverse statistical characteristics and simulations pertinent to the GamL distribution. Shanker, Sharma, and Shanker (2013) formulated a two-parameter Lindley distribution by merging an exponential (θ) and a gamma (2,θ) distribution. Subsequently, Messaadia and Zeghdoudi (2017) advanced the gamma-Lindley distribution, elucidating its characteristics and recognizing the standard Lindley distribution as a particular instance.

In a more recent contribution, Abdi et al. presented an innovative gamma-Lindley distribution by integrating the gamma and Lindley distributions. This new framework exhibits both decreasing and unimodal shapes of hazard rates, with its probability density function defined as follows:

$$f(x) = \frac{\alpha\beta^2}{1+\beta}\frac{(1+\alpha+\beta+x)e^{\alpha-1}}{(\beta+x)^{2+\alpha}}$$

Many current distributions are formulated by combining two other distributions. This research seeks to present a new lifetime distribution derived from the combination of exponential and Lindley distributions. This innovative model signifies a particular instance of the gamma-Lindley distribution (GaL) put forth by Abdi, Asgharzadeh, Bakouch, and Alipour in 2019, providing a significant framework for modeling lifetime data within the realms of biological and actuarial sciences.

Several other significant advancements have been made in recent years. Nedjar and Zeghdoudi (2020) and Seghier et al. (2020) presented the compound Poisson distribution and the zero-truncated Poisson quasi-Lindley distribution, respectively.

Segheir and Zeghdoudi (2021) further advanced a discrete distribution termed PXLD, achieved through the compounding of the Poisson and X-Lindley distributions. Its probability density function is defined as follows:

$$f_{PXL}(x,\theta) = \frac{\theta^2(\theta^2 + 3\theta + 3 + x)}{(\theta + 1)^{x+2}} \quad x = 0,1,2,\dots,\theta > 0$$

Belhamra et al. (2022) presented a novel compound Exponential-Lindley distribution. Furthermore, Chouia and Zeghdoudi (2021) put forward the X-Lindley distribution, which is derived from the amalgamation of exponential and Lindley distributions. Its probability density function is specified as follows:

$$f_{XL}(x,\theta) = \frac{\theta^2(2 + \theta + x)e^{-\theta x}}{(\theta + 1)^2} \quad x,\theta > 0$$

More recently, Khodja et al. (2023) presented the X-Lindley (XLindley) distribution, which is developed from a particular combination of exponential and Lindley distributions. The corresponding probability density function is delineated as follows:

$$f(x,\gamma) = \frac{\gamma}{2}(1 + \gamma x)e^{-\gamma x}; \qquad x,\gamma > 0.$$

The Compound Exponential New X-Lindley Distribution (CENXLD) has been proposed as a statistical distribution that demonstrates a decreasing hazard rate pattern, rendering it appropriate for representing datasets that exhibit such characteristics. This examination is organized as follows: Section 2 formulates the theoretical foundation, while Section 3 explores the shape properties of the CENXLD. In Section

4, we investigate quantiles and extreme order statistics. Section 5 addresses estimation methods, focusing on maximum likelihood and least squares approaches. The evaluation of the stress-strength parameter is elaborated in Section 6. Section 7 introduces generation algorithms and findings derived from Monte Carlo simulations. The results are compiled and analyzed in Section 8, along with their practical implications.

2. Generation of the Data

2.1. Theoretical model

Let X/λ follows the Exponential distribution with pdf

$$f(x/\lambda) = \lambda e^{-\lambda x}, \quad x > 0\,; \quad \lambda > 0,$$

And $\lambda\,/\beta$ having NXLD (Khodja et al.2023) with pdf

$$f(\lambda/\beta) = \frac{\beta}{2}(1+\beta\lambda)e^{-\beta\lambda}, \quad \lambda > 0\,; \quad \beta > 0,$$

The marginal distribution of $X$is calledCENXLD. The pdf of X is obtained by

$$f(x) = \frac{\beta}{2}\int_0^\infty (1+\beta\lambda)\lambda e^{-(\beta+x)\lambda}d\lambda$$

