# Some fixed point results in ordered complete dislocated quasi $G_d$ metric space

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*Abstract:* In this paper, we discuss the fixed points of mappings satisfying contractive type condition on a closed ball in an ordered complete dislocated quasi G metric space. The notion of dominated mappings is applied to approximate the unique solution of non linear functional equations. An example is given to show the validity of our work. Our results improve/generalize several well known recent and classical results.

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## 1 Introduction and Preliminaries

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of $T$ if $x = Tx$. Let $x_0$ be an arbitrary chosen point in $X$. Define a sequence $\{x_n\}$ in $X$ by a simple iterative method given by $x_{n+1} = Tx_n,$ where $n \in \{0, 1, 2, 3, ...\}$. Such a sequence is called a picard iterative sequence and its convergence plays a very important role in proving existence of fixed point of a mapping $T$. A self mapping $T$ on a metric space $X$ is said to be a Banach contraction mapping if,

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. Recently, many results appeared in literature related to fixed point results in complete metric spaces endowed with a partial ordering . Ran and Reurings [17] proved an analogue of Banach's fixed point theorem in metric space endowed with partial order and gave applications to matrix equations. Subsequently, Nieto et. al. [12] extended the results of [17] for non decreasing mappings and applied this results obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. On the other hand in 2005, Mustafa and Sims in [14] introduce the notion of a generalized metric space as generalization the usual metric space. Mustafa and others studied fixed point theorems for mappings satisfying different contractive conditions for further useful results can be seen in [3, 8, 9, 10, 15, 16, 21]. Recently, Arshad et. al. [4] proved a result concerning the existence of fixed points of a mapping satisfying a contractive condition on closed ball in a complete dislocated metric space. For further results on closed ball we refer the reader to [5, 6, 7, 13, 20] and references their in. The dominated mapping [2] which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems . For example $x$ denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

In this paper we have obtained fixed point results for dominated self- mappings in an ordered complete dislocated quasi $G_d$ metric space on a closed ball under contractive condition to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exists even yet in metric spaces. An example shows how this result can be used when the corresponding results cannot.

**Definition 1** *Let $X$ be a nonempty set and let $G_d : X \times X \times X \to R^+$ be a function satisfying the*

following axioms:
(i) If $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) = 0$,then $x = y = z$,
(ii) $G_d(x, y, z) \leq G_d(x, a, a) + G_d(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair $(X, G_d)$ is called the dislocated quasi $G_d$-metric space. It is clear that if

$G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) = 0$ then from (i) $x = y = z$. But if $x = y = z$ then $G_d(x, y, z)$ may not be 0. It is observed that if $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y)$ for all $x, y, z \in X$, then $(X, G_d)$ becomes a dislocated $G_d$-metric space.

**Example 2** *If $X = R^+ \cup \{0\}$ then $G_d(x, y, z) = x + \max\{x, y, z\}$ defines a dislocated quasi metric $G$ on $X$.*

**Definition 3** *Let $(X, G_d)$ be a $G_d$-metric space, and let $\{x_n\}$ be a sequence of points in $X$, a point $x$ in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G_d(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is $G_d$-convergent to $x$.Thus, if $x_n \to x$ in a $G_d$-metric space $(X, G_d)$, then for any $\in > 0$, there exists $n, m \in \mathbb{N}$ such that $G_d(x, x_n, x_m) < \in$, for all $n, m \geq N$.*

**Definition 4** *Let $(X, G_d)$ be a $G_d$-metric space. A sequence $\{x_n\}$ is called $G_d$-Cauchy sequence if, for each $\in > 0$ there exists a positive integer $n^\star \in \mathbb{N}$ such that $G_d(x_n, x_m, x_l) < \in$ for all $n, l, m \geq n^\star$; i.e. if $G_d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

***Definition 5*** *$G_d$-metric space $(X, G_d)$ is said to be $G_d$-complete if every $G_d$-Cauchy sequence in $(X, G_d)$ is $G_d$-convergent in $X$.*

**Proposition 6** *Let $(X, G_d)$ be a $G_d$-metric space, then the following are equivalent:*

(1) $\{x_n\}$ is $G_d$ convergent to $x$.
(2) $G_d(x_n, x_n, x) \to 0$ as $n \to \infty$.
(3) $G_d(x_n, x, x) \to 0$ as $n \to \infty$.
(4) $G_d(x_n, x_m, x) \to 0$ as $m\ n \to \infty$.

