J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Daubechies Wavelet Method for Second Kind Fredholm Integral Equations with Weakly Singular Kernel *

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Abstract

In this paper, the weakly singular Fredholm integral equations of the second kind are solved by the periodized Daubechies wavelets method. In order to obtain a good degree of accuracy of the numerical solutions, the Sidi-Israeli quadrature formulae are used to construct the approximation of the singular kernel functions. By applying the asymptotically compact theory, we prove the convergence of approximate solutions. In addition, the sidi transformation can be used to degrade the singularities when the kernel function is non-periodic. At last, numerical examples show the method is efficient and errors of the numerical solutions possess high accuracy order $O\left(h^{3+\alpha}\right)$, where h is the mesh size.

Keyword: Daubechies wavelets; weakly singular kernel; Fredholm integral equation of the second; linear and nonlinear integral equations; convergence rate.

1 Introduction

Many problems in science and engineering such as Lapalace's equation, problems in elasticity, conformal mapping, free surface flows and so on, result in Fredholm integral equations with singular or weakly singular (in general logarithmic) and periodic kernels [11]. Therefore, singular or weakly Singular Fredholm linear equations and its nonlinear counterparts are most frequently studied for decades.

*This work is supported by Project (NO. KYTZ201505) Supported by the Scientific Research Foundation of CUIT

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Generally, the weakly singular Fredholm integral equation of the second can be converted into the following form

$$u(x) - \int_0^1 k(x,y)g(u(y))\mathrm{d}y = f(x), \quad x \in [0,1], \tag{1.1}$$

where

$$k(x,y) = H_1(x,y)|x-y|^{\alpha}(\ln|x-y|)^{\beta} + H_2(x,y),\ \alpha > -1,\ \beta \geq 0, \tag{1.2}$$

$u(x)$ is an unknown function and $f \in L^2[0,1]$, and $H_j(x,y)$ $(j=1,2)$ are continuous on $[0,1]$. The integral equation (1.1) is linear when $g(u(y)) = u(y)$, and when $g(u(y)) \neq u(y)$ the equation is nonlinear.

As is known, several different orthonormal basis functions, for example, Chebyshev polynomial [8], Fourier functions [2], and wavelets [3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 17], can be used to approximate the solutions of integral equations. However, for large scale problems, the most attractive one among them may be the wavelet bases, in which the kernel can be transformed to a sparse matrix after discretization. This is mainly due to functions with fast oscillations, or even discontinuities, in localized regions may be approximated well by a linear combination of relatively few wavelets [3].

This paper is organized as follows: in Section 2, the periodized Daubechies wavelets is introduced for solving weakly singular Fredholm integral equations of the second in detail. In Section 3, the convergence and error analysis are investigated. In Section 4, numerical examples are provided to verify the theoretical results. Some useful conclusions are made in Section 5.

2 Periodized Daubechies wavelets method

2.1 Multiresolution analysis and function expansions

Wavelets are attractive for the numerical solution of integrations, because their vanishing moments property leads to operator compression. Especially, Daubechies wavelets [12, 15] have many good properties and can deal with some types of kernels arising from boundary integral formulation of elliptic PDEs, and the coefficient are often numerically sparse. In fact, there are only $O\ (n\log n)$ significant elements. Supposed that ψ and ϕ be the the wavelet of genus N and Daubechies scaling function respectively. Thus their support are $\mathrm{supp}(\phi) = \mathrm{supp}(\psi) = [0, N-1]$. For any $j,k \in Z$, we introduce the notations $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, then their periodic kin with period-1 can be described by

$$\tilde{\phi}_{j,k}(x) = \sum_{n\in Z}\phi_{j,k}(x+n), \quad \tilde{\psi}_{j,k}(x) = \sum_{n\in Z}\psi_{j,k}(x+n), \quad x \in R, \quad 0 \leq k < 2^j. \tag{2.1}$$