By simplifying, we getCENXLD pdf as

$$f(x) = \frac{\beta(x+3\beta)}{2(x+\beta)^3}, \qquad x > 0; \;\; \beta > 0, \tag{1}$$

Moreover, the cumulative distribution function (cdf) ofCENXLD is

$$F(x) = 1 - \frac{\beta(2\beta+x)}{2(x+\beta)^2} = \frac{x(3\beta+2x)}{2(\beta+x)^2}$$

Hence, the corresponding reliability (survival) function is given by

$$R(x) = 1 - F(x) = \frac{\beta(2\beta+x)}{2(x+\beta)^2} \tag{2}$$

3. Shape characteristics

In this section, we discuss the shape characteristics of pdf, hrf and rhrfofCENXLD.

3.1 Shape of pdf

We can see from (1) that

$$\lim_{x\to 0} f(x) = \frac{3}{2\beta}$$

and $\lim_{x\to\infty} f(x) = 0$. Figure 1 shows the pdf of the Exponential New X-Lindley distribution for some selected choices of β.

Theorem 3.1. *The pdf of the New Exponential-Lindley distribution given by* (1) *is decreasing for* $\beta > 0$.

Proof.

We have;

$$\frac{d}{dx}f(x) = -\frac{\beta(x+4\beta)}{(x+\beta)^4} \tag{3}$$

We observe that$f(x) < 0$

So in this case, $f(x)$ is decreasing for all x.

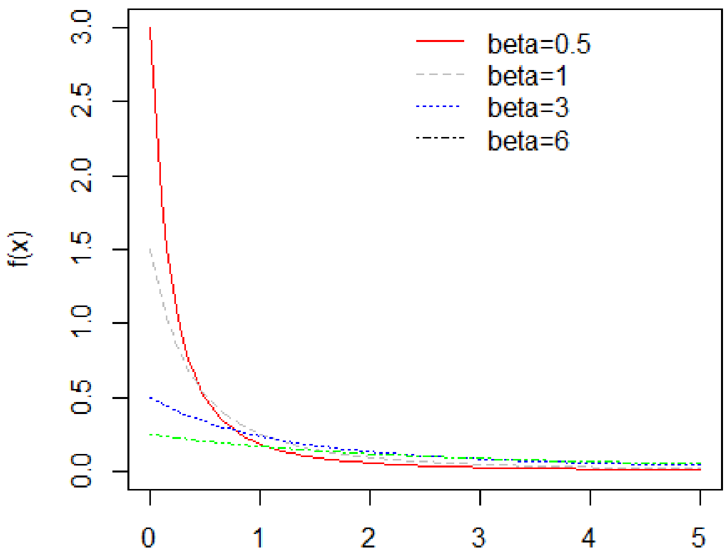

Figure 1.Shapecharacteristicof Compound Exponential New XLindley distribution

We have;

$$\frac{d}{dx}f(x) = -\frac{\beta(x+4\beta)}{(x+\beta)^4} \tag{3}$$

We observe that$f(x)' < 0$

So in this case, $f(x)$ is decreasing for all x.

3.2 Shapes of hazard rate and reversed hazard rate functions

The hrf and rhrf corresponding to (1) and (2) are given, respectively by

$$h(x) = \frac{f(x)}{R(x)} = \frac{(x+3\beta)}{(x+\beta)(2\beta+x)} \qquad (4)$$

and

$$r(x) = \frac{f(x)}{F(x)} = \frac{\beta(x + 3\beta)}{x(x + \beta)(2x + \beta)} \quad (5)$$

The behavior of $h(x)$ when $x \to 0$and $x \to \infty$, respectively, are given by

$$\lim_{x\to 0} h(x) = \frac{3}{2\beta} \text{and } \lim_{x\to \infty} h(x) = 0$$

Also, we can find the $\lim_{x\to 0} r(x) = \infty$and $\lim_{x\to \infty} r(x) = 0$. From figure 2 and figure 3 the $hrf\ h(x)$ and$rhrf\ r(x)$ of CENXLD for some choices of β is shown.

Theorem 3.2. The hazard rate and reversed hazard rate functions of the Compound New Exponential-Lindley distribution given by (4) are decreasing for $\beta > 0$.

Proof.