**Definition 7** *Let $(X, G_d)$ be a $G_d$-metric space then for $x_0 \in X$, $r > 0$, the closed ball with centre $x_0$ and radius $r$ is,*

$$\overline{B(x_0, r)} = \{y \in X : G_d(x_0, y, y) \leq r\}.$$

**Definition 8** *[2] Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

**Definition 9** *[2] Let $(X, \preceq)$ be a partially ordered set. A self mapping $f$ on $X$ is called dominated if $fx \preceq x$ for each $x$ in $X$.*

**Example 10** *[2] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore $f$ is a dominated map.*

## 2 Fixed Points of Contractive Mapping

**Theorem 11** *Let $(X, \preceq, G_d)$ be an ordered complete dislocated quasi $G_d$ metric space, and $T : X \to X$ be a dominated mapping. Suppose there exists $a, b$ such that $a + 3b < 1$ and for all comparable elements $x, y$ and $z$ in $\overline{B(x_0, r)}$, with $x_0 \in \overline{B(x_0, r)}$, $r > 0$,.*

$$\begin{aligned} G_d(Tx, Ty, Tz) &\leq a\ G_d(x, y, z) + b\ [G_d(x, Tx, Tx) \\ &\quad + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)] \end{aligned} \tag{2.1}$$

$$\text{where } \lambda = \frac{a + b}{1 - 2b}$$

$$\text{and } G_d(x_0, Tx_0, Tx_0) \leq (1 - \lambda)r. \tag{2.2}$$

*If for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and*

$$\begin{aligned} & G(x_0, Tx_0, Tx_0) + G(v, Tv, Tv) + G(v, Tv, Tv) \\ \leq\ & G(x_0, v, v) + G(Tx_0, Tv, Tv) + G(Tx_0, Tv, Tv) \end{aligned} \tag{2.3}$$

*then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $G_d(x^\star, x^\star, x^\star) = 0$ and $x^\star = Tx^\star$.*

***Proof.*** Consider a picard sequence $x_{n+1} = Tx_n$ with initial guess $x_0$. As $x_{n+1} = Tx_n \preceq x_n$ for all $n \in \{0\} \cup \mathbb{N}$. By inequality (2.2), $G_d(x_0, x_1, x_1) \leq r$. It implies that $x_1 \in \overline{B(x_0, r)}$. Similarly $x_2 \ldots x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$.

$$\begin{aligned} G_d(x_j, x_{j+1}, x_{j+1}) &= G_d(Tx_{j-1}, Tx_j, Tx_j) \leq a\ G_d(x_{j-1}, x_j, x_j) \\ &\quad + b[G_d(x_{j-1}, Tx_{j-1}, Tx_{j-1}) + G_d(x_j, x_{j+1}, x_{j+1}) \\ &\quad + G_d(x_j, x_{j+1}, x_{j+1})] \\ (1 - 2b)G_d(x_j, x_{j+1}, x_{j+1}) &\leq (a + b)G_d(x_{j-1}, x_j, x_j) \\ G_d(x_j, x_{j+1}, x_{j+1}) &\leq \frac{(a + b)}{(1 - 2b)} G_d(x_{j-1}, x_j, x_j) \\ &\ \ \vdots \\ G_d(x_j, x_{j+1}, x_{j+1}) &\leq \lambda^j G_d(x_0, x_1, x_1). \end{aligned} \tag{2.4}$$

