Finite Sum Representation of Partial Derivatives of Multivariable Incomplete Aleph Functions

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Abstract

Newly discovered, incomplete forms of special functions are increasing the interest of both pure and applied mathematicians. The main purpose of this work is to derive four theorems on partial derivatives with incomplete Aleph functions of two variables and generalize them up to r-variables. In addition to these theorems, we also established some novel formulae on the partial derivatives that play a key role in deriving the main results in terms of finite sum. Further, we generalize the result and obtain the finite sum for the incomplete Aleph functions with r-variables. Here, we also established some particular cases that are in most general character and including the results given earlier by Buschman and Deshpande and may prove significant in numerous interesting situations appearing in the literature on mathematical analysis, applied mathematics and mathematical physics.

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1 Introduction and Preliminaries

Recently, Tadesse et al. [7], Kumar et al. [5] and Oli et al. [3] have derived some results on partial derivatives and fractional order derivatives of multivariate Aleph function. Earlier to that, Buschman [11, 12] and Deshpande [18, 19] have generated some important results on partial derivatives of special functions of one variable, two variables and r-variables.
In the 18th century, some fundamental research was initiated in the field of special functions when Prym (1877) introduced incomplete gamma functions that were further studied by several authors. Sdland et al. [10] introduced and investigated the Aleph function in 1998. In 2020, the incomplete forms of the Aleph function were introduced by Bansal et al. [8].
The incomplete Aleph function with r-variables [6] defined using the Mellin Barnes type contour integral as given below:

$$ {}^{(\Gamma)}\aleph^{0,n:m_1,n_1,m_2,n_2;\ldots;m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)},\ldots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R_{(r)}}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{cc} W & W' \\ Z & Z' \end{array}\right] $$

$$ = \frac{1}{(2\pi\omega)^r}\int_{L_1}\int_{L_2}\cdots\int_{L_r}\varphi(s_1,s_2,\ldots,s_r,y)\prod_{k=1}^{r}\left[\phi_k(s_k){z_k}^{s_k}\right]ds_1 ds_2\ldots ds_r, \quad (1) $$

where $\omega=\sqrt{-1}$,

$W=\left[a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},y\right],\left[a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right]$,

$W'=\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\ldots,\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r}$,
$\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]$,

$Z=\left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]$,

$Z'=\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\ldots,\left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r}$,
$\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]$,

$\varphi(s_1,s_2,\ldots,s_r,y)=\frac{\Gamma(1-a_1+\sum_{k=1}^{r}\alpha_1^{(k)}s_k,y)\prod_{j=2}^{n}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}\tau_i\left[\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]}$,

$\phi_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\zeta_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}\tau_{i^{(k)}}\left[\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\zeta_{ji^{(k)}}^{(k)}s_k\right)\right]}$,
$k=1,2,\ldots,r.$

Similarly, another form of incomplete Aleph function with r-variables is defined

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by

$$^{(\gamma)}\aleph^{0,n:m_1,n_1,m_2,n_2,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i(1)},q_{i(1)},\tau_{i(1)};R_{(1)},\ldots,p_{i(r)},q_{i(r)},\tau_{i(r)};R_{(r)}}\left[\begin{array}{c|cc} z_1 & W & W' \\ \vdots & Z & Z' \\ z_r & & \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\int_{L_2}\cdots\int_{L_r}\varphi'(s_1,s_2,\ldots,s_r,y)\prod_{k=1}^{r}\left[\phi_k(s_k){z_k}^{s_k}\right]ds_1ds_2\ldots ds_r, \tag{2}$$

where

$$\varphi'(s_1,s_2,\ldots,s_r,y) = \frac{\gamma(1-a_1+\sum_{k=1}^{r}\alpha_1^{(k)}s_k,y)\prod_{j=2}^{n}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}\tau_i\left[\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]}.$$

Decomposition formula satisfying for incomplete Aleph functions with r-variables defined in (1) and (2) as $^{(\Gamma)}\aleph^{0,n:m_1,n_1,m_2,n_2,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i(1)},q_{i(1)},\tau_{i(1)};R_{(1)},\ldots,p_{i(r)},q_{i(r)},\tau_{i(r)};R_{(r)}}[z_r] + {}^{(\gamma)}\aleph^{0,n:m_1,n_1,m_2,n_2,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i(1)},q_{i(1)},\tau_{i(1)};R_{(1)},\ldots,p_{i(r)},q_{i(r)},\tau_{i(r)};R_{(r)}}[z_r]$ $= \aleph^{0,n:m_1,n_1,m_2,n_2,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i(1)},q_{i(1)},\tau_{i(1)};R_{(1)},\ldots,p_{i(r)},q_{i(r)},\tau_{i(r)};R_{(r)}}[z_r]$.

Here, a_j $(j = 1,\ldots,p)$; b_j $(j = 1,\ldots,q)$; $c_j^{(k)}$ $(j = 1,\ldots,n_k)$; $c_{ji^{(k)}}^{(k)}$ $(j = n_k+1,\ldots,p_{i^{(k)}})$; $d_j^{(k)}$ $(j = 1,\ldots,m_k)$; $d_{ji^{(k)}}^{(k)}$ $(j = m_k+1,\ldots,q_{i^{(k)}})$; $k = 1,\ldots,r$; $i = 1,\ldots,R$ and $i^{(k)} = 1,\ldots,R^{(k)}$ are complex numbers.

$$\Omega_i^{(k)} = \sum_{j=1}^{n}\alpha_j^{(k)} + \tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\zeta_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\zeta_{ji^{(k)}}^{(k)}$$
$$-\tau_i\sum_{j=1}^{q_i}\beta_{ji^{(k)}}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leq 0,$$

τ_i $(i = 1,\ldots,R)$, $\tau_{i^{(k)}}$ $(i^{(k)} = 1,\ldots,R^{(k)})$ are positive real numbers. The integral path $L_{il:\infty}$ is a contour starting from $l - i\infty$ to $l + i\infty$ and the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$, $j = 1,\ldots,m_k$ are separated from those of $\Gamma(1 - a_j + \sum_{k=1}^{r}\alpha_j^{(k)}s_k)$, $j = 1,\ldots,n$ and $\Gamma(1 - c_j^{(k)} + \zeta_j^{(k)}s_k)$, $j = 1,\ldots,n_k$ to the left of the contour L_k. The existence conditions for multiple Mellin-Barnes contours (1) can be obtained with the bits of help of multivariable H-function as $|arg z_k| < \frac{\pi}{2}\bar{\Omega}_i^{(k)}$, where

$$\bar{\Omega}_i^{(k)} = \sum_{j=1}^{n}\alpha_j^{(k)} - \tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\zeta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\zeta_{ji^{(k)}}^{(k)}$$
$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0.$$

We can reduce incomplete Aleph functions with r-variables defined in (1) or (2) to the other well-known special functions by establishing values as given below:

(i) When we set $\tau_i = \tau_{i^{(k)}} = 1$ $(k = 1, \ldots, r)$, the multivariable incomplete Aleph functions reduce to the multivariable incomplete I-functions [16].
(ii) When we set $r = 1$, the incomplete Aleph functions with r-variables reduce to the incomplete Aleph-functions [8, 20, 17].
(iii) When we set $r = 1$ and $\tau_i = \tau_{i^{(k)}} = 1$ $(k = 1, \ldots, r)$, the multivariable incomplete Aleph functions reduce to the incomplete I-functions [9, 15].
(iv) By setting $y = 0$ and $\tau_i = \tau_{i^{(k)}} = 1$ $(k = 1, \ldots, r)$, the multivariable incomplete Aleph functions reduce to the multivariable I-function defined by Sharma et al. [4].
(v) By setting $y = 0,\ \tau_i = \tau_{i^{(k)}} = 1$ and $R = R^{(k)} = 1$ $(k = 1, \ldots, r)$, the multivariable incomplete Aleph-functions reduce to the multivariable H-function [14].
(vi) When we set $y = 0$ and $r = 1$, the multivariable incomplete Aleph functions reduce to the Aleph-function [10].

