Algorithm for new lower bounds for reliability : F systems and numerical examples

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Abstract

This paper deals around the hk -within-connected-out-of-(m,n) : F systems consist in rectangular sheets of components, of size $m \times n$, such that the system fails if all components in a submatrix of size $h \times k$ are failing. In $[5]$ has been given the computation of a lower bound for the reliability for such systems, in case of square matrix. We use a method presented in $[11]$ and based on the technic of replication of components to propose new lower bounds, which, in case of square sheets, are better than these in $[5]$. For this purpose, we rely on an algorithm, called "consecutive columns algorithm" under the additive condition $n \leq 2k$. Finally, numerical examples are provided to evaluate the new technique.

Keywords: Consecutive-k-out of-n systems hk-within-connected-out-of-(m, n) : F systems Lower Bound of Reliability Algorithm.

1. Introduction

System, namely consecutive- k -out-of- n : F system (or (one-dimensional) $C(k;\ddot{n}:F)$ system), are of increasing interest all engineering systems and equipment's since 1980 due to the followings $[2]$. As we know the reliability of series system is not high, especially of a large series system, and the reliability of parallel system is high, but tends to be very expensive. The two-dimensional $C(k;\ddot{n}:F)$ system was first introduced by Salvia and Lasher$[8]$. The system consists of n^2 components in a square grid of side n and it fails if and only if there is at least one square of side $k(2 \leq k \leq n-1)$ that contains all failed components. A more general system called t -within-connected-(h,k)-out-of-(m,n): F system has been proposed and investigated, in which the system of size $m \times n$ is failed if and only if there are at least t , $(1 \leq t \leq h.k)$ failed components within any sub matrix of size $h \times k$. Such a system generalizes the mentioned two-dimensional $C(k;\ddot{n}:F)$ system when $h=k, m=n$, and $t=hk$. For the case of $t=hk$, Yamamoto and Miyakawa [12] proposed a recursive algorithm for the exact system reliability of hk -within-connected-$(h;k)$-out-of- $(m;n)$: F systems with computing time complexity $O(k^{m-h}m^2hn)$.

We propose for our hk-within-connected-out-of-(m,n) F: systems, in case all components are independent and have the same reliability p, a recursive formula for the calculation of the exact reliability and a new lower bounds which are better than those in [5] ; the computation of these bounds, for any dimension of the system, makes use of the computation in auxiliary systems submitted to a technical condition, i.e. $n \leq 2k$ (or symmetrically $m \leq 2h$). In order to present the principle for computation of such lower bounds, we consider, in section 2, a general class of systems, which we call and covered-up systems, which include as well the well-known one-dimensional-consecutive-k-out of-n systems, the two-dimensional contiguous-(h,k)-out of-(m,n) systems and their generalizations in higher dimensions. Within this framework, we present the technique, which we call duplication of components, which allows us to compute our lower bounds. In section 3 we use the one-dimensional consecutive-k-out of-n systems with independent components, for which an exact computation of reliability is easy, as a benchmark for the adequacy of our lower bounds. In this case the replication of components is reduced to an easier technique, called duplication and we give, some numerical examples of these bounds, varying with some parameters which intervene in the duplication. In section 4 we explicit the technique of replication of components for two- dimensional hk-within-connected-$(h;k)$-out-of-(m,n): F systems. We show why the computation of lower bounds, h and k being fixed, may rest on the computation of exact reliability for systems of lower size, i.e. $m^{'} \times n^{'}$, with $m^{'} \leq m$ and $n^{'} \leq n$; such an exact computation is available if is satisfied condition $n^{'} \leq 2k$ (or symmetrically $m^{'} \leq 2h$) and we present the algorithm to perform it. We give numerical examples of realizations of this algorithm and of lower bound which they allow to compute for systems not complying with this restrictive condition; we show that these bounds are better than the ones obtained by using the technique in [5].

2. Covered-up systems

Let us consider a system ζ, whose set of components is S ; for each $i \in S$, r.v. X_i takes value 0 if component i failing, and 1 if not. If I a subset of S, we note $X_I = (X_t)_{t \in T}$.

Definition 1. System ζ is said to be" covered-up" if it exists a sequence $S_1, S_2, ..., S_m$ of subsets of S such that $\cup_{j=1}^{m} S_j = S$ and the whole system ζ fails if and only if there exists a subset S_j whose all components are failing (i.e. $X_{S\,j} = 0_{S\,j}$). $(S_1, S_2, ..., S_m)$ is called the characteristic sequence of system S.