*Now by using the inequality* (2.2) *and* (2.4) *we have*

$$\begin{aligned}
G_d(x_j,x_{j+1},x_{j+1}) &\leq G_d(x_0,x_1,x_1)+G_d(x_1,x_2,x_2)+\cdots+G_d(x_j,x_{j+1},x_{j+1})\\
G_d(x_j,x_{j+1},x_{j+1}) &\leq (1-\lambda)r+\lambda(1-\lambda)r+\cdots+\lambda^j(1-\lambda)r\\
G_d(x_j,x_{j+1},x_{j+1}) &\leq r(1-\lambda)[1+\lambda+\lambda^2+\cdots+\lambda^j]\\
G_d(x_j,x_{j+1},x_{j+1}) &\leq r(1-\lambda)\frac{(1-\lambda^{j+1})}{(1-\lambda)}\leq r\\
&\Rightarrow G_d(x_j,x_{j+1},x_{j+1})\leq r.
\end{aligned}$$

*Thus* $x_{j+1}\in\overline{B(x_0,r)}$. *Hence* $x_n\in\overline{B(x_0,r)}$ *for all* $n\in\mathbb{N}$. *Now inequality* (2.4) *can be written as in the form of*

$$G_d(x_n,x_{n+1},x_{n+1})\leq\lambda^n G_d(x_0,x_1,x_1)\text{ for all }n\in\mathbb{N}. \tag{2.5}$$

*By using inequality* (2.5) *we get*

$$\begin{aligned}
G_d(x_n,x_{n+i},x_{n+i}) &\leq G_d(x_n,x_{n+1},x_{n+1})+\cdots+G_d(x_{n+i-1},x_{n+i},x_{n+i})\\
G_d(x_n,x_{n+i},x_{n+i}) &\leq \frac{\lambda^n(1-\lambda^i)}{(1-\lambda)}\,G_d(x_0,x_1,x_1)\to 0\text{ as }n\to\infty \qquad (2.6)
\end{aligned}$$

*Notice that the sequence* $\{x_n\}$ *is Cauchy sequence in* $(\overline{B(x_0,r)}\ ,G_d)$. *Therefore there exist a point* $x^\star\in\overline{B(x_0,r)}$.

$$\begin{aligned}
\lim_{n\to\infty}G_d(x_n,x^\star,x^\star) &= \lim_{n\to\infty}G_d(x^\star,x^\star,x_n)=0\\
G_d(x^\star,Tx^\star,Tx^\star) &\leq G_d(x^\star,x_n,x_n)+G_d(x_n,Tx^\star,Tx^\star)
\end{aligned}$$

*By assumption* $x^\star\preceq x_n\preceq x_{n-1}$, *therefore,*

$$\begin{aligned}
G_d(x^\star,Tx^\star,Tx^\star) &\leq G_d(x^\star,x_n,x_n)+G_d(Tx_{n-1},Tx^\star,Tx^\star)\\
G_d(x^\star,Tx^\star,Tx^\star) &\leq G_d(x^\star,x_n,x_n)+a\ G_d(x_{n-1},x^\star,x^\star)\\
&\quad +b[G_d(x_{n-1},Tx_{n-1},Tx_{n-1})+G_d(x^\star,Tx^\star,Tx^\star)\\
&\quad G_d(x^\star,Tx^\star,Tx^\star)]\\
G_d(x^\star,Tx^\star,Tx^\star) &\leq G_d(x^\star,x_n,x_n)+a\ G_d(x_{n-1},x^\star,x^\star)\\
&\quad +b[G_d(x_{n-1},Tx_{n-1},Tx_{n-1})+2G_d(x^\star,Tx^\star,Tx^\star)\\
(1-2b)G_d(x^\star,Tx^\star,Tx^\star) &\leq G_d(x^\star,x_n,x_n)+a\ G_d(x_{n-1},x^\star,x^\star)\\
&\quad +b\ G_d(x_{n-1},x_n,x_n)
\end{aligned}$$

*Taking* $\lim_{n\to\infty}$ *both sides and using* (2.6) *we have*

$$\begin{aligned}
(1-2b)G_d(x^\star,Tx^\star,Tx^\star) &\leq 0+a(0)+b(0)\\
&\Rightarrow G_d(x^\star,Tx^\star,Tx^\star)\leq 0\\
&\Rightarrow x^\star=Tx^\star. \qquad (2.7)
\end{aligned}$$