In the upcoming sections, we explore three formulas which serve as valuable tools for solving the theorems of sections 3 and 4. Within section 4, we generalized Theorem 1 and obtained the finite sum pertaining to the incomplete Aleph functions with r-variables. Additionally, some particular cases given by several authors are also discussed in section 5.

2 Formulas

In this particular section, we have developed three formulas that are intended to assist in resolving the theorems presented in sections 3 and 4.
The incomplete Aleph function can be expressed with two variables ${}^{(\Gamma)}\aleph_Q^P\left[\begin{array}{c} Z_1 \\ Z_2 \end{array}\right]$ as:

$$
{}^{(\Gamma)}\aleph_Q^P\left[\begin{array}{c|c} z_1 & X \\ z_2 & Y \end{array}\right] = \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \varphi\left(s_1, s_2, y\right)\phi_1(s_1) z_1^{s_1}\phi_2(s_2) z_2^{s_2} ds_1 ds_2, \quad (3)
$$

where
$\omega = \sqrt{-1},\ P = 0, n : m_1, n_1, m_2, n_2,$
$Q = p_i, q_i, \tau_i; R; p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}}, q_{i^{(2)}}, \tau_{i^{(2)}}; R_{(2)},$

$\varphi\left(s_1, s_2, y\right) = \frac{\Gamma(1-a_1+\sum_{k=1}^{2}\alpha_1^{(k)}s_k, y)\prod_{j=2}^{n}\Gamma(1-a_j+\sum_{k=1}^{2}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}\tau_i\left[\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{2}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{2}\beta_{ji}^{(k)}s_k\right)\right]},$
$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\zeta_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}\tau_{i^{(k)}}\left[\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\zeta_{ji^{(k)}}^{(k)}s_k\right)\right]},$
$k = 1, 2,$

$$
X = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],
$$

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$\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$
$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y=\left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2}, \left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$

Similarly, the lower form of the incomplete Aleph function of two variables ${}^{(\gamma)}\aleph_Q^P\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ as follows:

$$ {}^{(\gamma)}\aleph_Q^P\left[\begin{array}{c|c} z_1 & X \\ z_2 & Y \end{array}\right] = \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\varphi'\left(s_1,s_2,y\right)\phi_1(s_1)z_1^{s_1}\phi_2(s_2)z_2^{s_2}ds_1ds_2, \quad (4)$$

where
$\varphi'\left(s_1,s_2,y\right)=\frac{\gamma(1-a_1+\sum_{k=1}^{2}\alpha_1^{(k)}s_k,y)\prod_{j=2}^{n}\Gamma(1-a_j+\sum_{k=1}^{2}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}\tau_i\left[\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{2}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{2}\beta_{ji}^{(k)}s_k\right)\right]}.$

Now, we derived the following formulas for the incomplete Aleph function with two variables that will be used for the proof of upcoming theorems.

Formula 1: We derive a formula for the incomplete Aleph function with two variables as follows:

$$ z_1^{\rho_1}z_2^{\rho_2}\,{}^{(\Gamma)}\aleph_Q^P\left[\begin{array}{c|c} z_1 & X \\ z_2 & Y \end{array}\right] = {}^{(\Gamma)}\aleph_Q^P\left[\begin{array}{c|c} z_1 & X_1 \\ z_2 & Y_1 \end{array}\right], \quad (5)$$

where
$P=0,n:m_1,n_1,m_2,n_2,$
$Q=p_i,q_i,\tau_i;R;p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)};p_{i^{(2)}},q_{i^{(2)}},\tau_{i^{(2)}};R_{(2)},$
$X_1=\left[a_1+\rho_1\alpha_1^{(1)}+\rho_2\alpha_1^{(2)};\alpha_1^{(1)},\alpha_1^{(2)},y\right], \left[a_j+\rho_1\alpha_j^{(1)}+\rho_2\alpha_j^{(2)};\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},$
$\left[\tau_i\left(a_{ji}+\rho_1\alpha_{ji}^{(1)}+\rho_2\alpha_{ji}^{(2)};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right], \left[c_j^{(1)}+\rho_1\zeta_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},$
$\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}+\rho_1\zeta_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)}+\rho_2\zeta_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$
$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)}+\rho_2\gamma_{ji^{(2)}}^{(2)},\gamma_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y_1=\left[\ldots,\tau_i\left(b_{ji}+\rho_1\beta_{ji}^{(1)}+\rho_2\beta_{ji}^{(2)};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[d_j^{(1)}+\rho_1\delta_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}+\rho_1\delta_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[d_j^{(2)}+\rho_2\delta_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},$
$\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)}+\rho_2\delta_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$

To prove this formula, substitute $s_k + \rho_k = \mathfrak{s}_k$ ($\rho_k \in \mathbb{C}$) and use (3). After a small simplification, we get the desired result.

Formula 2: For $z_1 = \Lambda_1^{\gamma_1}$ and $z_2 = \Lambda_2^{\gamma_2}$, the partial derivatives of the incomplete Aleph function two variables with respect to Λ_1, Λ_2 defined as:

$$\Lambda_1^s \Lambda_2^t \frac{\partial^s}{\partial \Lambda_1^s} \frac{\partial^t}{\partial \Lambda_2^t} {}^{(\Gamma)}\aleph_Q^P \left[\begin{array}{c} z_1 \\ z_2 \end{array} \middle| \begin{array}{c} X \\ Y \end{array} \right] = {}^{(\Gamma)}\aleph_{Q_2}^{P_2} \left[\begin{array}{c} z_1 \\ z_2 \end{array} \middle| \begin{array}{c} X_2 \\ Y_2 \end{array} \right], \tag{6}$$

where

$P_2 = 0, n : m_1 + 1, n_1, m_2 + 1, n_2,$

$Q_2 = p_i, q_i, \tau_i; R; p_{i^{(1)}} + 1, q_{i^{(1)}} + 1, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}} + 1, q_{i^{(2)}} + 1, \tau_{i^{(2)}}; R_{(2)},$

$$X_2 = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$\left[c_j^{(1)}, \zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}} \left(c_{ji^{(1)}}^{(1)}, \zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], [0, \gamma_1], \left[c_j^{(2)}, \zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}} \left(c_{ji^{(2)}}^{(2)}, \zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right], [0, \gamma_2],$$
$$Y_2 = \left[\ldots, \tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \beta_{ji}^{(2)}\right)_{m+1,q_i}\right], [s, \gamma_1], \left[d_j^{(1)}, \delta_j^{(1)}\right]_{1,m_1},$$
$$\left[\tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], [t, \gamma_2], \left[d_j^{(2)}, \delta_j^{(2)}\right]_{1,m_2},$$
$$\left[\tau_{i^{(2)}} \left(d_{ji^{(2)}}^{(2)}, \delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$$