The derivative of (4) is

$$\frac{d}{dx} h(x) = -\frac{(x^2 + 6x\beta + 7\beta^2)}{(x^2 + 3x\beta + 2\beta^2)^2}$$

The derivative of (5) is

$$\frac{d}{dx} r(x) = -\frac{\beta(4x^3 + 21x^2\beta + 18x\beta^2 + 3\beta^3)}{x^2(2x^2 + 3x\beta + \beta^2)^2}$$

For all β. Therefore, the hazard rate and the reversed hazard rate function are decreasing.

4. Quantiles

The p^{th} quantile x_p ofCENXLD distribution defined by $F(x_p) = p$, is the root of the equation

$$x_p = p\left(1 + \frac{\beta}{x_p}\right)^2 - \frac{3}{2}\beta$$

The median of CENXLD is obtained by using the above equation for $p = \frac{1}{2}$, and $x_p$which can be used to generate the Compound Exponential New XLindley Distribution (CENXLD) random variables.

Remark 4.1. *The moment and the negative moment are undefined.*

We cannot find the exact solution but we can make numerical solution. See table 1.

Table 1. Some values of quantiles for variation values of β

	β

p	0.1	0.5	1	3
0.1	0.00732	0.0366	0.0732	0.2196
0.25	0.0215	0.10764	0.21525	0.64575
0.5	0.0618	0.309	0.618	1.854
0.75	0.17321	0.86603	1.7321	5.1962
0.9	0.485	2.4271	4.8541	14.562

4.1. Extreme order statistics and their limiting distributions

Let $X_{1:n},\ldots,X_{n:n}$ be the order statistics of a random sample size n from CENXLD(x,β)distribution with distribution function $F(x)$ given as in(2). The cdf of minimum order statistics $X_{1:n}$ is given by

$$F_{X_{1:n}}(x) = [F(x)]^n = \left[1-\frac{\beta(2\beta+x)}{2(x+\beta)^2}\right]^n$$

The minimum (maximum) order statistics represents the lifetime of a series (parallel) system in reliability studies.

From the following proposition 1, the limiting distribution of $X_{1:n}$ and $X_{n:n}$for the CENXLD(x,β) model is provided.

Proposition4.1. Let $X_{1:n},\ldots,X_{n:n}$ be the order statistics of a random sample of size n from the CENXLD(x,β) then

1. $\lim_{n\to\infty} p(\frac{X_{1:n}-a_n^*}{b_n^*} \le t) = 1-e^{-t}, t>0,$
2. $\lim_{n\to\infty} p(\frac{X_{n:n}-a_n}{b_n} \le t) = e^{-t^{-1}}, t>0,$

$a_n^* = 0,\quad b_n^* = F^{-1}(n^{-1}) a_n = 0,\quad b_n = F^{-1}(1-n^{-1})$

4.2. Stochastic orders

Stochastic ordering of positive continuous random variables is an important tool to judge the comparative behavior of such variables. For this purpose, we shall recall some basic definitions.

A random variable X_1 is said to be smaller than a random variable X_2 in the

(i) Stochastic order $(X_1 <_{st} X_2)$ if $F_{X_1}(x) < F_{X_2}(x)$ for all x,

(ii) Hazard rate order ($X_1 <_{hr} X_2$) if $h_{X_1}(x) \geq h_{X_2}(x)$ for all x,

(iii) Likelihood ratio order ($X_1 <_{lr} X_2$) if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreases in x.

It is well known that likelihood ratio order implies hazard rate order which in turn implies stochastic order, see (ShakedandShanthikumar,1994)for additional details.

Theorem 4.1.Let$X_i \sim \mathrm{CENXLD}(x_i; \beta_i), i = 1,2,$ *be random variables. if* $\beta_1 \leq \beta_2$,

then $X_1 <_{lr} X_2 \Rightarrow X_1 <_{hr} X_2 \Rightarrow X_1 <_{st} X_2$.