Here $\{\tilde{\phi}_{j,k}(x)\}_{k\in Z}$ and $\{\tilde{\psi}_{j,k}(x)\}_{k\in Z}$ are orthogonal [17]. Defining the periodic spaces $\tilde{V}_j = \mathrm{span}\{\tilde{\phi}_{j,k}\}_{k=0}^{2^j-1}$ and $\tilde{W}_j = \mathrm{span}\{\tilde{\psi}_{j,k}\}_{k=0}^{2^j-1}$. A chain of spaces $\tilde{V}_0 \subset \tilde{V}_1 \cdots \subset$

$L^2[0,1]$ can be constructed, which subject to the following conditions: (a) $\cup_{j\geq 0} \tilde{V}_j = L^2[0,1]$, $\cap_{j\in Z}\tilde{V}_j = \{0\}$; (b) $h(x) \in \tilde{V}_j \Leftrightarrow h(2x) \in \tilde{V}_{j+1}$; (c) $\tilde{V}_j \oplus \tilde{W}_j = \tilde{V}_{j+1}$, $\tilde{W}_j \perp \tilde{V}_j$. The Daubechies wavelets and scaling functions described above result in the wavelet theory (i.e., multiresolution analysis (MRA)) of $L^2[0,1]$.

Supposed that function $p(x) \in L^2[0,1]$ be approximated by scaling series at resolution J as

$$p(x) = \sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(x) = \Phi^t(x)c, \quad x \in [0,1], \tag{2.2}$$

where

$$\Phi(x) = [\tilde{\phi}_{J,0}(x), \tilde{\phi}_{J,1}(x), \cdots, \tilde{\phi}_{J,2^{J-1}}(x)], \tag{2.3}$$

and

$$c = (c_{J,0}, c_{J,1}, \cdots, c_{J,2^{J-1}})^t, \quad c_{J,k} = \int_0^1 p(x)\tilde{\phi}_{J,k}(x)\mathrm{d}x. \tag{2.4}$$

First, we calculate the wavelet coefficient $c_{J,k}$ for nonsingular function $p(x) \in L^2[0,1]$. Let $x_i = i/2^J$, $i = 0, 1, \cdots, 2^{J-1}$. Substituting $x = x_i$ into Eq. (2.2), we have

$$p(x) = \sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(i/2^J) = 2^{J/2}\sum_{k=0}^{2^J-1} c_{J,k}\sum_{n\in Z}\phi_{j,k}(2^J n + i - k). \tag{2.5}$$

By using the relationship between $\mathrm{supp}(\phi)$ and $[0,1]$, we know when $J \geq J_0$ only finite terms of the inner summation in (2.5) contribute the following result

$$n = \begin{cases} 0 \text{ or } 1, & \text{if } \ 2^J - N + 2 \leq N - 1, \\ 0, & \text{if } \ 0 \leq k \leq 2^J s - N + 1. \end{cases}$$

Now we write (2.5) as the matrix form

$$p = Tc, \tag{2.6}$$

where $p = [p(0), p(1/2^J), \cdots, p((2^J-1)/2^J)]^t$, T is the nonsingular matrix which entries are the function values of $\phi(x)$ at integers (i.e., $\phi(0)$,$\phi(1)$,$\cdots$,$\phi(N-2)$) appear in it, and hence it satisfies

$$\sum_{i=1}^{2^J} T_{ij} = \sum_{j=1}^{2^J} T_{ij} = 2^{J/2}, \quad i,j = 1, 2, \cdots, 2^J. \tag{2.7}$$

Consequently, the function $k(x,y) \in L^2([0,1]\times[0,1])$ in Eq.(1.1) can be approximated at resolution J as

$$k(x,y) = \Phi^t(x)Q\Phi(y), \tag{2.8}$$

where Q is the $2^J \times 2^J$ coefficient matrix. Eq. (2.8) can be written as the following form

$$Q = T^{-1}KT^{-t}, \tag{2.9}$$

where K is the $2^J \times 2^J$ kernel matrix with $K_{i,j} = k(i/2^J, j/2^J)$.