Formula 3: For $z_1 = \Lambda_1^{-\gamma_1}$ and $z_2 = \Lambda_2^{-\gamma_2}$, the partial derivatives of the incomplete Aleph function two variables with respect to Λ_1, Λ_2 is as follows:

$$(-1)^{s+t} (\Lambda_1)^s (\Lambda_2)^t \frac{\partial^s}{\partial \Lambda_1^s} \frac{\partial^t}{\partial \Lambda_2^t} {}^{(\Gamma)}\aleph_Q^P \left[\begin{array}{c} z_1 \\ z_2 \end{array} \middle| \begin{array}{c} X \\ Y \end{array} \right] = {}^{(\Gamma)}\aleph_{Q_3}^{P_3} \left[\begin{array}{c} z_1 \\ z_2 \end{array} \middle| \begin{array}{c} X_3 \\ Y_3 \end{array} \right], \tag{7}$$

where

$P_3 = 0, n : m_1, n_1 + 1, m_2, n_2 + 1,$

$Q_3 = p_i, q_i, \tau_i; R; p_{i^{(1)}} + 1, q_{i^{(1)}} + 1, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}} + 1, q_{i^{(2)}} + 1, \tau_{i^{(2)}}; R_{(2)},$

$$X_3 = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$[1 - s, \gamma_1], \left[c_j^{(1)}, \zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}} \left(c_{ji^{(1)}}^{(1)}, \zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], [1 - t, \gamma_2],$$
$$\left[c_j^{(2)}, \zeta_j^{(2)}\right]_{1,n_2}, \left[\tau_{i^{(2)}} \left(c_{ji^{(2)}}^{(2)}, \zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$$
$$Y_3 = \left[\ldots, \tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[d_j^{(1)}, \delta_j^{(1)}\right]_{1,m_1},$$
$$\left[\tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], [1, \gamma_1], \left[d_j^{(2)}, \delta_j^{(2)}\right]_{1,m_2},$$
$$\left[\tau_{i^{(2)}} \left(d_{ji^{(2)}}^{(2)}, \delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right], [1, \gamma_2].$$

Similarly, we can derive all three formulas for the lower form of the incomplete Aleph function with two variables ${}^{(\gamma)}\aleph_{Q}^{P}\left[\begin{array}{c|c} z_1 & X \\ z_2 & Y \end{array}\right]$.

3 Main Results

In this section, we produce four theorems concerning the partial derivatives of incomplete Aleph functions with two variables, each having distinct parameters. Subsequently, we express these functions in terms of finite sums.

If we put $z = N \in Z^{+}$ in formula (30) of Erdelyi et al. ([1], P.19), we obtain the result as given:

$$\psi(e+N) - \psi(e) = \sum_{l=1}^{N} (-1)^{l-1} \frac{N!}{l(N-l)!} \frac{\Gamma(e)}{\Gamma(e+l)}, \tag{8}$$

where function $\psi(z) = \frac{d}{dz} log\,[\Gamma(z)]$.

Theorem 1: We derived the partial derivatives of the incomplete Aleph function with two variables in terms of finite sum for the given values of $\Lambda_1 = N_1,\ \Lambda_2 = N_2$ as:

$$\frac{\partial}{\partial \Lambda_1} \frac{\partial}{\partial \Lambda_2} {}^{(\Gamma)}\aleph_{Q_4}^{P_4}\left[\begin{array}{c|c} z_1 & X_4 \\ z_2 & Y_4 \end{array}\right]_{\Lambda_1=N_1,\ \Lambda_2=N_2}$$
$$= \frac{N_1! N_2!}{4} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{1}{k_1! k_2! (N_1-k_1)(N_2-k_2)} {}^{(\Gamma)}\aleph_{Q_4}^{P_4}\left[\begin{array}{c|c} z_1 & X_5 \\ z_2 & Y_5 \end{array}\right], \tag{9}$$

where
$P_4 = 0, n : m_1, n_1 + 2, m_2, n_2 + 2,$
$Q_4 = p_i, q_i, \tau_i; R; p_{i^{(1)}} + 2, q_{i^{(1)}}, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}} + 2, q_{i^{(2)}}, \tau_{i^{(2)}}; R_{(2)},$

$X_4 = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$
$\left(1-\rho_1 \pm \frac{\Lambda_1}{2}, \gamma_1\right), \left[c_j^{(1)}, \zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}}, \left(c_{ji^{(1)}}^{(1)}, \zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],$
$\left(1-\rho_2 \pm \frac{\Lambda_2}{2}, \gamma_2\right), \left[c_j^{(2)}, \zeta_j^{(2)}\right]_{1,n_2}, \left[\tau_{i^{(2)}} \left(c_{ji^{(2)}}^{(2)}, \zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y_4 = \left[\ldots, \tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[d_j^{(1)}, \delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[d_j^{(2)}, \delta_j^{(2)}\right]_{1,m_2}, \left[\tau_{i^{(2)}} \left(d_{ji^{(2)}}^{(2)}, \delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$
$X_5 = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$

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$\left[1-\rho_1+\frac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\frac{N_1}{2}-k_1,\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},$
$\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\left[1-\rho_2+\frac{N_2}{2},\gamma_2\right],\left[1-\rho_2+\frac{N_2}{2}-k_2,\gamma_2\right],$
$\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y_5=\left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$

Proof: In the first step, express the left-hand side of (9) in the form of the Mellin-Barnes integral as given in (3). Further, by using the chain rule of derivatives and result defined in (8), we have

$$\frac{\partial}{\partial\Lambda_1}\Gamma\left(\rho_1\pm\frac{\Lambda_1}{2}+\gamma_1 s_1\right)\Big|_{\Lambda_1=N_1}=\frac{1}{2}\Gamma\left(\rho_1\pm\frac{N_1}{2}+\gamma_1 s_1\right)\times$$
$$\left[\psi\left(\rho_1+\frac{N_1}{2}+\gamma_1 s_1\right)-\psi\left(\rho_1-\frac{N_1}{2}+\gamma_1 s_1\right)\right]=\frac{1}{2}\Gamma\left(\rho_1\pm\frac{N_1}{2}+\gamma_1 s_1\right)$$
$$\sum_{L_1=1}^{N_1}\frac{(-1)^{L_1-1}N_1!\,\Gamma\left(\rho_1-\frac{N_1}{2}+\gamma_1 s_1\right)}{L_1(N_1-L_1)!\,\Gamma\left(\rho_1-\frac{N_1}{2}+L_1+\gamma_1 s_1\right)}. \quad (10)$$

Similarly, we can write

$$\frac{\partial}{\partial\Lambda_2}\Gamma\left(\rho_2\pm\frac{\Lambda_2}{2}+\gamma_2 s_2\right)\Big|_{\Lambda_2=N_2}=\frac{1}{2}\Gamma\left(\rho_2\pm\frac{N_2}{2}+\gamma_2 s_2\right)\times$$
$$\left[\psi\left(\rho_2+\frac{N_2}{2}+\gamma_2 s_2\right)-\psi\left(\rho_2-\frac{N_2}{2}+\gamma_2 s_2\right)\right]=\frac{1}{2}\Gamma\left(\rho_2\pm\frac{N_2}{2}+\gamma_2 s_2\right)$$
$$\sum_{L_2=1}^{N_2}\frac{(-1)^{L_2-1}N_2!\,\Gamma\left(\rho_2-\frac{N_2}{2}+\gamma_2 s_2\right)}{L_2(N_2-L_2)!\,\Gamma\left(\rho_2-\frac{N_2}{2}+L_2+\gamma_2 s_2\right)}. \quad (11)$$