Proof

We have

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\beta_1(x+3\beta_1)2(x+\beta_2)^3}{\beta_2(x+3\beta_2)2(x+\beta_1)^3} \quad (6)$$

This equation can have the following form

$$g(x) = \frac{\beta_1(x+3\beta_1)}{\beta_2(x+3\beta_2)}\left(\frac{x+\beta_2}{x+\beta_1}\right)^3$$

Then, we can write

$$\frac{dlogg(x)}{dx} = \left(\frac{3}{x+\beta_2} - \frac{1}{x+3\beta_2}\right) - \left(\frac{3}{x+\beta_1} - \frac{1}{x+3\beta_1}\right)$$
$$= q(\beta_2) - q(\beta_1)$$

Where

$$q(\beta) = \left(\frac{3}{x+\beta} - \frac{1}{x+3\beta}\right)$$

Note that

$$\frac{dq(\beta)}{dx} = \frac{-3}{(x+\beta)^2} + \frac{1}{(x+3\beta)^2} < 0,$$

So, X_1 is stochastically smaller than X_2 with respect to the likelihood ratio if and only if $\beta_1 \leq \beta_2$.

5. Estimation

In this section, we consider estimation of the unknown parameters ofCENXLD$(x; \beta)$ by maximum likelihood, least squares and weighted least squares methods.

5.1. Maximum likelihood estimation

Let $x_1, x_2, \dots, x_n$ be the CENXLD $(x; \beta)$ distribution.

Then, the log-likelihood function is given by

$$L(x, \beta) = \left(\frac{\beta}{2}\right)^n \frac{\prod_{i=1}^{n}(x_i + 3\beta)}{\left[\prod_{i=1}^{n}(x_i + \beta)\right]^3}$$

The log-likelihood function of the parameter β is

$$Ln\ L(\beta,x) = nln\beta - nln(2) + \sum_{i=1}^{n} \ln(x_i + 3\beta) - 3\sum_{i=1}^{n} \ln(x_i + \beta) \quad (7)$$

$$\frac{dln\ L(\beta;x)}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left(\frac{3}{x_i + 3\beta}\right) - 3\sum_{i=1}^{n}\left(\frac{1}{x_i + \beta}\right) = 0 \quad (8)$$

$$\frac{d^2 ln\ L(\beta;x)}{d\beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n}\left(\frac{9}{(x_i + 3\beta)^3}\right) + 3\sum_{i=1}^{n}\left(\frac{1}{(x_i + \beta)^3}\right) < 0,$$

To obtain the MLE of β, we can maximize (7) directly with respect to β or we can solve the non-linear equation given in(8). Note that MLE of β cannot be solved analytically, however, numerical iteration techniques, such as the Newton-Raphson algorithm, or Fisher scoring method can be used.

5.2. Least squares and weighted least squares estimators

We present a regression based method estimators for the unknown parameters of the CENXLD. The method of ordinary least squares and method weighted least squares were originally proposed by (Swain, Venkatraman, and Wilson, 1988) to estimate the parameters of Beta distributions.

Suppose $X_1, \dots, X_n$ is a random sample of size n from a distribution function $F(.)$ and $X_{1:n} < \cdots < X_{n:n}$ be the ordered statistics of the sample. The least squares estimators (LSEs) can be obtained by minimizing

$$\sum_{i=1}^{n}\left[F(X_{i:n}) - \frac{i}{n+1}\right]^2,$$

With respect to the unknown parameter of $F(.)$. Therefore in the case of CENXLD distribution, the least square estimator of β, say $\hat{\beta}_{LSE}$, can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\frac{1}{2}\frac{X_{i:n}(3\beta + 2X_{i:n})}{(\beta + X_{i:n})^2} - \frac{i}{n+1}\right]^2,$$

With respect to β.

While, the weighted least squares estimators (WLSEs) of the unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n} w_i \left[F(X_{i;n}) - \frac{i}{n+1}\right]^2,$$

With respect to the unknown parameters, where

$$w_i = \frac{1}{Var\left[F(X_{i;n})\right]} = \frac{(1+n)^2(n+2)}{i(n-i+1)}.$$

Therefore, in case of CENXLD, the weighted least square of β, say $\hat{\beta}_{WLSE}$, can be obtained by minimizing

$$\sum_{i=1}^{n} w_i \left[\frac{1}{2}\frac{X_{i:n}(3\beta + 2X_{i:n})}{(\beta + X_{i:n})^2} - \frac{i}{n+1}\right]^2,$$

With respect to β

6. Estimation of the stress-strength parameter $R = P(X > Y)$

In reliability, the stress-strength model describes the life of a component which has a random strength X subjected to a random stress Y. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function whenever $X > Y$.