Secondly, if the function $p(x)$ is singular on $[0,1]$, some values of $p(x)$ at the dyadic points $x_i = i/2^J$, $i = 0, 1, \cdots, 2^J - 1$ may be unbounded and then c can not be immediately solved from the Eq.(2.6). In order to avoid the Eq.(2.6) being invalid, we can use the method in the literature [17] to compute the values of $p(x)$. Without loss of generality, we assumed that function $p(x) \in L^2[0,1]$ has only one singular point $x_i = i/2^J$, $i \in \{0, 1, \cdots, 2^J - 1\}$. Then the function value $p(x_i)$ in Eq.(2.6) can be computed via on the following (see [17])

$$p(i/2^J) = 2^J \int_0^1 p(x)\mathrm{d}x - \sum_{j=0,j\neq i}^{2^J-1} p(j/2^J), \quad i \in \{0, 1, \cdots, 2^J - 1\}, \tag{2.10}$$

where integration $\int_0^1 p(x)\mathrm{d}x$ can be calculated by Sidi-Israeli quadrature formulae [11].

2.2 Kernel function approximation and discretization of singular integral equation

Motivated by the Eq.(2.10) and by thinking $k(x, y)$ as a one-dimensional function of variable x and y respectively, we also have

$$k(x, m/2^J) = 2^J \int_0^1 k(x, y)\mathrm{d}y - \sum_{j=0,j\neq m}^{2^J-1} k(x, j/2^J), \quad m \in \{0, 1, \cdots, 2^J - 1\}. \tag{2.11}$$

The following Theorem 2.1 can be used to construct the kernel approximation of Eq.(1.1).

Theorem 2.1 [11] Assume that the functions $H_1(x, y)$ and $H_2(x, y)$ are 2ℓ times differentiable on $[a, b]$. Assume also that the functions $k(x, y)$ are periodic with period $T = b - a$, and that they are 2ℓ times differentiable on $\widetilde{R} = (-\infty, \infty) \backslash \{x + jT\}_{j=-\infty}^{\infty}$. If $k(x, y) = H_1(x, y)|x - y|^\alpha (\ln |x - y|)^\beta + H_2(x, y)$, $s > -1$, $\beta = 0, 1$, then the quadrature rules of the following integral

$$I[k(x, y)] = \int_a^b k(x, y)\mathrm{d}y, \tag{2.12}$$

are

$$I_n[k(x, y)] = h \sum_{j=1,y_j\neq x}^{n} k(x, y_j) + 2[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x, x)h^{\alpha+1} + H_2(x, x)h, \tag{2.13}$$

and the quadrature errors are

$$E_n[k(x, y)] = 2\sum_{\mu=1}^{\ell-1} [\beta\zeta'(-\alpha - 2\mu) - \zeta(-\alpha - 2\mu)(\ln h)^\beta] \frac{H_1^{(2\mu)}(x, y_j)}{(2\mu)!} h^{2\mu+\alpha+1} + o\left(h^{2\ell}\right), \tag{2.14}$$

where $E_n[k(x,y] = I[k(x,y)] - I_n[k(x,y)]$, and the mesh size is $h = (b-a)/n$.