Now, we can write the left-hand side of (9) by using the above-given results in (10) and (11). We have

$$=\frac{N_1!N_2!}{4}\sum_{L_1=0}^{N_1}\sum_{L_2=0}^{N_2}\frac{(-1)^{L_1+L_2-2}}{L_1L_2(N_1-L_1)!(N_2-L_2)!}\times$$
$$(\Gamma)\aleph_{p_i,q_i,\tau_i;R;p_{i^{(1)}}+3,q_{i^{(1)}}+1,\tau_{i^{(1)}};R^{(1)};p_{i^{(2)}}+3,q_{i^{(2)}}+1,\tau_{i^{(2)}};R^{(2)}}^{0,n:m_1,n_1+3,m_2,n_2+3}\left[\begin{array}{c}z_1\\ z_2\end{array}\middle|\begin{array}{l}\left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right],\\ \ldots,\end{array}\right.$$

$$\left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],\left[1-\rho_1\pm\frac{N_1}{2},\gamma_1\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

$$\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],$$
$$\left[1-\rho_1+\tfrac{N_1}{2}-L_1,\gamma_1\right],\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},$$

$$\left[1-\rho_2\pm\tfrac{N_2}{2},\gamma_2\right],\left[1-\rho_2+\tfrac{N_2}{2},\gamma_2\right],\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$$
$$\left.\begin{matrix}\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right]\\ \left[1-\rho_2+\tfrac{N_2}{2}-L_2,\gamma_2\right]\end{matrix}\right]. \tag{12}$$

Now, consider the incomplete Aleph function with two variables portion of (12) and for the sake of convenience denote it by T. Thus we can write it as follows:

$$T={}^{(\Gamma)}\aleph^{0,n:m_1,n_1+3,m_2,n_2+3}_{p_i,q_i,\tau_i;R;p_{i(1)}+3,q_{i(1)}+1,\tau_{i(1)};R_{(1)};p_{i(2)}+3,q_{i(2)}+1,\tau_{i(2)};R_{(2)}}\left[\begin{matrix}z_1\\ z_2\end{matrix}\ \middle|\ \begin{matrix}\left[a_1;\alpha_1^{(1)},\right.\\ \ldots,\end{matrix}\right.$$

$$\alpha_1^{(2)},y\Big],\left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},$$
$$\left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],$$

$$\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],\left[1-\rho_1\pm\tfrac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right]$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

$$\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\left[1-\rho_2\pm\tfrac{N_2}{2},\gamma_2\right],$$
$$\left[1-\rho_1+\tfrac{N_1}{2}-L_1,\gamma_1\right],\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$$

$$\left.\begin{matrix}\left[1-\rho_2+\tfrac{N_2}{2},\gamma_2\right],\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right]\\ \left[1-\rho_2+\tfrac{N_2}{2}-L_2,\gamma_2\right]\end{matrix}\right]. \tag{13}$$

Now, evaluate the value of $Z_1^{-\frac{1}{\gamma_1}\left(\frac{N_1}{2}-\rho_1-L_1\right)}Z_2^{-\frac{1}{\gamma_2}\left(\frac{N_2}{2}-\rho_2-L_2\right)}\times T$, by using the formula 1 and formula 3, and after little simplification, we arrive at

$$(-1)^{N_1+N_2-L_1-L_2}(\Lambda_1)^{N_1/2+\rho_1}(\Lambda_2)^{N_2/2+\rho_2}\frac{\partial^{N_1-L_1}}{\partial\Lambda_1^{N_1-L_1}}\frac{\partial^{N_2-L_2}}{\partial\Lambda_2^{N_2-L_2}}\times\Lambda_1^{(N_1/2-\rho_1-L_1)}\times$$

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$$\Lambda_2^{(N_2/2-\rho_2-L_2)(\Gamma)}\aleph^{0,n:m_1,n_1+2,m_2,n_2+2}_{p_i,q_i,\tau_i;R;p_{i(1)}+2,q_{i(1)},\tau_{i(1)};R_{(1)};p_{i(2)}+2,q_{i(2)},\tau_{i(2)};R_{(2)}}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle| \begin{array}{l}[a_1; \\ \dots,\end{array}\right.$$

$$\left[\alpha_1^{(1)},\alpha_1^{(2)},y\right],\left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right]^2,$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],$$

$$\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right]^2,\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

$$\left.\begin{array}{l}\left[1-\rho_2+\tfrac{N_2}{2},\gamma_2\right]^2,\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right] \\ \left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right]\end{array}\right], \tag{14}$$

where $z_1=\Lambda_1^{-\gamma_1}$ and $z_2=\Lambda_2^{-\gamma_2}$.
Now applying the Leibniz formula

$$\frac{\partial^{N-L}}{\partial\Lambda^{N-L}}\left(\Lambda^{-L}\Lambda^{N/2-\rho}\aleph\left[\begin{array}{c}\Lambda^{-\gamma} \\ z_2\end{array}\right]\right)$$
$$=\sum_{l=0}^{N-L}\binom{N-L}{l}\frac{\partial^{N-L-l}}{\partial\Lambda^{N-L-l}}(\Lambda^{-L})\frac{\partial^l}{\partial\Lambda^l}(\Lambda^{n/2-\rho})\aleph\left[\begin{array}{c}\Lambda^{-\gamma} \\ z_2\end{array}\right] \tag{15}$$

in (14) and after simplifying it, we have

$$=\sum_{l_1=0}^{N_1-L_1}\sum_{l_2=0}^{N_2-L_2}\frac{(N_1-L_1)!(N_2-L_2)!\Gamma(N_1-l_1)\Gamma(N_2-l_2)}{l_1!l_2!(N_1-L_1-l_1)!(N_2-L_2-l_2)!\Gamma(L_1)\Gamma(L_2)}\times$$

$$\aleph^{0,n:m_1,n_1+2,m_2,n_2+2}_{p_i,q_i,\tau_i;R;p_{i(1)}+2,q_{i(1)},\tau_{i(1)};R_{(1)};p_{i(2)}+2,q_{i(2)},\tau_{i(2)};R_{(2)}}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle| \begin{array}{l}\left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right],[a_j; \\ \dots,\end{array}\right.$$

$$\left.\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\tfrac{N_1}{2}\right.$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$$

$$\left.-k_1,\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\left[1-\rho_2+\tfrac{N_2}{2},\gamma_2\right],$$
$$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

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$$\left.\begin{matrix} \left[1-\rho_2+\frac{N_2}{2}-k_2,\gamma_2\right], \left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2}, \left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right] \\ \left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2}, \left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right] \end{matrix}\right]. \quad (16)$$

By using (13) and change the order of summations in $\sum_{L_1=1}^{N_1}\sum_{l_1=0}^{N_1-L_1}$ and $\sum_{L_2=1}^{N_2}\sum_{l_2=0}^{N_2-L_2}$ and inner sums then simplify to $L_1 = L_2 = 1$. Consequently, we get the desired result.