*Similarly* $G_d(Tx^\star, Tx^\star, x^\star) = 0$ *and* $G_d(Tx^\star, x^\star, Tx^\star) = 0$ *and hence* $x^\star = Tx^\star$*.Now*

$$\begin{aligned} G_d(x^\star, x^\star, x^\star) &= G_d(Tx^\star, Tx^\star, Tx^\star) \le a\ G_d(x^\star, x^\star, x^\star) \\ &\quad +3bG_d(x^\star, Tx^\star, Tx^\star) \\ (1-a-3b)G_d(x^\star, x^\star, x^\star) &\le 0 \\ &\Rightarrow G_d(x^\star, x^\star, x^\star) \le 0. \end{aligned}$$

*This implies that* $G_d(x^\star, x^\star, x^\star) = 0$.

***Uniqueness:***

*Let* $y^\star$ *be another point in* $\overline{B(x_0, r)}$ *such that*

$$\begin{aligned} y^\star &= Ty^\star. &\text{(2.8)} \\ G_d(y^\star, y^\star, y^\star) &= G_d(Ty^\star, Ty^\star, Ty^\star) \le a\ G_d(y^\star, y^\star, y^\star) \\ &\quad +3b[G_d(y^\star, Ty^\star, Ty^\star)] \\ (1-a-3b)G_d(y^\star, y^\star, y^\star) &\le 0 \\ &\Rightarrow G_d(y^\star, y^\star, y^\star) \le 0. \\ &\Rightarrow G_d(y^\star, y^\star, y^\star) = 0. \end{aligned}$$

*If* $x^\star$ *and* $y^\star$ *are comparable then*

$$\begin{aligned} G_d(x^\star, y^\star, y^\star) &= G_d(Tx^\star, Ty^\star, Ty^\star) \le a\ G_d(x^\star, y^\star, y^\star) \\ &\quad +b[G_d(x^\star, Tx^\star, Tx^\star) + 2G_d(y^\star, Ty^\star, Ty^\star)] \\ (1-a)G_d(x^\star, y^\star, y^\star) &\le 0 \\ &\Rightarrow G_d(x^\star, y^\star, y^\star) = 0. \end{aligned}$$

*Similarly,* $G_d(y^\star, y^\star, x^\star) = 0$*. This shows that* $x^\star = y^\star$.

*If* $x^\star$ *and* $y^\star$ *are not comparable then there exist a point* $v \in \overline{B(x_0, r)}$ *which is a lower bound of both* $x^\star$ *and* $y^\star$*. Now we will to prove that* $T^n v \in \overline{B(x_0, r)}$*. Moreover by assumptions* $v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0$*. Now by using* (2.1)*, we have,*

$$G_d(Tx_0, Tv, Tv) \le a\ G_d(x_0, v, v) + b\ [G_d(x_0, x_1, x_1) + 2G_d(v, Tv, Tv)].$$

*By using* (2.3)*, we have*

$$\begin{aligned} G_d(Tx_0, Tv, Tv) &\le a\ G_d(x_0, v, v) + b\ [G_d(x_0, v, v) + 2G_d(x_1, Tv, Tv)] \\ (1-2b)G_d(Tx_0, Tv, Tv) &\le (a+b)\ G_d(x_0, v, v) \\ G_d(Tx_0, Tv, Tv) &\le \frac{(a+b)}{(1-2b)}\ G_d(x_0, v, v) \\ G_d(Tx_0, Tv, Tv) &\le \lambda\ G_d(x_0, v, v). &\text{(2.9)} \end{aligned}$$

*Now,*

$$\begin{aligned} G_d(x_0, Tv, Tv) &\le G_d(x_0, x_1, x_1) + G_d(x_1, Tv, Tv) \\ G_d(x_0, Tv, Tv) &\le G_d(x_0, x_1, x_1) + \lambda\ G_d(x_0, v, v)\ \textit{by}\ (2.9) \\ G_d(x_0, Tv, Tv) &\le (1-\lambda)r + \lambda r \\ G_d(x_0, Tv, Tv) &\le r. \end{aligned}$$