In this paper we suppose that all the r.v. $X_i (i \in S)$ are independent ; our results could be easily extented without demanding independance for couples of r.v. r.v. X_i and X_i', such that, for all j $(1 \leq j \leq m)$, either $i \in S_j$ and $i^{'} \in S_j$, or $i \notin S_j$ and $i^{'} \notin S_j$.

Let us note M_S (resp. F_S) the event" system S is failing" (resp. Is not failing).

If $T \subset S$, subset S, with components in T inherits the structure of system with covering by considering subsets $S_1 \cap T, S_2 \cap T, ..., S_m \cap T$.

The classical unidimensional k-consecutive-out-of-n system is obviously a covered-up system with $S = \{1, ..., n\}$ and for all $j\,(1 \leq j \leq n - k + 1)$, $S_j = \{j, ..., j + k - 1\}$. So are also the two-dimensional $k-$consecutive-out-of- n systems studied in$[5]$; this two- dimensional situation will be studied

under more general conditions in section 3.
The following theorem is a generalization of an inequality used in [5].

Theorem 1. Let ζ be a covered-up system with independent components and characteristic sequence $S_1, S_2, ..., S_m$. Then,

$$P(M_S) \geq \prod_{j=1}^{m} P(M_{S_j}) \quad .$$

Comments.

1. *For all j we get* $P(M_S j) = 1 - \prod_{i \in Sj} P(Y_i = 0)$.
2. $\prod_{i=1}^{m} P(M_S)$ is the reliability of a system, named ζ^1, built as follows:

ζ^1 is a series system whose elements are parallel systems $\zeta^1{}_j$, isomorphic to the ζ_j in the following sense: the components of $\zeta^1{}_j$ *are* noted i_j for all i such that $i \in S_j$ and the probability law of families of X_{S_j} and $X_{S'_j}$ are identical. We shall say that component i has been replicated in new components i_j Sketch of the proof (for details, see [11]).

It makes use of the following *lemma1*, we recall that a system U is coherent if, when it fails for $(x_u)_{u \in U}$, it also fails each time we modify $x_u = 1$ into $x_u = 0$.

Lemma 1. Let ζ be a series system made of two sub-systems ξ_1 et ξ_2, with $T_1 \cup T_2 = T$, $T_1 \cap T_2 \neq \varnothing$ $X_{T_1 - T_2}$, and $X_{T_2 - T_1}$ are independent; T_1 is a are independent; ξ_1 is a parallel system and ξ_2 is Coherent. Then, if M1 (resp. M2) is the event "T1 is not failing" (resp. "T2 is not failing"), we get:

P (M1 | M2) ≥ P (M1), i.e P (M) = P (M1 ∩ M2) ≥ P (M1)P (M2).

The proof of the theorem is recursive, by applying the lemma successively to couples

$$(S_1 \bigcup_{j=2}^{m}), (S_2 \bigcup_{j=3}^{m}) .... (S_{m-1}, S_m).$$

We shall now proceed to the presentation of better lower bound for reliability - The idea is as follows:

we give an interpretation of theorem 1 by using the replication of elements $i(\in S)$ in new elements i_j

(with j such that $i \in S_j$; we shall now introduce "partial replications", giving birth to systems ζ^* which is same sens loy between ζ and $\zeta^{'}$: we can conjecture that their reliability is worse than the one of ζ but better than the one of $\zeta^{'}$.

Definition 2. Let ζ be a covered-up system, with characteristic sequence $(S_1, S_2, ...S_m)$; a system ζ^* is said to be deduced from ζ by replication of elements of if its characteristic sequence $(S_1^*, S_2^*, ...S_m^*)$ satisfies the following conditions :

1. For all $j(1 \leq j \leq m)$, S_j^* is isomorphic to S_j (i.e. there is a bijection between S_j and S_j^* such that the laws of X_{S_j} and $X_{S_j^*}$ become identical) .
2. Every element of S_j^* comes from the duplication of an element i of S_j in the following sense : n_i is the number of j such that $i \in S_j$, there exist k ($1 \leq k \leq n_i$), a partition $\left(L_1, L_2, ...L_k\right)$ in $\{j; i \in S_j\}$ and k elements of , noted $\left(i_{L_1}, ..., i_{L_h}\right)$, such as for all $h\left(1 \leq h \leq k\right) i_{L_h}$ belonging to

S_j^* such that and to any of the other elements $j \in L_h$ and to any of the other elements $S_j^*(i.e$ those such $j \notin \bigcup_{h' \neq h} L_{h'}$) , if $n_i = 1$ m, there is only one element derived from the duplication of i m, which will keep the notation i ; we also keep the notation i if $k = 1$ (no duplication).