Theorem 2: The incomplete Aleph function with two variables in terms of finite sum for the given values of $\Lambda_1 = N_1,\ \Lambda_2 = N_2$ as:

$$\frac{\partial}{\partial\Lambda_1}\frac{\partial}{\partial\Lambda_2}{}^{(\Gamma)}\aleph_{Q_5}^{P_5}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\;\middle|\;\begin{matrix} X_6 \\ Y_6 \end{matrix}\right]_{\Lambda_1=N_1,\ \Lambda_2=N_2}$$

$$= \frac{N_1!N_2!}{4}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\frac{1}{k_1!k_2!(N_1-k_1)(N_2-k_2)}{}^{(\Gamma)}\aleph_{Q_5}^{P_5}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\;\middle|\;\begin{matrix} X_7 \\ Y_7 \end{matrix}\right], \quad (17)$$

where
$P_5 = 0, n : m_1+2, n_1, m_2+2, n_2,$
$Q_5 = p_i, q_i, \tau_i; R; p_{i^{(1)}}, q_{i^{(1)}}+2, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}}, q_{i^{(2)}}+2, \tau_{i^{(2)}}; R_{(2)},$

$$X_6 = \left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right], \left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n}, \left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$$
$$Y_6 = \left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[\rho_1 \pm \tfrac{\Lambda_1}{2},\delta_1\right], \left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$$
$$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[\rho_2 \pm \tfrac{\Lambda_2}{2},\delta_2\right], \left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},$$
$$\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$$
$$X_7 = \left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right], \left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n}, \left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$$
$$Y_7 = \left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[\rho_1-\tfrac{N_1}{2},\delta_1\right], \left[\rho_1-\tfrac{N_1}{2}+k_1,\delta_1\right],$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1}, \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[\rho_2-\tfrac{N_2}{2},\delta_2\right], \left[\rho_2-\tfrac{N_2}{2}+k_2,\delta_2\right],$$

$\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$

Proof: The proof for Theorem 2 follows a similar approach as that used for Theorem 1.

Theorem 3: A new variant of the partial derivatives of the incomplete Aleph function with two variables in terms of finite sum for the given values of $\Lambda_1 = N_1$ as:

$$\frac{\partial}{\partial\Lambda_1}{}^{(\Gamma)}\aleph_{Q_6}^{P_6}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{c} X_8 \\ Y_8 \end{array}\right]_{\Lambda_1=N_1} = \frac{N_1!}{2}\sum_{k_1=0}^{N_1-1}\frac{1}{k_1!(N_1-k_1)}{}^{(\Gamma)}\aleph_{Q_6}^{P_6}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{c} X_9 \\ Y_9 \end{array}\right], \quad (18)$$

where
$P_6 = 0, n : m_1, n_1+2, m_2, n_2,$
$Q_6 = p_i, q_i, \tau_i; R; p_{i^{(1)}}+2, q_{i^{(1)}}, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}}, q_{i^{(2)}}, \tau_{i^{(2)}}; R_{(2)},$

$X_8 = \left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right],\left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$
$\left[1-\rho_1\pm\frac{\Lambda_1}{2},\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$
$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y_8 = \left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$
$X_9 = \left[a_1;\alpha_1^{(1)},\alpha_1^{(2)},y\right],\left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$
$\left[1-\rho_1+\frac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\frac{N_1}{2}-k_1,\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},$
$\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)},\zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$
$Y_9 = \left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$
$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$

Proof: The proof for Theorem 3 can be obtained using the same method employed in establishing Theorem 1.

Theorem 4: The partial derivatives of the incomplete Aleph function with

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two variables in terms of finite sum for the given values of $\Lambda_1 = N_1$ as:

$$\frac{\partial}{\partial\Lambda_1} {}^{(\Gamma)}\aleph^{P_7}_{Q_7}\left[\begin{array}{c|c} z_1 & X_{10} \\ z_2 & Y_{10} \end{array}\right]_{\Lambda_1=N_1} = \frac{N_1!}{2}\sum_{k_1=0}^{N_1-1}\frac{1}{k_1!(N_1-k_1)} {}^{(\Gamma)}\aleph^{P_7}_{Q_7}\left[\begin{array}{c|c} z_1 & X_{11} \\ z_2 & Y_{11} \end{array}\right] \tag{19}$$

where

$P_7 = 0, n : m_1 + 2, n_1, m_2, n_2,$

$Q_7 = p_i, q_i, \tau_i; R; p_{i^{(1)}}, q_{i^{(1)}} + 2, \tau_{i^{(1)}}; R_{(1)}; p_{i^{(2)}}, q_{i^{(2)}}, \tau_{i^{(2)}}; R_{(2)},$

$$X_{10} = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$\left[c_j^{(1)}, \zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}}, \left(c_{ji^{(1)}}^{(1)}, \zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)}, \zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)}, \zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$$
$$Y_{10} = \left[\ldots, \tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[\rho_1 \pm \tfrac{\Lambda_1}{2}, \delta_1\right], \left[d_j^{(1)}, \delta_j^{(1)}\right]_{1,m_1},$$
$$\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[d_j^{(2)}, \delta_j^{(2)}\right]_{1,m_2}, \left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)}, \delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],$$
$$X_{11} = \left[a_1; \alpha_1^{(1)}, \alpha_1^{(2)}, y\right], \left[a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right]_{2,n}, \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \alpha_{ji}^{(2)}\right)_{n+1,p_i}\right],$$
$$\left[c_j^{(1)}, \zeta_j^{(1)}\right]_{1,n_1}, \left[\tau_{i^{(1)}}, \left(c_{ji^{(1)}}^{(1)}, \zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right], \left[c_j^{(2)}, \zeta_j^{(2)}\right]_{1,n_2},$$
$$\left[\tau_{i^{(2)}}\left(c_{ji^{(2)}}^{(2)}, \zeta_{ji^{(2)}}^{(2)}\right)_{n_2+1,p_i^{(2)}}\right],$$
$$Y_{11} = \left[\ldots, \tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \beta_{ji}^{(2)}\right)_{m+1,q_i}\right], \left[\rho_1 - \tfrac{N_1}{2}, \delta_1\right], \left[\rho_1 - \tfrac{N_1}{2} + k_1, \delta_1\right],$$
$$\left[d_j^{(1)}, \delta_j^{(1)}\right]_{1,m_1}, \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right], \left[d_j^{(2)}, \delta_j^{(2)}\right]_{1,m_2},$$
$$\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)}, \delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right].$$

Proof: The proof for Theorem 4 can be obtained using the same method employed in establishing Theorem 1.