*It follows that* $Tv \in \overline{B(x_0, r)}$. *Now we will prove that* $T^n v \in \overline{B(x_0, r)}$. *By using mathematical induction to apply inequality* (2.1). *Let* $T^2 v,\ T^3 v, \cdots T^j v \in \overline{B(x_0, r)}$ *for some* $j \in N$. *As*

$$T^j v \preceq T^{j-1} v \preceq \cdots \preceq v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0.$$

*Then,*

$$\begin{aligned}
G_d(T^j v, T^{j+1} v, T^{j+1} v) &= G_d(T(T^{j-1} v), T(T^j v), T(T^j v)) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq a\ G_d(T^{j-1} v, T^j v, T^j v) + b\ [G_d(T^{j-1} v, T^j v, T^j v) \\
&\quad +2G_d(T^j v, T^{j+1} v, T^{j+1} v)] \\
(1-2b)G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq (a+b)G_d(T^{j-1} v, T^j v, T^j v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda G_d(T^{j-1} v, T^j v, T^j v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^2 G_d(T^{j-2} v, T^{j-1} v, T^{j-1} v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^3 G_d(T^{j-3} v, T^{j-2} v, T^{j-2} v) \\
&\ \ \vdots \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^j G_d(T^{j-j} v, T^{j-(j-1)} v, T^{j-(j-1)} v) \\
G_d(T^j v, T^{j+1} v, T^{j+1} v) &\leq \lambda^j G_d(v, Tv, Tv) \qquad (2.10)
\end{aligned}$$

*Now,*

$$\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq G_d(Tx_j, T(T^j v), T(T^j v)) \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\ G_d(x_j, T^j v, T^j v) \\
&\quad +b\ [G_d(x_j, Tx_j, Tx_j) + 2G_d(T^j v, T^{j+1} v, T^{j+1} v)].
\end{aligned}$$

*By using* (2.4) *and* (2.10)

$$\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\quad +b[\lambda^j G_d(x_0, x_1, x_1) + 2\lambda^j G_d(v, Tv, Tv)] \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\quad +b\lambda^j [G_d(x_0, x_1, x_1) + 2G_d(v, Tv, Tv)]
\end{aligned}$$

*By using the condition* (2.3)

$$\begin{aligned}
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq a\lambda^j G_d(x_0, v, v) \\
&\quad +b\lambda^j [G_d(x_0, v, v) + 2\lambda G_d(x_0, v, v)] \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq \lambda^j (a + b + 2b\lambda) G_d(x_0, v, v) \\
G_d(x_{j+1}, T^{j+1} v, T^{j+1} v) &\leq \lambda^{j+1} G_d(x_0, v, v) \qquad (2.11)
\end{aligned}$$

*Now ,*

$$\begin{aligned}
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le G_d(x_0,x_{j+1},x_{j+1})+G_d(x_{j+1},T^{j+1}v,T^{j+1}v)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le G_d(x_0,x_1,x_1)+\cdots+G_d(x_j,x_{j+1},x_{j+1})\\
&\quad +G_d(x_{j+1},T^{j+1}v,T^{j+1}v)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le G_d(x_0,x_1,x_1)+\lambda G_d(x_0,x_1,x_1)\\
&\quad +\cdots+\lambda^{j+1}G_d(x_0,v,v) \text{ by } (2.5) \text{ and } (2.11)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le G_d(x_0,x_1,x_1)[1+\lambda+\lambda^2+\cdots+\lambda^j]+\lambda^{j+1}r \text{ as } v\in\overline{B(x_0,r)}\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le (1-\lambda)r\frac{(1-\lambda^{j+1})}{(1-\lambda)}+\lambda^{j+1}r=r\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\le r.
\end{aligned}$$

*It follows that* $T^{j+1}v\in\overline{B(x_0,r)}$ *and hence* $T^jv\in\overline{B(x_0,r)}$. *Now the inequality* (2.10) *can be written as*

$$G_d(T^nv,T^{n+1}v,T^{n+1}v)\le\lambda^nG_d(v,Tv,Tv)\to 0 \text{ as } n\to\infty \tag{2.12}$$