3. The r.v $X_{L_1}, ..., X_{L_h}$ are independents.
4. ζ^* is failed if are failed all the elements of one of the S_j^*

Such replication can be observed as the following (Figure 1), showing systems obtained by successive replications, the last are being $\zeta\ '$.

Theorem 2. Let ζ^* with the characteristic sequence $(S_1^*, S_2^*, ...S_m^*)$, bet obtained from system ζ by some replications of elements, then $P(M_S) \geq P(M_{S^*}) \geq P(M_{S'}) = \prod_{j=1}^{m} P(M_{S'_j})$.

The proof of the theorem is recursive, introducing a sequence of systems $(\zeta_1^*, \zeta_2^*, ...\zeta_l^*)$, with $\zeta^* = \zeta$ and $\zeta_l^* = \zeta^*$, each $\zeta_h^*(1 \leq h \leq l)$ being deduced from ζ^*_{h-1} by duplication (i.e splitting into new elements) of one element ζ^*_{h-1} ; the fact that $P(M^*_{S_h} \leq M^*_{S_{h-1}})$ results from the lemma which lead to theorem 1 (for more details, see [11]).

3. One-dimensional k-consecutive-out of-n systems with independent components

A *k*-consecutive-out of-*n* system S can be viewed on a covered-up system of characteristic sequence $(S_1, S_2, ...S_{n-k+1})$ with $S_j = j, ..., j+k-1$. The reliability of such a system, for independent and of component reliability p , let $Q_{k,n,p}$, is well known and can be obtained by the recursive formula:

$Q_{k,n,p} = (1-p)^k + p[\sum_{i=1}^{k}(1-p)^{i-1}Q_{k,n-i,p}],\ for\ n > k$

(for $n < k, Q_{k,n,p} = 0\ and\ for\ n = k, Q_{k,n,p} = (1-p)^k$.

4. *r.s*-within-connected (h,k) – out-of – (m,n) systems with independent components

Let $R_{m,n} = \{1,...,m\} \times \{1,...,n\}$. An $(h,k)-$ rectangle, with $1 \le h \le m$ and $1 \le k \le n$, is any subset of R -m , n of shape $\{u,...,u+h-1\} \times \{v,...,v+k-1\}$, with $1 \le u \le m-h+1$ and $1 \le v \le n-h+1$.

A $(1,n)$ rectangle is called a column; in other words, it is a subset of shape $\{u\} \times \{1,...,n\}$, with $1 \le u \le m$. A sub column is any subset "without holes" of a column, i.e. a subset of shape $\{u\} \times J$, where J is an interval in $\{1,...,n\}$.

Let us consider the family of Bernoulli r.v. $X_{i,j}$, with $(i,j) \in R_{m,n}$; they are independent and identically distributed; for any (i,j) we have $P(X_{i,j} = 1) = p$.

Our aim is to compute the probability $Q_{h,m;k,n}$ that there exists an (*h*, *k*)-rectangle, let R , called a null-rectangle, such that

$$\forall (i,j) \in R, X_{i,j} = 0.$$

If $\{u\} \times J$ is a null column (i.e. $\forall j \in J, X_{u,j} = 0$.) we shall also say that the interval J is null for column u .

It is clear that the rectangle $\{u,...u+h-1\} \times \{v,...v+k-1\}$ in null iff there exist h successive null sub columns, let $\{u\} \times J_u, \{u+1\} \times J_{u+1},...,\{u+h-1\} \times J_{u+h-1}$, such that the meet of all intervals $J_u, J_{u+1}...J_{u+h-1}$ is non empty and of length at least equal to k .

A way to obtain such a null-rectangle, if it exists, is to proceed from left to right starting with $u = 1$, until we find an adequate sequence $J_u, J_{u+1}...J_{u+h-1}$. To keep at each step of tentative construction of such a sequence the maximum of chances, if any, to proceed to the following columns, we make use of the notion of maximal null interval : a subinterval J' of interval J is said to be null maximal in J for column u iff it is null for column u and any sub interval of J which strictly contains J' is not; J itself, if it is null for column u, is also said maximal in J for this column.

It results that we shall inductively try to build a sequence of h successive null sub columns, let
$\{u\} \times J_u, \{u+1\} \times J_{u+1}...\{u+h-1\} \times J_{u+h-1}$, such that :

- $J_u \supset J_{u+1} \supset ... \supset J_{u+h-1}$,
- J_u is null maximal in $\{1...n\}$ for column u ,
- for any $u' > u$, J_u ' is null maximal in J_u '-1 for column u' ,

- the length of J_{u+h-1} is at least equal to k.