Let $n = 2^J$ and by (2.13), we can get the Nyström approximation for the kernel function $k(x,y)$

$$k_D(x_i, y_j) = \begin{cases} 2[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x_i,x_i)h^\alpha + H_2(x_i,x_i), & \text{if } i = j, \\ k(x_i,y_j), \quad \text{if } i \neq j. \end{cases} \tag{2.15}$$

Supposed that the kernel function $k(x,y)$, $u(x)$ and $f(x)$ be approximated at resolution J as

$$k(x,y) = \Phi^t(x)Q\Phi(y), \quad f(x) = \Phi^t(x)b \ \text{ and } u(x) = \Phi^t(x)c, \tag{2.16}$$

where $c = [c(0), c(1/2^J), \cdots, c((2^J-1)/2^J)]^t$ is the expansion coefficient vector of $u(x)$. By the orthonormality of periodized wavelets, the integration of the product of the same two scaling function vectors is achieved as

$$\int_0^1 \Phi(x)\Phi^t(x)\mathrm{d}x = I, \tag{2.17}$$

where I is the 2^J by 2^J identity matrix. For the linear integral equation, we have

$$\Phi^t(x)c - \int_0^1 \Phi^t(x)Q\Phi(y)\Phi^t(y)c\mathrm{d}y = \Phi^t(x)b. \tag{2.18}$$

Substituting (2.15), (2.16) and (2.17) into (2.18), and by invoking (2.9), we get a linear system

$$(I - K_D(T^tT)^{-1})u_D = f, \tag{2.19}$$

where $f = [f(0), f(1/2^J), \cdots, f((2^J-1)/2^J)]^t$ and $K_D = T^tQT$. Similarly, the nonlinear case for Eq. (1.1) can be transformed into the following by the wavelet method

$$u_D - K_D(TT^t)^{-1}g(u_D) = f, \tag{2.20}$$

Eq. (2.20) is a system of nonlinear equations about u and can be computed by Newton iteration method.

3 Convergence and error analysis

In this section, we mainly study the convergence and error for the linear case of (1.1) by wavelet method.

We write Eq. (1.1) as the operate form

$$(I - \tilde{K})u = f, \tag{3.1}$$

where

$$(\tilde{K}u)(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y, \tag{3.2}$$

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with the kernel

$$k(x,y) = H_1(x,y)|x-y|^\alpha(\ln|x-y|)^\beta + H_2(x,y),\ \alpha > -1,\ \beta \geq 0, \tag{3.3}$$

and the approximation of $\tilde{K}$ is defined by

$$(\tilde{K}_n u)(x) = h \sum_{j=1, y_j \neq x}^{n} k(x,y_j)u(y_j) + \omega_n(x)u(x), \tag{3.4}$$

where the weight function

$$\omega(x) = 2[\beta\zeta^{'}(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x,x)h^{\alpha+1} + H_2(x,x)h. \tag{3.5}$$

Supposed that the approximation of (3.1) is

$$(I - \tilde{K}_n)u_n(x) = g. \tag{3.6}$$

Lemma 3.1 Supposed the the operator $\tilde{K}_n$ is defined by (3.4), then the operator sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n \overset{a.c}{\to} \tilde{K}, \tag{3.7}$$

where $\overset{a.c}{\to}$ denotes the asymptotically compact convergence.

Proof. Let the continuous kernel approximation of $\tilde{K}$ be defined by

$$k_n^c(x,y) = \begin{cases} k(x,y), & \text{if } |x-y| \geq h, \\ H_1(x,x)h^\alpha(\ln h)^\beta + H_2(x,x), & \text{if } |x-y| < h, \end{cases} \tag{3.8}$$

and the corresponding operator approximation be

$$(K_n^c u)(x) = h\sum_{j=1}^{n} k_n^c(x,y_j)u(y_j). \tag{3.9}$$

For any $v \in C[0,1]$, we have

$$\begin{aligned} \|(\tilde{K} - K_n^c)v\| &= \sup_{\|v\|_\infty \leq 1} \int_0^1 |(k(x,y) - k_n^c(x,y))v(y)|\mathrm{d}y \\ &\leq \int_0^1 |k(x,y) - k_n^c(x,y)|\mathrm{d}y\|v\|_\infty \\ &\leq \int_{|x-y|\leq h} |H_1(x,y)||\,|x-y|^\alpha(\ln|x-y|)^\beta - h^\alpha(\ln h)^\beta|\mathrm{d}y\|v\|_\infty \\ &\leq \max_{x,y\in C[0,1]} |H_1(x,y)| \int_{|x-y|\leq h} ||x-y|^\alpha(\ln|x-y|)^\beta - h^\alpha(\ln h)^\beta|\mathrm{d}y\|v\|_\infty \\ &= O((\ln h)^\beta h^\alpha)\|v\|_\infty, \end{aligned} \tag{3.10}$$