4 Generalization of Main Result

In this section, we extend the primary outcome established in Theorem 1 to encompass the incomplete Aleph function considering r-variables ${}^{(\Gamma)}\aleph^{P}_{Q}[z_r]$ as follows:

$$\prod_{k=1}^{r}\frac{\partial}{\partial\Lambda^{k}}{}^{(\Gamma)}\aleph^{0,n:m_1,n_1+2,m_2,n_2+2,\dots,m_r,n_r+2}_{p_i,q_i,\tau_i;R:p_{i^{(1)}}+2,q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)},\dots,p_{i^{(r)}}+2,q_{i^{(r)}},\tau_{i^{(r)}};R_{(r)}}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{l} [a_1; \\ \dots, \end{array}\right.$$

$$\alpha_1^{(1)},\dots,\alpha_1^{(r)},y\Big],\left[1-\rho_1\pm\tfrac{\Lambda_1}{2},\gamma_1\right],\left[a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right]_{2,n},$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\dots\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

$$\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\dots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],$$
$$\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},\left[\tau_{i^{(2)}}\left(d_{ji^{(2)}}^{(2)},\delta_{ji^{(2)}}^{(2)}\right)_{m_2+1,q_i^{(2)}}\right],\dots,$$

$$\left.\begin{array}{l}\dots,\left[1-\rho_r\pm\frac{\Lambda_r}{2},\gamma_r\right],\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\ \left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r},\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\end{array}\right]_{\Lambda_k=N_k}$$

$$=\frac{\prod_{k=1}^{r}(N_k)!}{2^k}\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_r=0}^{N_r-1}\frac{1}{\prod_{l=1}^{r}\left[k_l!(N_k-k_l)\right]}\times$$

$${}^{(\Gamma)}\aleph^{0,n:m_1,n_1+2,m_2,n_2+2,\dots,m_r,n_r+2}_{p_i,q_i,\tau_i;R;p_{i^{(1)}}+2,q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)},\dots,p_{i^{(r)}}+2,q_{i^{(r)}},\tau_{i^{(r)}};R_{(r)}}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{l} \left[a_1;\alpha_1^{(1)},\dots,\alpha_1^{(r)},\right. \\ \dots, \end{array}\right.$$

$$y],\left[a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\dots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\dots\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],$$

$$\left[1-\rho_1+\tfrac{N_1}{2}-k_1,\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\dots,$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\dots,$$

$$\left[1-\rho_r+\tfrac{N_r}{2},\gamma_r\right],\left[1-\rho_r+\tfrac{N_r}{2}-k_r,\gamma_r\right],\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r},$$
$$\left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r},$$

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$$\left.\begin{matrix}\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]\\ \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\end{matrix}\right]. \tag{20}$$

Proof: Initially, we express the left-hand side of (20) in the form of the Mellin-Barnes integral as given in (1). Further, by applying the chain rule of derivatives and the result defined in (8), we get

$$\frac{\partial}{\partial\Lambda_1}\Gamma\left(\rho_1\pm\frac{\Lambda_1}{2}+\gamma_1 s_1\right)\Big|_{\Lambda_1=N_1}=\frac{1}{2}\Gamma\left(\rho_1\pm\frac{N_1}{2}+\gamma_1 s_1\right)\times$$
$$\left[\psi\left(\rho_1+\frac{N_1}{2}+\gamma_1 s_1\right)-\psi\left(\rho_1-\frac{N_1}{2}+\gamma_1 s_1\right)\right]=\frac{1}{2}\Gamma\left(\rho_1\pm\frac{N_1}{2}+\gamma_1 s_1\right)$$
$$\sum_{L_1=1}^{N_1}\frac{(-1)^{L_1-1}N_1!\Gamma\left(\rho_1-\frac{N_1}{2}+\gamma_1 s_1\right)}{L_1(N_1-L_1)!\Gamma\left(\rho_1-\frac{N_1}{2}+L_1+\gamma_1 s_1\right)}. \tag{21}$$

Similarly, we can generalize (21) for $\frac{\partial}{\partial\Lambda_k}\Gamma\left(\rho_k\pm\frac{\Lambda_k}{2}+\gamma_k s_k\right)\Big|_{\Lambda_k=N_k}$ $(k=1,\ldots,r-1)$. And the last term of the sequence is defined below by:

$$\frac{\partial}{\partial\Lambda_r}\Gamma\left(\rho_r\pm\frac{\Lambda_r}{2}+\gamma_r s_r\right)\Big|_{\Lambda_r=N_r}=\frac{1}{2}\Gamma\left(\rho_r\pm\frac{N_r}{2}+\gamma_r s_r\right)\times$$
$$\left[\psi\left(\rho_r+\frac{N_r}{2}+\gamma_r s_r\right)-\psi\left(\rho_r-\frac{N_r}{2}+\gamma_r s_r\right)\right]=\frac{1}{2}\Gamma\left(\rho_r\pm\frac{N_r}{2}+\gamma_r s_r\right)$$
$$\sum_{L_r=1}^{N_r}\frac{(-1)^{L_r-1}N_r!\Gamma\left(\rho_r-\frac{N_r}{2}+\gamma_r s_r\right)}{L_r(N_r-L_r)!\Gamma\left(\rho_r-\frac{N_r}{2}+L_r+\gamma_r s_r\right)}. \tag{22}$$

Now, we can write the left-hand side of (20) by using the above-given results in (21) and (22). After a bit of simplification, We have

$$=\frac{\prod_{k=1}^{r}(N_k)!}{2^k}\sum_{L_1=0}^{N_1}\cdots\sum_{L_r=0}^{N_r}\frac{(-1)^{L_1+L_2+\cdots+L_r-r}}{\prod_{k=1}^{r}L_k(N_k-L_k)!}\times$$

$$(\Gamma)\aleph_{p_i,q_i,\tau_i;R;p_{i(1)}+3,q_{i(1)}+1,\tau_{i(1)};R_{(1)},\ldots,p_{i(r)}+3,q_{i(r)}+1,\tau_{i(r)};R_{(r)}}^{0,n:m_1,n_1+3,\ldots,m_r,n_r+3}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}\left[a_1;\alpha_1^{(1)},\ldots,\right.\\ \ldots,\end{matrix}\right.$$

$$\left.\alpha_1^{(r)},y\right],\left[a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

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$$\begin{matrix}\left[1-\rho_1 \pm \frac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\frac{N_1}{2},\gamma_1\right],\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\\ \left[1-\rho_1+\frac{N_1}{2}-L_1,\gamma_1\right],\ldots,\left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r},\end{matrix}$$

$$\begin{matrix}\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\ldots,\left[1-\rho_r \pm \frac{N_r}{2},\gamma_r\right],\\ \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right],\end{matrix}$$

$$\begin{matrix}\left[1-\rho_r+\frac{N_r}{2},\gamma_r\right],\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]\\ \left[1-\rho_r+\frac{N_r}{2}-L_r,\gamma_r\right]\end{matrix}\Bigg]. \tag{23}$$

Now, consider the incomplete Aleph function with two variables portion of (23) and for the sake of convenience denote it by L. So, we can write it as follows:

$$L={}^{(\Gamma)}\aleph^{0,n:m_1,n_1+3,m_2,n_2+3}_{p_i,q_i,\tau_i;R;p_{i^{(1)}}+3,q_{i^{(1)}}+1,\tau_{i^{(1)}};R_{(1)},\ldots,p_{i^{(r)}}+3,q_{i^{(r)}}+1,\tau_{i^{(r)}};R_{(r)}}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}\left[a_1;\alpha_1^{(1)},\right.\\ \ldots,\end{matrix}\right.$$

$$\begin{matrix}\left.\ldots,\alpha_1^{(r)},y\right],\left[a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],\\ \left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],\end{matrix}$$

$$\begin{matrix}\left[1-\rho_1 \pm \frac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\frac{N_1}{2},\gamma_1\right]\\ \left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],\end{matrix}$$

$$\begin{matrix}\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\ldots,\left[1-\rho_r \pm \frac{N_r}{2},\gamma_r\right],\\ \left[1-\rho_1+\frac{N_1}{2}-L_1,\gamma_1\right],\ldots,\left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r},\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right],\end{matrix}$$

$$\begin{matrix}\left[1-\rho_r+\frac{N_r}{2},\gamma_r\right],\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]\\ \left[1-\rho_r+\frac{N_r}{2}-L_r,\gamma_r\right]\end{matrix}\Bigg]. \tag{24}$$