In this section, we consider the problem of estimating $R = P(X > Y)$, under the assumption that $X \sim CENXLD(\beta_1), Y \sim CENXLD(\beta_2), X$and Y are independently distributed. Then it can be easily seen that

$$\begin{aligned} R = P(X > Y) &= \int_0^{\infty} P\left(X > Y \,\middle|\, Y = y\right) f_Y(y)dy \\ &= \int_0^{\infty} [1 - F_X(y)] f_Y(y)dy \\ &= \int_0^{\infty} \frac{\beta_2}{4} \times \frac{(3\beta_1 + 2y)(y + 3\beta_2)}{(\beta_1 + y)^2(y + \beta_2 +)^3} dy \\ &= S(\beta_1, \beta_2) \end{aligned}$$

To compute the maximum likelihood estimator (MLE) of R, let us first obtain the MLEs of $\beta_1$and β_2. Suppose $x_1, x_2, \dots, x_n$be the observed values of a random sample of size n from CENXLD(β_1) and $y_1, y_2, \dots, y_m$ be the observed values of a random sample of size m from CENXLD(β_2). Therefore, the log-likelihood function of β_1 and β_2 is given by

$$\begin{aligned} Ln\, L(\beta_1, \beta_2) = nln\beta_1 - nln(2) + \sum_{i=1}^{n} \ln(x_i + 3\beta_1) - 3\sum_{i=1}^{n} \ln(x_i + \beta_1) \\ + nln\beta_2 - nln(2) + \sum_{i=1}^{m} \ln(y_i + 3\beta_2) - 3\sum_{i=1}^{m} \ln(y_i + \beta_2) \end{aligned}$$

It follows that the MLEs of β_1 and β_2 say $\hat{\beta}_1$and $\hat{\beta}_2$, are the simultaneous solutions of the following equations:

$$\frac{n}{\beta_1} + \sum_{i=1}^{n}\left(\frac{3}{x_i + 3\beta_1}\right) - 3\sum_{i=1}^{n}\left(\frac{1}{x_i + \beta_1}\right) = 0$$

$$\frac{m}{\beta_2}+\sum_{i=1}^{m}\left(\frac{3}{y_i+3\beta_2}\right)-3\sum_{i=1}^{m}\left(\frac{1}{y_i+\beta_2}\right)=0$$

Once we obtain $\hat{\beta}_1$and $\hat{\beta}_2$, then, we compute the MLE of R as

$\hat{R}=S(\hat{\beta}_1,\hat{\beta}_2)$

Here the maximum likelihood approach does not give an explicit estimator for the MLEs of the parameters and hence the MLEs of R. In practice, one has to use numerical methods to find the MLEs, such methods are well implemented in MATLAB and R packagers.

7. Generation algorithms and Monte Carlo simulation study

In this section, we propose two different algorithms for generating the random data$x_1,x_2,\dots,x_n$from the CENXLD. Further, a simulation study is given to compare the performances of different estimators using the different estimation methods.

7.1. Algorithms

Two proposed algorithms for generating the random data $x_1,x_2,\dots,x_n$from the CENXLD are as follows.

(i) The first algorithm is based on generating the random data from the LD and conditional gamma distribution.

(ii) The second algorithm is based on generation random data from the inverse cdf of the CENXLD.

Algorithm 1 (conditional distribution).

1. Generate $\lambda_i \sim NXLindley(\beta), i=1,2,\dots,n$;
2. Generate $X_i \backslash \lambda_i \sim \exp(\alpha,\lambda_i), i=1,2,\dots,n$.

Using the following algorithm 1, we can see (Ghitany, Ateih, and Nadarajah 2008) to generate $\lambda_i$from the Lindley (β) distribution:

1. Generate $U_i \sim Uniform(0,1),\ i=1,\dots,n$;
2. Generate $V_i \sim Exponential(\beta),\ i=1,2,\dots,n$;
3. Generate $W_i \sim Gamma(2,\beta),\ i=1,2,\dots,n$;
4. If $U_i \le p=\frac{1}{2}$, then set $\lambda_i=V_i$, otherwise, set $\lambda_i=W_i, i=1,2,\dots,n$.