*Now,*

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &= G_d(Tx^\star,Ty^\star,Ty^\star)\\
G_d(x^\star,y^\star,y^\star) &\le G_d(Tx^\star,T^{n+1}v,T^{n+1}v)+G_d(T^{n+1}v,Ty^\star,Ty^\star)\\
G_d(x^\star,y^\star,y^\star) &\le a\ G_d(x^\star,T^nv,T^nv)+b\ [G_d(x^\star,Tx^\star,Tx^\star)\\
&\quad +2G_d(T^nv,T^{n+1}v,T^{n+1}v)]+a\ G_d(T^nv,Ty^\star,Ty^\star)\\
&\quad +b\ [G_d(T^nv,T^{n+1}v,T^{n+1}v)+2G_d(y^\star,Ty^\star,Ty^\star)]
\end{aligned}$$

*By using* (2.7), (2.8) *and* (2.12) *we have*

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &\le a\ G_d(x^\star,T^nv,T^nv)+a\ G_d(T^nv,y^\star,y^\star)\\
G_d(x^\star,y^\star,y^\star) &\le a\ [G_d(Tx^\star,T^nv,T^nv)+G_d(T^nv,Ty^\star,Ty^\star)]\\
G_d(x^\star,y^\star,y^\star) &\le a\ [a\ G_d(x^\star,T^{n-1}v,T^{n-1}v)+b\ G_d(x^\star,Tx^\star,Tx^\star)\\
&\quad +2b\ G_d(T^{n-1}v,T^nv,T^nv)+a\ G_d(T^{n-1}v,y^\star,y^\star)\\
&\quad +b\ G_d(T^{n-1}v,T^nv,T^nv)+2b\ G_d(y^\star,Ty^\star,Ty^\star)].
\end{aligned}$$

*By using* (2.7), (2.8) *and* (2.12) *we have*

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &\le a^2\ [G_d(x^\star,T^{n-1}v,T^{n-1}v)+G_d(T^{n-1}v,y^\star,y^\star)]\\
G_d(x^\star,y^\star,y^\star) &\le a^3\ [G_d(x^\star,T^{n-2}v,T^{n-2}v)+G_d(T^{n-2}v,y^\star,y^\star)]\\
&\ \ \vdots\\
G_d(x^\star,y^\star,y^\star) &\le a^n\ [G_d(x^\star,Tv,Tv)+G_d(Tv,y^\star,y^\star)]\\
G_d(x^\star,y^\star,y^\star) &\to 0 \text{ as } n\to\infty\\
G_d(x^\star,y^\star,y^\star) &= 0\\
x^\star &= y^\star.
\end{aligned}$$

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*This proves the uniqueness of the fixed point.* ■

Now we give an example of an ordered complete dislocated quasi $G_d$-metric space in which the contraction does not hold on the whole space rather it holds on a closed ball only.

**Example 12** *Let $X = \mathbb{R}^+ \cup \{0\}$ be endowed with usual order and $G_d : X \times X \times X \to X$ be a complete dislocated quasi $G_d$ metric space defined by,*

$$G_d(x, y, z) = \left\{ \begin{array}{c} 0 \text{ if } x = y = z \\ \max\ \{2x, y, z\} \text{ otherwise.} \end{array} \right\}$$

Then $(X, G_d)$ is a $G_d$ complete $G$ dislocated quasi metric space.