The construction of maximal null subintervals of length at least k, together with the computation of the probability of their existence, though natural, gives rise to a combinatorial complexity rapidly increasing with n. This is why we propose an algorithm under the additive condition $n \le 2k$. In this case, there is at most one maximal null subinterval of length at least k in $\{1...n\}$ and, a fortiori, in any of its subintervals. This restriction, which will be enforced from now on in this section, is admissible because our final purpose in this paper is to make use of this algorithm only for auxiliary subsystems of the system for which we will give lower bounds of its reliability.

The first step in this algorithm will be to compute the probability $q_{s,s}'$, with $k \le s' \le s \le n$ that there exists, in J of length s, a maximal subinterval J' (necessarily unique) of length s' :

- for $s = s'$, $q_{s,s}' = (1-p)^s$,
- for $s < s'$, $q_{s,s}' = (1-p)^s[2p + (s-s'-1)p_2]$.

Then the recursive formula is as follows:

Let $S_{l;k,n}$, ($for\ 1 \le l \le n-1$) be the set of all sequences $(s_1,...,s_l)$ such that $k \le s_l \le s_{l-1} \le ... \le s_1 \le n$ and, let, $s_0 = n$. Then, for, $m > n$,

$$Q_{h,m;k,n} = p_n Q_{h,m-1;k,n} + \sum_{l=1}^{h-1}[\sum_{(s_1,...,s_l)\in\zeta_{l;k,n}} (\prod_{t=0}^{l-1} q_{s_t,s_{t+1}}) p_{s_l}] Q_{h,m-l-1;k,n} + \sum_{(s_1,...,s_{h-1})\in\zeta_{h-1;k,n}} (\prod_{t=0}^{h-2} q_{s_t,s_{t+1}})(1-p_{s_{h-1}}).$$

The initialization is as follows:

- if $m < h$ $Q_{h,m;k,n} = 0$,
- if $m = h$ $Q_{h,h;k,n} = \sum_{s=k}^{n-1}(1-r)^s[2r + (n-s-1)r^2] + (1-r)^n$,

with $r = 1-(1-p)^h$.

The algorithm performing this computation, called" consecutive columns algorithm" has been written in language C, by Tanguy Briançon, professor at Lycée Agora (France)and is available from the author. We sketch here its main lines.

5. Notation have been modified

p is noted x (in order to avoid confusion with the p_s) x and integers n,k and h are being fixed,

we note: $z_m = Q_{h,m;k,n.}$

we set $y = 1-(1-x)^h$ and notations $q_{s,s'}$ and p_s are kept.

the central computation is the one (with $1 \le l \le h-1$) of

$$a_l = \sum_{(s_1,...,s_l)\in\zeta_{l;k,n}} (\prod_{t=0}^{l-1} q_{s_t,s_{t+1}}) p_{s_l} = \sum_{s_1=k}^{n} q_{n},_{s_1}[\sum_{s_2}^{S_1} q_{s_t,s_2.....} | \sum_{s_l=k}^{S_{l-1}} q_{s_{l-1,s_l}}, p_{S_l}].....]$$

It is suitable, for induction on *l*, to reorder the sequence, $(s_1,...,s_l)$, i.e to set

$t_1 = s_l, t_2 = s_{l-1}, ..., t_{l-1} = s_2, t_l = s_1$ $(and\ t_{l+1} = n)$, Here

$$a_l = \sum_{t_l}^{n} q_{n,t_1} [\sum_{t_{l-1=k}}^{t_l} q_{t_l,t_{l-1}} ......[\sum_{t_1=k}^{t_2} q_{t_2,t_1 p t_1}]....]$$

Let, $w_1(t_1) = p_{t_1}$ et, pour tout $l(1 \le l \le h)$

$$w_l(t_l) = \left\lfloor \sum_{t_l-1=k}^{t_l} q_{t_l,t_l-1} ....\,...[\sum_{t_1=k}^{t_2} q_{t_l,pt_1}].... \right\rfloor$$

we get $a_l = w_{l+1}(n)$.

the recursive formula for computing the w_t is:

$$w_l(t_l) = \sum_{t_{l-1}=k}^{t_l} qt_l, t_{l-1} .. w_{l-1}(t_l - 1)$$

where initialization, for t_l such that $k \le t_l \le n$, is:

$w_1(t_1) = p_{t_1}$

We proceed in an analogue way for computing

$$b = \sum_{(s_1,...,s_{h-1}) \in \zeta_{h-1;k,n}} \left(\prod_{t=0}^{h-2} q_{s_t,s_{t-1}} \right) \left(1 - p_{s_{h-1}}\right)$$

$$= \sum_{t_{h-1}}^{n} q_{n,t_{h-1}} \left| \sum_{t_{h-2=k}}^{t_{h-1}} q_{t_{h-1},t_{h-2}} ...... \left[\sum_{t_1=k}^{t_2} q_{t_2,t_1} (1 - p_{t_1}) \right] .... \right|$$