hence, we can obtain

$$\|\tilde{K} - K_n^c\| = O((\ln h)^{\beta} h^{\alpha}) \to 0, \quad \text{as } h \to 0. \tag{3.11}$$

On the other hand, we know $\omega(x) \to 0$ as $h \to 0$ by (3.5), then

$$\|K_n^c - \tilde{K}_n\| \to 0, \quad \text{as } h \to 0. \tag{3.12}$$

First, there exists a subsequence in $\{K_n^c y_n\}$ for any $y_n \subset C[0,1]$ by (3.11),. Without loss of generality, assumed that $K_n^c y_n \to z$ and by (3.12), then

$$\begin{aligned} \|\tilde{K}_n y_n - z\| &\leq \|\tilde{K}_n y_n - K_n^c y_n\| + \|K_n^c y_n - z\| \\ &\leq \|\tilde{K}_n - K_n^c\|\|y_n\| + \|K_n^c y_n - z\| \to 0, \end{aligned} \tag{3.13}$$

that is to say, the sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent. Secondly, for any $y \in C[0,1]$, we have

$$\|\tilde{K}_n y - \tilde{K} y\| \leq \|\tilde{K}_n - K_n^c\|\|y\| + \|K_n^c y - \tilde{K} y\| \to 0. \tag{3.14}$$

The proof of Lemma 3.1 is completed. $\square$

Corollary 3.2 The operator sequence $\{\tilde{K}_n(I - o(h)E)\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n(I - o(h)E) \overset{a.c}{\to} \tilde{K}. \tag{3.15}$$

where E is a matrix and every element in it is one.

Proof. By Lemma 3.1, we know

$$\tilde{K}_n \overset{a.c}{\to} \tilde{K}, \tag{3.16}$$

that is,

$$\|\tilde{K}_n - \tilde{K}\| \to 0. \tag{3.17}$$

Hence, we immediately have

$$\|\tilde{K}_n(I - o(h)E) - \tilde{K}\| \leq \|\tilde{K}_n - \tilde{K}\| + \|\tilde{K}_n\|\|o(h)E\| \to 0. \tag{3.18}$$

The proof of Corollary 3.2 is completed. $\square$

Let $x = (i-1)h,\ i = 1, 2, \cdots, 2^J$, where $h = 1/2^J$. Using the trapezoidal rule to approximate Eq.(2.17), then we have $hTT^t = I + o(h)E$, where E is a matrix and every element in it is one. By $(hTT^t)^{-1} = I + o(h)E$, (2.15) and (2.19), we get

$$(I - \tilde{K}_n(hT^{ht}T)^{-1})u_D = f, \tag{3.19}$$

which is equivalent to

$$(I - \tilde{K}_n(I - o(h)E)u_D = f. \tag{3.20}$$

Hence, by the Corollary 3.2 we get the following remark.

Remark 1 According to the Corollary 3.2, the solutions u_D of Eq.(3.20) by

Daubechies wavelet method are convergent to the solutions u_n of Eq.(3.6) when $h \to 0$.

Theorem 3.3 The solutions of Eq.(3.6) have asymptotic expansions hold at nodes

$$u_n(x) = u(x) + \sigma_1(x)h^{\alpha+3} + \sigma_2(x)h^{\alpha+3}\ln h + o(h^{\alpha+5}\ln h), \tag{3.21}$$

where $\sigma_j(x) \in C[0,1],\ j = 1, 2$ are independent of h, and $\sigma_2 = 0$ when $\beta = 0$ and $\alpha > -1$, or $\beta = 1$ and $\alpha = 0$.