Now, evaluate the value of $\prod_{k=1}^{r} Z_k^{-\frac{1}{\gamma_k}\left(\frac{N_k}{2}-\rho_k-L_k\right)} \times L$, by using the generalized form of formula 1 and 3, and after simplification. We obtain

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$$(-1)^{\sum_{k=1}^{r}(N_k-L_k)}\prod_{k=1}^{r}\left[(\Lambda_k)^{N_k/2+\rho_k}\frac{\partial^{N_k-L_k}}{\partial\Lambda_k^{N_k-L_k}}\Lambda_1^{(N_k/2-\rho_k-L_k)}\right]\times$$

$$(\Gamma)\aleph^{0,n:m_1,n_1+2,\dots,m_r,n_r+2}_{p_i,q_i,\tau_i;R;p_{i^{(1)}}+2,q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)},\dots,p_{i^{(r)}}+2,q_{i^{(r)}},\tau_{i^{(r)}};R_{(r)}}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{l}\left[a_1;\alpha_1^{(1)},\dots,\right. \\ \dots,\end{array}\right.$$

$$\left.\alpha_1^{(r)},y\right],\left[a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\dots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\dots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],$$

$$\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right]^2,\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\zeta_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right],\dots,$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right],$$

$$\left.\begin{array}{l}\left[1-\rho_r+\tfrac{N_r}{2},\gamma_r\right]^2,\left[c_j^{(r)},\zeta_j^{(r)}\right]_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\zeta_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\ \left[d_j^{(r)},\delta_j^{(r)}\right]_{1,m_r},\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\end{array}\right], \tag{25}$$

where $z_k=\Lambda_k^{-\gamma_k}$ $(k=1,\dots,r)$.
Now applying the Leibniz formula (15) in (25) and simplify it, we have

$$=\sum_{l_1=0}^{N_1-L_1}\sum_{l_2=0}^{N_2-L_2}\cdots\sum_{l_r=0}^{N_r-L_r}\left[\prod_{k=1}^{r}\frac{(N_k-L_k)!\Gamma(N_k-l_k)}{l_k!(N_k-L_k-l_k)!\Gamma(L_k)}\right]\times$$

$$\aleph^{0,n:m_1,n_1+2,\dots,m_r,n_r+2}_{p_i,q_i,\tau_i;R:p_{i^{(1)}}+2,q_{i^{(1)}},\tau_{i^{(1)}};R_{(1)};\dots,p_{i^{(r)}}+2,q_{i^{(r)}},\tau_{i^{(r)}};R_{(r)}}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle|\begin{array}{l}\left[a_1;\alpha_1^{(1)},\dots,\alpha_1^{(r)},y\right], \\ \dots,\end{array}\right.$$

$$\left[a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\dots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right],\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\dots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right],\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$$

$$\left[1-\rho_1+\tfrac{N_1}{2}-k_1,\gamma_1\right]\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},$$
$$\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},$$

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$$\begin{matrix}\left[\tau_{i^{(1)}}\left(c^{(1)}_{ji^{(1)}},\zeta^{(1)}_{ji^{(1)}}\right)_{n_1+1,p^{(1)}_i}\right],\left[1-\rho_r+\frac{N_r}{2},\gamma_r\right],\\ \left[\tau_{i^{(1)}}\left(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}}\right)_{m_1+1,q^{(1)}_i}\right],\end{matrix}$$

$$\left.\begin{matrix}\left[1-\rho_r+\frac{N_r}{2}-k_r,\gamma_r\right],\left[c^{(r)}_j,\zeta^{(r)}_j\right]_{1,n_r},\left[\tau_{i^{(r)}}\left(c^{(r)}_{ji^{(r)}},\zeta^{(r)}_{ji^{(r)}}\right)_{n_r+1,p^{(r)}_i}\right]\\ \left[d^{(r)}_j,\delta^{(r)}_j\right]_{1,m_r},\left[\tau_{i^{(r)}}\left(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}}\right)_{m_r+1,q^{(r)}_i}\right]\end{matrix}\right]. \quad (26)$$

By using (24) and change the order of summations in $\sum_{L_k=1}^{N_k}\sum_{l_k=0}^{N_k-L_k}$ $(k = 1,\ldots,r)$ and inner sums then simplify to $L_k = 1$ $(k = 1,\ldots,r)$. Finally, we obtain the desired result.
Similarly, We can derive all the theorems for the incomplete Aleph function with r-variables ${}^{(\gamma)}\aleph_Q^P[z_r]$ as we derived for ${}^{(\Gamma)}\aleph_Q^P[z_r]$ and generalized it also as given in section 4.

5 Particular Cases

In this part, we discuss some important cases of Theorem 1 which can also comfortably obtain the results identically to Theorem 2, 3 and 4. Further, if we assign specific values to the parameters in the incomplete Aleph functions of two variables then we have the following cases.

5.1 In terms of H-function with two variables

By setting $y = 0, \tau_i = \tau_{i^{(1)}} = \tau_{i^{(2)}} = 1$ and $R = R^{(1)} = R^{(2)} = 1$ in Theorem 1, we get a known result derived by Deshpande [19] as follows:

$$\frac{\partial}{\partial\Lambda_1}\frac{\partial}{\partial\Lambda_2}H^{P^*}_{Q^*}\left[\begin{matrix}z_1\\z_2\end{matrix}\;\middle|\;\begin{matrix}X^*\\Y^*\end{matrix}\right]_{\Lambda_1=N_1,\ \Lambda_2=N_2}$$
$$=\frac{N_1!N_2!}{4}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\frac{1}{k_1!k_2!(N_1-k_1)(N_2-k_2)}H^{P^*}_{Q^*}\left[\begin{matrix}z_1\\z_2\end{matrix}\;\middle|\;\begin{matrix}X^{**}\\Y^{**}\end{matrix}\right], \quad (27)$$

where
$P^* = 0, n : m_1, n_1 + 2, m_2, n_2 + 2,$
$Q^* = p, q : p_{(1)} + 2, q_{(1)}, p_{(2)} + 2, q_{(2)},$

$X^* = \left[a_j;\alpha^{(1)}_j,\alpha^{(2)}_j\right]_{1,n},\left[1-\rho_1\pm\frac{\Lambda_1}{2},\gamma_1\right],\left[c^{(1)}_j,\zeta^{(1)}_j\right]_{1,n_1},\left[1-\rho_2\pm\frac{\Lambda_2}{2},\gamma_2\right],$
$\left[c^{(2)}_j,\zeta^{(2)}_j\right]_{1,n_2},$

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$$Y^* = \left[b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right]_{m+1,q_i},\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2},$$

$$X^{**} = \left[a_j;\alpha_j^{(1)},\alpha_j^{(2)}\right]_{1,n},\left[1-\rho_1+\tfrac{N_1}{2},\gamma_1\right),\left(1-\rho_1+\tfrac{N_1}{2}-k_1,\gamma_1\right],$$

$$\left[c_j^{(1)},\zeta_j^{(1)}\right]_{1,n_1},\left[1-\rho_2+\tfrac{N_2}{2},\gamma_2\right],\left[1-\rho_2+\tfrac{N_2}{2}-k_2,\gamma_2\right],\left[c_j^{(2)},\zeta_j^{(2)}\right]_{1,n_2},$$

$$Y^{**} = \left[b_{ji};\beta_{ji}^{(1)},\beta_{ji}^{(2)}\right]_{m+1,q_i},\left[d_j^{(1)},\delta_j^{(1)}\right]_{1,m_1},\left[d_j^{(2)},\delta_j^{(2)}\right]_{1,m_2}.$$

Similarly, we get more results by setting $y = 0, \tau_i = \tau_{i^{(1)}} = \tau_{i^{(2)}} = 1$ and $R = R^{(1)} = R^{(2)} = 1$, thus convert Multivariable incomplete Aleph function to Multivariable H-function in all theorems of section 3.