We let $v_1 = 1 - p(t_1)$ and, for all l $(1 \le l \le h)$

$$v_l(t_l) = \left\lfloor \sum_{t_{l-1}=k}^{t_l} q_{t_l t_{l-1}} ...... \left[\sum_{t_1=k}^{t_2} q_{t_2 t_1} (1 - p_{t_1}) \right] ...... \right\rfloor ;$$

Then $b = v_h(n)$.

the recursive formula for computing the v_l is

$$v_l(t_l) = \sum_{t_{l-1}=h}^{t_l} q_{t_l,t_{l-1}} v_{l-1}(t_{l-1})$$

(identical at the one for the w_l) where initialization for t_1 such that $k \le t_1 \le n$, is

$$v_1(t_1) = 1 - p_{t_1}$$

The recursive computation of the z_m is then as follows: initialization (for $m \le h$)

- $if \;\; m < h, z_m = 0$

- $z_h = \sum_{s=k}^{n-1} (1-y)^s [2y + (n-s-1)y^2 + (1-y)^n;$

- recursion (for $m > h$):

-

6. Numerical examples

We give the value of $Q_{h,m;k,n} = z_m$ *for* $h = k = 3;$

$n = 2k = 6; m = 10$ *and* $m = 50; p = 0,9, p = 0,95$ *and* $p = 0,99.$

We first give the values of the failure probability $z_m = Q_{h,m;k,n}$ *for* $h = k = 3, m = 10$ and

$m = 50, n = 2k = 6$ and some values of p .

	$p = 0,9$	$p = 0,95$	$p = 0,99$
$m = 10$	$3,18728.10^{-8}$	$6,2468.10^{-11}$	$3,19996.10^{-17}$
$m = 50$	$1,911163.10^{-7}$	$3,74789.10^{-10}$	$1,91997.10^{-16}$

We deduce the lower bounds of the failure probability $Q_{h,m;k,n}$ for $h = k = 3; m = 10$ and $m =$ 50 ; $n =$ 10 and $n =$ 50 , using for b the value 6 from which results the number a of intervals 2−link-up of length 6 in $\{1,2,..,n\}$:

- if $n = 10,$ $a = 2$ and $c = 0$, hence $Q_{3,m;3,10} \geq \left(Q_{3,m;3,6}\right)^2$
- if $n = 50, a = 12$ and $c = 0$, hence $Q_{3,m;3,50} \geq \left(Q_{3,m;3,6}\right)^{12}$

Tables of lower bounds:

$p = 0.99$

	$n = 10$	$n = 50$
$m = 10$	6.3746×10^{-8}	3.8247×10^{-7}
$m = 50$	3.8233×10^{-7}	2.2940×10^{-6}

$p = 0.95$

	$n = 10$	$n = 50$
$m = 10$	1.2494×10^{-10}	7.4962×10^{-10}
$m = 50$	7.4958×10^{-10}	4.4975×10^{-9}

$p = 0.99$

	$n = 10$	$n = 50$
$m = 10$	6.3999×10^{-17}	3.8400×10^{-16}
$m = 50$	3.8399×10^{-16}	2.3040×10^{-15}

Note that in each line the lower bound increases when n increases, which was expected because the reliability of the system decreases as the size increases. In addition, the results in the boxes (10, 50) and (50, 10) are very close, which is natural because the systems are the same, just one is symmetrical to the other.

7. Conclusion

In this work , we study the hk -within-connected-out-of-(m,n) : F systems consist in rectangular sheets of components, of size $m \times n$, such that the system fails if all components in a submatrix of size $h \times k$ are failing. Some properties are gjiven in covered-up systems section. Also, we use the resultats in usinjg square matrix. We employ a technique introduced in and grounded in the component replication method to suggest novel lower bounds, which, in the scenario of square sheets, surpass those found in [11] . To achieve this objective, we utilize an algorithm referred to as the "consecutive columns algorithm" under the additive condition $n \leq 2k$. Finally, numerical illustrations are provided to evaluate the new methodology.

AUTHORS CONTRIBUTIONS

Zineb AZOUZ: Investigation; formal analysis; supervision; methodology, writing—original draft; simulation, interpretation of results; writing—review and editing.

Abdelfateh Beghriche: methodology, writing—original draft ; Software; formal analysis; supervision; validation.

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