Algorithm 2(Inverse CDF).

1. Generate $U_i \sim Uniform(0,1), i=1,\dots,n$;
2. Set $X_i=p\left(1+\frac{\beta}{x_p}\right)^2-\frac{3}{2}\beta$

Note that X_i is the root of the equation $F(X_i) = U_i$.

n	β	Method	$bias(\hat{\beta})$	Ranks	$MSE(\hat{\beta})$	Ranks
40	0.5	MLE	0.0500	3	0.1705	3
		LSE	0.0025	1	0.0308	1
		WLSE	0.0123	2	0.0552	2
80	0.5	MLE	0.0204	3	0.0566	3
		LSE	-0.0004	1	0.0148	1
		WLSE	0.0121	2	0.0192	2
150	0.5	MLE	0.0097	3	0.0243	3
		LSE	-0.0018	1	0.0075	1
		WLSE	0.0059	2	0.0087	2
40	1.5	MLE	0.0553	2	0.2511	1
		LSE	0.0298	1	0.3655	2
		WLSE	0.1645	3	0.5422	3
80	1.5	MLE	0.0256	2	0.0698	1
		LSE	0.0186	1	0.1588	2
		WLSE	0.0722	3	0.2055	3
150	1.5	MLE	0.0075	1	0.0321	1
		LSE	0.0144	2	0.0754	2
		WLSE	0.0377	3	0.0988	3
40	2.5	MLE	0.0732	1	0.3566	1
		LSE	0.0875	2	0.7095	2
		WLSE	0.2488	3	0.8785	3
80	2.5	MLE	0.0252	1	0.0834	1
		LSE	0.0777	3	0.2245	2
		WLSE	0.1133	2	0.3632	3

150	2.5	MLE	0.0072	1	0.0386	1
		LSE	0.0246	3	0.1058	2
		WLSE	0.0904	2	0.1084	3

Table 2. Average biases and MSEs of the simulated estimates for MLE, LSE and WLSE methods

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7.2 Monte Carlo simulation study

A simulation studies to compare the performances of maximum likelihood, least squares and weighted least squares an estimator of the unknown parameter β via Monte Carlo simulation is given here. For a given n and β, we have generated the sample $x_1, x_2, \dots, x_n$ from the $CENXLD(\beta)$ model and then obtain the estimates using the preceding estimation methods.

We used Algorithm 1 to generate data from theCENXLD. The simulation experiment was repeated $N = 10{,}000$ times each with sample sizes $n = 40, 80, 150$and $\beta = 0.5, 1.5, 2.5$.Note that the selected values of β given (0.1, 0.5, 1, 2, 6) for shape parameter of density as displayed in Figure 1. Two quantities were examined in this Monte Carlo study:

(i) Average bias of $MLE\ \hat{\theta}$ of the parameter $\hat{\theta} = \beta$:

$$\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta} - \theta\right),$$

(ii) Mean square error (MSE) of the $MLE\ \hat{\theta}$of the parameter $\hat{\theta} = \beta$:

$$\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta} - \theta\right)^2.$$

The results of this study are reported in Table 1. The following conclusions can be noted:

i) Table 1 shows that for $n = 40 and\ n = 80$the LSE is the best average for $\beta < 1$.

ii) The MLE is the best compared to the other methods for all values of n.

iii) The results of WLSE are not satisfactory for all values of n.