Let $T : X \to X$ be defined by,

$$Tx = \left\{ \begin{array}{c} \frac{x}{5} \text{ if } x \in [0, \frac{3}{2}] \\ x - \frac{1}{3} \text{ if } x \in [\frac{3}{2}, \infty) \end{array} \right\}.$$

Clearly, $T$ is a dominated mappings. Take $x_0 = \frac{1}{3}$, $r = \frac{3}{2}$, $\overline{B(x_0, r)} = [0, \frac{3}{2}]$ and $\lambda = \frac{1}{4}$, $a + 3b < 1$, where $a = \frac{1}{10}$, and $b = \frac{1}{10}$.

$$\begin{aligned} G_d(x_0, Tx_0, Tx_0) &\leq (1 - \lambda)r \\ G_d(\frac{1}{3}, T\frac{1}{3}, T\frac{1}{3}) &= \max\{\frac{2}{3}, \frac{1}{15}, \frac{1}{15}\} = \frac{2}{3} \\ \text{Since } (1 - \lambda)r &= (1 - \frac{1}{4})\frac{3}{2} = \frac{9}{8} \\ &\Rightarrow \frac{2}{3} \leq \frac{9}{8} \\ &\Rightarrow 16 \leq 27 \end{aligned}$$

Also if $x, y$ and $z \in [\frac{3}{2}, \infty)$. We assume that $x > y$, and $y > z$, then

$$\begin{aligned} \max\{2x - \frac{2}{3}, y - \frac{1}{3}, z - \frac{1}{3}\} &\geq \frac{1}{10} \max\{2x, y, z\} \\ &\quad \frac{1}{10}[\max\{2x, x - \frac{1}{3}, x - \frac{1}{3}\} \\ &\quad + \max\{2y, y - \frac{1}{2}, y - \frac{1}{2}\} \\ &\quad + \max\{2z, z - \frac{1}{2}, z - \frac{1}{2}\}] \\ G_d(Tx, Ty, Tz) &\geq a\ G_d(x, y, z) + b\ [G_d(x, Tx, Tx) \\ &\quad + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)] \end{aligned}$$

So the contractive conditions does not holds in $X$. Now if $x, y$ and $z \in \overline{B(x_0, r)}$

then,

$$\begin{aligned} G_d(Tx,Ty,Tz) &= \max\{\frac{2x}{5},\frac{y}{5},\frac{z}{5}\} \le \frac{1}{10}\{2x,y,z\} \\ &+\frac{1}{10}[\max\{2x,\frac{x}{5},\frac{x}{5}\}+\max\{2y,\frac{y}{5},\frac{y}{5}\} \\ &+\max\{z,\frac{z}{5},\frac{z}{5}\}] \\ &\Rightarrow G_d(Tx,Ty,Tz) \le a\ G_d(x,y,z) + b\ [G_d(x,Tx,Tx) \\ &+G_d(y,Ty,Ty)+G_d(z,Tz,Tz)]. \end{aligned}$$

Hence it satisfies all the requirements of Theorem11. If we take $b = 0$ in inequality (2.1) then we obtain the following corollary.

**Corollary 13** *Let $(X, \preceq, G)$ be an ordered complete dislocated quasi $G-$metric space, $T : X \to X$ be a dominated mapping and $x_0$ be any arbitrary point in $X$. Suppose there exists $a \in [0, 1)$ with,*

$$G(Tx,Ty,Tz) \le a\ G(x,y,z),\ \textit{for all } x,\ y \textit{ and } z \in Y = \overline{B(x_0,r),}$$

*and*

$$G(x_0,Tx_0,Tx_0) \le (1-a)r.$$

*If for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0,r)}$ such that $x^\star = Sx^\star$ and $G(x^\star,x^\star,x^\star) = 0$. Moreover if for any three points $x, y$ and $z$ in $\overline{B(x_0,r)}$ such that there exists a point $v \in \overline{B(x_0,r)}$ such that $v \preceq x$, $v \preceq y$ and $v \preceq z$, that is, every three of elements in $\overline{B(x_0,r)}$ has a lower bound, then the point $x^\star$ is unique.*
*Similarly if we take $a = 0$ in inequality (2.1) then we obtain the following corollary.*