5.2 In terms of G-function with two variables

By substituting $\alpha_j^{(1)} = \alpha_j^{(2)} = \zeta_j^{(1)} = \zeta_j^{(2)} = \beta_{ji}^{(1)} = \beta_{ji}^{(2)} = \delta_j^{(1)} = \delta_j^{(2)} = 1$ in the result 5.1, we can transform that result in G-function [13, 2, 14] as follows:

$$\frac{\partial}{\partial\Lambda_1}\frac{\partial}{\partial\Lambda_2}G^{0,n:m_1,n_1+2,m_2,n_2+2}_{p,q:p_{(1)}+2,q_{(1)},p_{(2)}+2,q_{(2)}}\left[\begin{array}{c|l} z_1 & (a_j)_{1,n},\left(1-\rho_1\pm\frac{\Lambda_1}{2},\gamma_1\right),\left(c_j^{(1)}\right),\left(1-\rho_2\pm\frac{\Lambda_2}{2},\gamma_2\right),\left(c_j^{(2)}\right) \\ z_2 & (b_{ji})_{m+1,q_i},\left(d_j^{(1)}\right)_{1,m_1},\left(d_j^{(2)}\right)_{1,m_2} \end{array}\right]_{\Lambda_1=N_1,\ \Lambda_2=N_2}$$

$$= \frac{N_1!N_2!}{4}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\frac{1}{k_1!k_2!(N_1-k_1)(N_2-k_2)}\times$$

$$G^{0,n:m_1,n_1+2,m_2,n_2+2}_{p,q:p_{(1)}+2,q_{(1)},p_{(2)}+2,q_{(2)}}\left[\begin{array}{c|l} z_1 & (a_j)_{1,n},\left[1-\rho_1+\frac{N_1}{2},\gamma_1\right],\left[1-\rho_1+\frac{N_1}{2}-k_1,\gamma_1\right],(c_j^{(1)})_{1,n_1},\left[1-\rho_2+\frac{N_2}{2},\gamma_2\right],\left[1-\rho_2+\frac{N_2}{2}-k_2,\gamma_2\right],\left(c_j^{(2)}\right)_{1,n_2} \\ z_2 & (b_{ji})_{m+1,q_i},\left(d_j^{(1)}\right)_{1,m_1},\left(d_j^{(2)}\right)_{1,m_2} \end{array}\right]. \quad (28)$$

By putting suitable values to the parameters, we arrived at the known results given by Buschman and Despande [11, 12, 18].

6 Conclusions

We summed up this analysis by considering the utility and prospective applications of the newly derived special functions. Several known and novel outcomes involving special functions follow as specific cases of our main findings due to the most fundamental character of the functions involved in the present work. The significance of our results lies in many fold generality. Because of the generality of the incomplete Aleph functions with r-variables, on suitable specializing the various parameters and variables in these functions from our results, we can establish extensive varieties of useful results, which are expressible in terms of families of the incomplete H-functions, incomplete generalized hypergeometric functions, incomplete I-functions, I-function, Aleph function and many more. This work will remove the constraints of special functions and these results may be used to solve a variety of problems in mathematical analysis.

References

[1] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions Vol. 1*, McGraw-Hill Book Co., Inc., New York, 1953.

[2] A. Verma, A note on expansions involving Meijers G-functions, *Mathematics of Computation*, 21(97), 107-112, (1967).

[3] A. Oli, K. Tilahun, G. V. Reddy, The Multivariable Aleph-function involving the Generalized Mellin-Barnes Contour Integrals, *Cubo (Temuco)*, 22(3), 351-359, (2020).

[4] C. K. Sharma, S. S. Ahmad, On the multivariable I-function, *Acta ciencia Indica Math.*, 20(2), 113116, (1994).

[5] D. Kumar, F. Y. Ayant, S. D. Purohit, F. Ucar, On Partial Derivatives of the I-function of r-variables, *Azerbaijan Journal of Mathematics*, 10, 49-61, (2020).

[6] F. Ayant, P. Kumar, Incomplete Multivariable Aleph-function and Integral of Three Parameters Calculus, *The Mathematics Education*, 55. 12-27, (2021).

[7] H. Tadesse, D. L. Suthar, M. Ayalew, Finite Integral Formulas Involving Multivariable Aleph-Functions, *Journal of Applied Mathematics*, Article ID: 6821797, (2019).

[8] M. K. Bansal, D. Kumar, K. S. Nisar, J. Singh, Certain fractional calculus and integral transform results of incomplete Aleph-functions with applications, *Mathematical Methods in the Applied Sciences*, 43(8), 5602-5614, (2020).

[9] M. K. Bansal, D. Kumar, On the integral operators pertaining to a family of incomplete I-functions, *AIMS-Math.*, (2), 1247-1259, (2020).

[10] N. Sdland, B. Baumann, T. F. Nannenmacher, Open problem: who knows about the aleph-function?, *Appl Anal*, 1, 401402, (1998).

[11] R. G. Buschman, Finite sum representations for partial derivatives of special functions with respect to parameters, *Mathematics of Computation*, 28(127), 817-824, (1974).

[12] R. G. Buschman, Partial derivatives of the H-function with respect to parameters expressed as finite sum and integrals, *Univ. Nac. Tucuman, Rev. Ser. A*, 24, 149-155, (1974).

[13] R. P. Agarwal, An extention of Meijer's G-function, *Proc. Nat. Inst. Sei. India. Sect. A*, 32, (1965).

[14] R. Panda, H. M. Srivastava, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.*, 283/284, 265-274, (1976).

[15] R. Sharma, J. Singh, D. Kumar, Y. Singh, Certain Unified Integrals Associated with Product of the General Class of Polynomials and Incomplete I-Functions, *Int. J. Appl. Comput. Math.*, 8(7), (2021).

[16] R. Sharma, J. Singh, D. Kumar, Y. Singh, An Application of Incomplete I-Functions with Two Variables to Solve the Nonlinear Differential Equations Using S-Function, *J. Computational Analysis And Applications*, 31(1), 80-95, (2023).

[17] S. Momani, R. Sharma, J. Singh, Y. Singh, S. Hadid, Fractional Order Mathematical Modelling for Studying the Impact on the Emergence of Pollution and Biodiversity Pertaining to Incomplete Aleph Functions, *Progress in Fractional Differentiation and Applications*, 10(1), 15-22, (2024)

[18] V. L. Deshpande, Partial derivatives of hypergeometric functions of two variables with respect to parameters,*To appear in the J. Indian Acad. Math., Indore*, (1978).

[19] V. L. Deshpande, Finite sum representations of the H-function of two and more variables with respect to parameters,*Indian J. pure appl. Math*, 10, 1514-1524, (1979).

[20] Y. Singh, R. Sharma, R. Maanju, Beer-Lamberts law as an application of incomplete Aleph (ℵ) functions, *AIP Conf. Proc*, 2768 (1): 020038, (2023).