Proof. We construct the auxiliary equation

$$(I - \tilde{K})\sigma = P(x), \tag{3.22}$$

where

$$P(x) = [\beta\zeta'(-\alpha-2) - \zeta(-\alpha-2)(\ln h)^{\beta}](H_1 u)^{(2)}h^{3+\alpha}. \tag{3.23}$$

By invoking Eq.(2.14), we have

$$(\tilde{K}_n - \tilde{K})u(x) = -P(x) + o(h^{5+\alpha}\ln h). \tag{3.24}$$

Using (3.22), we get

$$\begin{aligned} &(I - \tilde{K}_n)(u_n - u - h^{3+\alpha}\sigma) = f - u + \tilde{K}_n u + h^{3+\alpha}(I - \tilde{K}_n)\sigma \\ &= (\tilde{K}_n - \tilde{K})u + h^{3+\alpha}(I - \tilde{K})\sigma + h^{3+\alpha}(\tilde{K}_n - \tilde{K})\sigma \\ &= o(h^{\alpha+5}\ln h), \end{aligned} \tag{3.25}$$

that is,

$$u_n - u - h^{3+\alpha}\sigma = o(h^{\alpha+5}\ln h). \tag{3.26}$$

From (3.22), we obtain

$$\sigma = -\sigma_1 - \sigma_2(\ln h)^{\beta}, \tag{3.27}$$

where

$$\sigma_1 = -\beta\zeta'(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}, \text{ and } \sigma_2 = \zeta(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}. \tag{3.28}$$

Substituting (3.28) into (3.26), and by $\zeta(-2) = 0$ (see [1]), we know that (3.21) holds. The proof of Theorem 3.3 is completed. $\square$

Remark 2 According to the Theorem 3.3 and Remark 1, the numerical solutions u_D of Eq.(3.19) possess high accuracy order $O\ (h^{3+\alpha})$ as $h \to 0$.

4 Numerical experiments

In this section, two numerical examples about the Fredholm equations are computed by Daubechies wavelet method. Let $err_n^u(x) = \left|u(x) - u_n(x)\right|$ be the errors by Daubechies wavelet method using n $(= 2^J\ J = 3, \cdots, 8)$ nodes, and let $EOC = \log(err_n/err_{2n})/\log 2$ be the estimated order of convergence.

If the kernel function $k(x,y)$ of Eq.(1.1) is not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernel be periodic. The Sidi transformation is defined by (see [18])

$$\psi_\gamma(t) = \int_0^t (\sin\pi\tau)^\gamma \mathrm{d}\tau \Big(\int_0^1 (\sin\pi\tau)^\gamma \mathrm{d}\tau \Big)^{-1} : [0,1] \to [0,1], \quad \gamma \geq 1.$$

In the following three examples, the errors and error ratio of numerical solutions at the selected points $x_1 = 0$, $x_2 = 0.25$ and $x_3 = 0.5$ by Daubechies wavelet method using transformation $\psi_6(t)$ are listed in tables.

Example 1. Consider the linear Fredholm equation of the first kind

$$u(x) + \int_0^1 \ln|x-y|u(y)\mathrm{d}y = g(x)$$

where $g(x) = x^2 \ln x/2 + (1-x^2)\ln(1-x)/2 + x/2 - 1/4$ and the exact solution is $u(x) = x$. We use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors for Example 1 using different resolutions. The plots of computed errors are shown in Figure 1 and the errors and error ratio of numerical solutions are listed in Table 1. From the results in Table 1, we can see $EOC \approx 3$.