**Corollary 14** *Let $(X, \preceq, G)$ be an ordered complete dislocated quasi G-metric space $T : X \to R$ be a mapping and $x_0$ be an arbitrary point in $X$. Suppose there exists $b \in \left[0, \frac{1}{3}\right)$ with*

$$G(Tx,Ty,Tz) \le b\ (G(x,Tx,Tx) + G(y,Ty,Ty) + G(z,Tz,Tz))$$

*for all comparable elements $x, y, z \in \overline{B(x_0,r)}$ and*

$$G(x_0,Tx_0,Tx_0) \le (1-\lambda)r,$$

*where $\lambda = \frac{b}{1-2b}$. If for non increasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0,r)}$ such that $x^\star = Sx^\star$and $G(x^\star,x^\star,x^\star) = 0$. Moreover, if for any three points $x, y, z \in \overline{B(x_o,r)}$, there exists a point $v$ in $\overline{B(x_0,r)}$ such that $v \preceq x$ and $v \preceq y$, $v \preceq z$.*

## References


[1] M. Abbas and B. E. Rhoades, Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Appl Maths and Computation, 215(2009), 262-269.

[2] M. Abbas and S .Z. Nemeth, Finding solutions of implict complementarity problems by isotonicty of metric projection, Nonlinear Anal, 75(2012), 2349-2361.

[3] R. Agarwal and E. Karapinar, Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory and Appli, 2013(2013),15pages.

[4] M. Arshad ,A. Shoaib, and I. Beg, Fixed Point of pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space accepted in Fixed Point Theory and Appl, 2013(2013),15pages.

[5] A. Azam, S. Hussain and M. Arshad, Common Fixed Points of Kannan Type Fuzzy Mappings on closed balls, Appl. Math. Inf. Sci. Lett. 1, 2 (2013), 7-10.

[6] A. Azam, S. Hussain and M. Arshad, Common fixed points of Chatterjea type fuzzy mappings on closed balls, Neural Computing & Appl, 21(2012), S313–S317.

[7] A. Azam, M. Waseem, M. Rashid, Fixed point theorems for fuzzy contractive mappings in quasi-pseudo-metric spaces, Fixed Point Theory Appl, 27(2013) 14pages.

[8] Lj. Gaji 'c and M. Stojakovi 'c, On Ciri 'c generalization of mappings with a contractive iterate at a point in G-metric spaces, Appl Maths and computation, 219(2012), 435-441.

[9] H. Hydi, W. Shatanawi, C. Vetro, On generalized weak G-contraction mappings in G-metric spaces, Compute. Math. Appl., 62 (2011), 4223-4229.

[10] M. Jleli and B. Samet, Remarks on G-metric spaces and fixed point theorems Fixed Point Theory appl, 210 (2012).

[11] H.K Nashine, Coupled common fixed point results in ordered G-metric spaces, J. Nonlinear Sc. Appl. 1(2012), 1-13.

[12] J. J. Nieto and R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22 (3) (2005), 223-239.

[13] M. A. Kutbi, J. Ahmad, N. Hussain and M. Arshad, Common Fixed Point Results for Mappings with Rational Expressions, Abstr. Appl. Anal, 2013, Article ID 549518, 11 pages.

[14] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear and Convex Anal, 7(2006) 289-297.

[15] Z. Mustafa, H. Obiedat, and F. Awawdeh, Some fixed point theorem for mappings on a complete G- metric space, Fixed point theory and appl, 2008(2008),12pages.

[16] H. Obiedat and Z. Mustafa, Fixed point results on a non symmetric G-metric spaces,Jordan Journal of Maths and Stats, 3(2010), 65-79.

[17] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132 (5) (2004), 1435-1443.

[18] B. Samet, C. Vetro, F. Vetro, Remarks on G-metric spaces, Int. J. Anal., 2013(2013), 6pages.

[19] W. Shatanawi, "Fixed point theory for contractive mappings satisfying $\Phi-$maps in G-metric spaces, Fixed Point Theory and Appl, 2010(2010), 9pages.

[20] A. Shoaib, M. Arshad and J. Ahmad, Fixed point results of locally co-tractive mappings in ordered quasi-partial metric spaces, The Sci World Journal, 2013(2013), 14pages.

[21] S. Zhou and F.Gu, Some new fixed points in G- metric spaces Journal of Hangzhou Normal University, 11(2010), 47-50.

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