8. Fuzzy reliability

Let T be a continuous random variable that represents a system's failure time (component). The fuzzy dependability can then calculated using the fuzzy probability in formula:

$$R_F(t) = P(T > t) = \int_t^{\infty} \mu(x) f_{CENXLD}(x)dx, \quad 0 \le t \le x < \infty,$$

Where $\mu(x)$ is a membership function that describes the degree to which each element of given universe belongs to a fuzzy set (We can see Chen et al (2021)). Now, assume that $\mu(x)$ is

$$\mu(x) = \begin{cases} 0 & , x \le t_1 \\ \dfrac{x - t_1}{t_0 - t_1} & , \quad t_1 < x < t_2, \ t_1 \ge 0 \\ 1 & , x \ge t_2 \end{cases}$$

For $\mu(x)$, by the computational analysis of the function of fuzzy numbers, the lifetime $x(\gamma)$ can be obtained corresponds to a certain value of $\gamma - Cut, \gamma \in [0,1]$, can be obtained by

$\mu(x) = \gamma \to \frac{x-t_1}{t_0-t_1} = \gamma$,then

$$\begin{cases} x(\gamma) \le t_1 & , \gamma = 0 \\ x(\gamma) = t_1 + \gamma(t_2 - t_1) & , 0 < \gamma < 1 \\ x(\gamma) \ge t_2 & , \gamma = 1 \end{cases}$$

As a result, the fuzzy reliability values may be determined for all γ values. The fuzzy dependability of the CENXLD is determined by the fuzzy reliability definition. The fuzzy reliability of the CENXLD can be defined as,

$$R_F(t) = \frac{\beta(2\beta + t_1)}{2(t_1 + \beta)^2} - \frac{\beta(2\beta + x(\gamma))}{2(x(\gamma) + \beta)^2}$$

Then $R_F(t) = 0$.

Numerical values of fuzzy reliability

In this subsection, we obtained comparison between traditional reliability (see Finkelstein (2008))and Fuzzy reliability, where the traditional reliability is a survival function as

$$R(x) = \frac{\beta(2\beta + x)}{2(x + \beta)^2}$$

Table 2 discussed the comparison. The following observations are based on findings:

- When the $\gamma - Cut$ is increased, the Fuzzy reliability increases.
- When the t_2 of interval of membership function is increased the Fuzzy reliability increases.

- When the t_1 is decreased the fuzzy reliability increases, and vice versa.

- The traditional reliability with t_2 is lower than the traditional reliability with t_1.

The fuzzy estimation algorithm produces a series of draws from CENXLD as in algorithm 1.

Algorithm 1: fuzzy estimation algorithm

- **Input**: initial values of β,interval time (t_1, t_2) and γ where $0 < \gamma < 1$.
- **Calculate**: $x(\gamma) = t_1 + \gamma(t_2 - t_1)$.
- **For** each method do

Set:i=1.

Estimate parameter as $\hat{\beta}$.

Calculate

$$\hat{R}_F(t) = \frac{\beta(2\beta + t_1)}{2(t_1 + \beta)^2} - \frac{\beta(2\beta + x(\gamma))}{2(x(\gamma) + \beta)^2}$$

- **End**

					R_F		
β	t_1	t_2	$R(t_1)$	$R(t_2)$	0.25	0.5	0.9
0.1	0.001	1	0.9852	0.0496	0.80201	0.88881	0.9302
	0.05	2	0.5556	0.0249	0.3724	0.4584	0.5006
1	0.01	1.5	0.9852	0.2800	0.3619	0.7189	0.8146
	0.2	3	0.7639	0.1563	0.3622	0.4976	0.5934
3	0.1	1	0.9521	0.6563	0.0939	0.1725	0.27415

Table 3. Traditional and fuzzy reliability with different values.

9. Application and comparison

In this section, we present the application of the CENXLD to real-life data set to illustrate its flexibility for Populations Recorded by the US Census data.

This data set gives the population of the United States (in millions) as recorded by the decennial census for the period 1790-1970.

Source

McNeil, D. R. (1977) Interactive Data Analysis. New York: Wiley.

3.93, 5.31, 7.24, 9.64, 12.90, 17.10, 23.20, 31.40, 39.80, 50.20, 62.90, 76.00, 92.00, 105.70, 122.80, 131.70, 151.30, 179.30, 203.20

Model	β	AIC	BIC	$-2L$	$ACIC$	KS
$Lindley$	0.028	207.6266	208.571	205.6266	207.8619	0.2749752
$XLindley$	0.0279	206.924	207.8684	204.924	207.1593	0.8518941
$ZLindley$	0.077	272.9751	273.9196	270.9751	273.2104	
$Zeghdoudi$	0.0426	220.9815	221.926	218.9815	221.2168	0.3386842
$CENXLD$	68.11	**206.2763**	**207.2208**	**204.2763**	**206.5116**	**0.1755424**

Table 4. Goodness of fit statistics of CENXLD

The values of $AIC, BIC, -2logL, K\text{-}S$ statistics in Table 3, indicate that CENXLD is a strong competitor to the other distributions commonly used in literature for fitting lifetime data, moreover the best fit measured the previous goodness of fit statistics.