Table 1: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	2.058-02	2.111-03	3.214-04	3.935-05	4.896-06	6.116-07
$EOC(x_1)$	–	3.2857	2.715	3.030	3.007	3.001
$err_n^u(x_2)$	2.328-02	2.485-03	3.607-04	4.401-05	5.481-06	6.847-07
$EOC(x_2)$	–	3.228	2.784	3.035	3.006	3.001
$err_n^u(x_3)$	2.278-02	9.764-05	1.752-05	2.222-06	2.778-07	3.474-08
$EOC(x_3)$	–	7.866	2.478	2.979	3.000	3.000

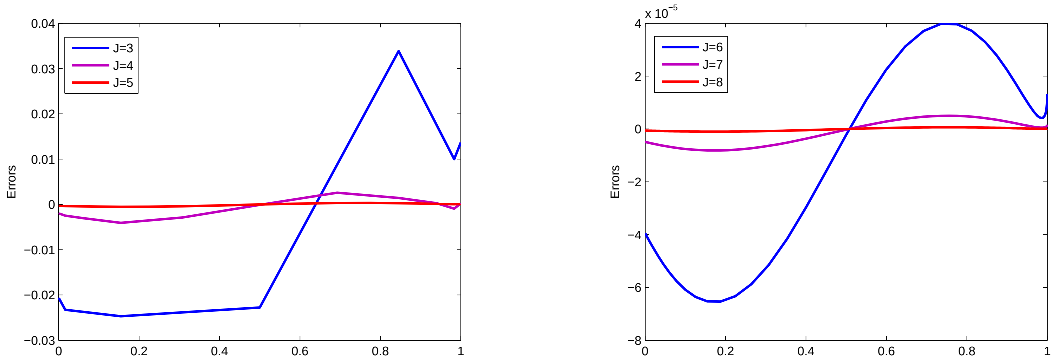

Figure 1: *The error distributions of Example 1 at different resolutions.*

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Example 2. Solving the following non-periodic second kind Fredholm integral equation with algebraic singular kernel

$$u(x) + \int_0^1 |x-y|^{-1/2} u(y)\mathrm{d}y = g(x),$$

where $g(x) = x + 2(x\sqrt(x) - x^{2/3}/3 + x\sqrt(1-x) + (1-x)^{2/3}/3)$ and the exact solution is $u(x) = x$. We also use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors using different resolutions. The plots of computed errors are shown in Figure 2 and the errors and error ratio of numerical solutions are listed in Table 2. From the results in Table 2, we can see $EOC \approx 2.5$.

Table 2: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	1.713-03	3.964-05	1.638-05	2.627-06	4.505-07	7.898-08
$EOC(x_1)$	–	5.433	1.275	2.640	2.544	2.512
$err_n^u(x_2)$	8.526-03	3.899-04	3.847-05	4.843-06	8.529-07	1.508-07
$EOC(x_2)$	–	4.451	3.341	2.990	2.505	2.500
$err_n^u(x_3)$	4.698-03	2.975-04	6.878-05	1.216-05	2.150-06	3.800-07
$EOC(x_3)$	–	3.981	2.113	2.499	2.500	2.500

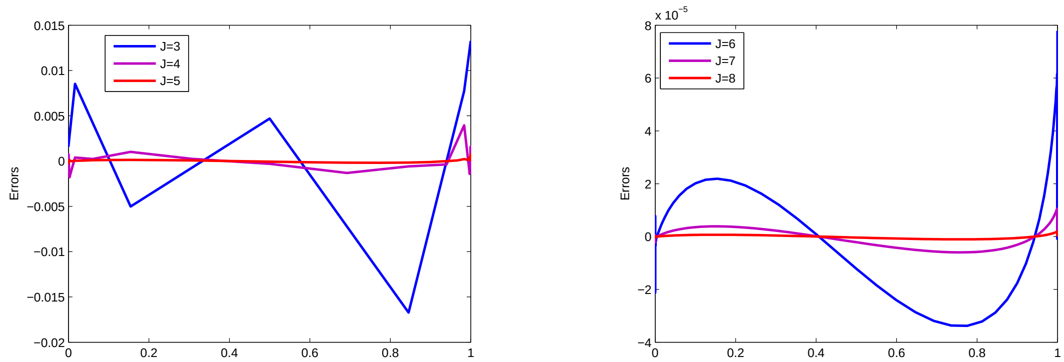

Figure 2: *The error distributions of Example 2 at different resolutions.*

Example 3. Solving the following nonlinear second kind Fredholm integral equation with weakly singular kernel