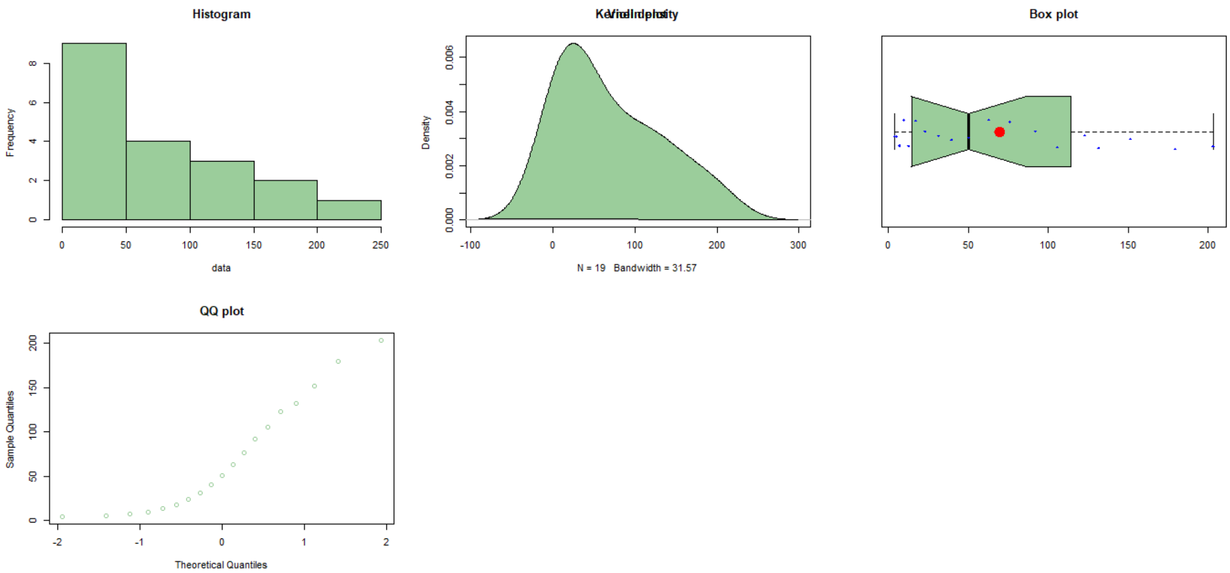

Figure 1: QQ plot and Box plot of Data set

10. Conclusions

In this investigation, a novel one-parameter lifetimedistribution, created through the combination of the exponential and new XLindley distributions, referred to as CENXLD, and has been introduced. This new distribution exhibits both decreasing hazard rate and reversed hazard rate characteristics, which are applicable in various contexts. Its pertinent mathematical attributes, encompassing the shape, characteristics of the probability density function (pdf), hazard rate function (hrf), reversed hazard rate function (rhrf), quintiles, moments, and stochastic ordering mean deviations, have been thoroughly explored.

The parameter of the CENXLD is estimated using the maximum likelihood, least squares, and weighted least squares methods. To evaluate the effectiveness of the proposed estimator under the outlined estimation techniques, a simulation approach has been employed. The efficacy of the new model relative to the contemporary iterations of the new XLindley and exponential distributions is demonstrated through a real-world dataset concerning failure rates from the decennial population census, utilizing goodness-of-fit statistics. Future research endeavors may focus on examining the Bayesian estimation of the CENXLD parameter and the introduction of a truncated version of the CENXLD to address the issue of infinite moments.

AUTHORS CONTRIBUTIONS

Imene Grabsia: Investigation; formal analysis; methodology, writing—original draft; simulation; interpretation of results; writing—review and editing.

Razika Grine: methodology, writing—original draft ; Software; formal analysis; validation.

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