$$u(x) + \int_0^1 \ln|x-y| g(u(y))\mathrm{d}y = f(x),$$

where

$$f(x) = (x-0.5)^{2/3} + \frac{1}{3}(x^2 - x + 1/3) - x(x^2/3 - x/2 + 0.25)\ln(\frac{x}{1-x}) - \frac{1}{12}\ln(1-x).$$

The exact solution is $u(x) = (x-0.5)^{2/3}$. The periodic Daubechies wavelets of genus $D = 12$ as basis functions are used to compute the errors. The Newton iteration method is used for solve Example 3 and the initial vector of u_0 is given by $u_0 = (1, 1, \cdots, 1)^t_{2^J \times 1}$. After 4 iterations the errors are shown in Fig.3. The errors and error ratio of numerical solutions are listed in Table 3. From Table 3, we can see $EOC \approx 3$.

Table 3: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	4.900-04	4.755-05	4.371-06	5.481-07	6.865-08	8.582-09
$EOC(x_1)$	–	3.365	3.443	2.996	2.997	3.000
$err_n^u(x_2)$	1.659-03	1.565-04	1.714-05	2.141-06	2.673-07	3.340-08
$EOC(x_2)$	–	3.407	3.190	3.001	3.002	3.001
$err_n^u(x_3)$	6.442-04	2.102-04	6.570-05	8.145-06	1.004-06	1.252-07
$EOC(x_3)$	–	1.616	1.678	3.012	3.019	3.003

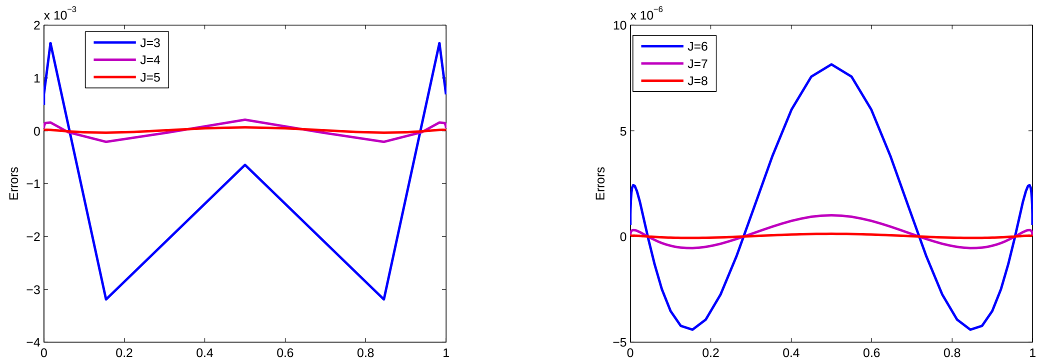

Figure 3: *The error distributions of Example 3 at different resolutions.*

5 Conclusions

In this paper, the Sidi-Israeli quadrature formula is used to construct the approximation of kernel functions and then the Daubechies wavelet method is used to solve Eq. (1.1). when the kernel functions are not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernels be periodic. Because the wavelet integrations are completely avoided and the expansion coefficients obtained here are exact, which

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makes the wavelets method has a good degree of accuracy. In addition, the Daubechies wavelets method is used for linear Fredholm integration equation, the discrete matrix of the associated linear system can be transformed into a very sparse and symmetrical one. Accordingly, many preconditioners can be used to reduce the computational cost.

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