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Completely monotonic functions involving Bateman's $G-$function

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Abstract

In this paper, we prove the complete monotonicity of some functions involving Bateman's $G-$function and show that

$$\frac{1}{2x^2+\alpha} < G(x) - \frac{1}{x} < \frac{1}{2x^2+\beta}, \qquad x > 0$$

where $\alpha = 1$ and $\beta = 0$ are the best possible constants, which is a refinement of a recent result. Then, we give a new proof of Slavić inequality about Wallis ratio W_m and provide a new inequality for W_m. Our new inequality improves some recent related works. We also present two inequalities for the hyperbolic tangent function.

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Key Words: Bateman's $G-$function, completely monotonic, best possible constant, Bernoulli numbers, Wallis ratio, hyperbolic tangent function.

1 Introduction

A function $H : J \to \mathbb{R}$ is said to be completely monotonic (see [45] and [11]), if $H^{(m)}(x)$ exists on J for all $m \geq 0$ and

$$(-1)^m H^{(m)}(x) \geq 0 \qquad x \in J;\ m \geq 0. \tag{1}$$

For $x > 0$, the necessary and sufficient condition for the function $H(x)$ to be completely monotonic is the convergence of the following integral

$$H(x) = \int_0^\infty e^{-xt} dv(t), \tag{2}$$

where $v(t)$ is a nonnegative measure on $t \geq 0$. The function $H(x)$ is said to be strictly completely monotonic if the inequality (1) is strict for all $x \in J$ and $m \geq 0$. The concept of completely monotonic function is the continuous analogue of the totally monotone sequence presented by Hausdorff in 1921 [15] (see also [45]). These functions find applications in several diverse fields such as in the theory of special functions, asymptotic analysis, probability, physics, and the list continues, see [2], [5], [6], [12] , [13], [32], [34], [35], [38], [44] and the references therein.

The Bateman's $G-$function is defined by (see Erdélyi [10])

$$G(t) = \psi\left(\frac{t}{2} + \frac{1}{2}\right) - \psi\left(\frac{t}{2}\right), \qquad t \neq 0, -1, -2, ... \tag{3}$$

where $\psi(t)$ is the digamma (Psi) function which is defined by

$$\psi(t) = \frac{d}{dt} \ln \Gamma(t)$$

and $\Gamma(z)$ is the classical Euler gamma function which is defined for $Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1} dw.$$

For more details on bounds, identities, properties and applications of Bateman's $G-$function, refer to [10], [21]-[25], [31], [39] and the references therein. The following relations hold for the function $G(x)$ [10]:

$$G(x+1) = -G(x) + 2x^{-1}, \tag{4}$$

$$G(x) = \int_0^\infty \frac{2e^{-xv}}{1+e^{-v}} dv, \qquad x > 0 \tag{5}$$

$$G(x) = x^{-1}\ {}_2F_1\left(1, 1; 1+x; \frac{1}{2}\right), \tag{6}$$

where

$${}_lF_m(v_1, ..., v_l; w_1, ..., w_m; z) = \sum_{k=0}^\infty \frac{(v_1)_k ... (v_l)_k}{(w_1)_k ... (w_m)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function [3] defined for $l, m \in \mathbb{N}$, $v_j, w_j \in \mathbb{C}$, $w_j \neq 0, -1, -2, ...$ and

$$(v)_0 = 1 \quad \text{and} \quad (v)_n = \frac{\Gamma(v+m)}{\Gamma(v)}, \qquad n \in \mathbb{N}.$$

Qiu and Vuorinen [39] established the inequality

$$\frac{(6 - 4\ln 4)}{x^2} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 1/2 \tag{7}$$

and Mortici [25] improved the inequality (7) to the double inequality

$$0 < \psi(x+h) - \psi(x) \leq \psi(h) + \gamma - h + h^{-1}, \qquad x \geq 1;\ h \in (0,1) \tag{8}$$

where γ is the Euler constant. Mahmoud and Agarwal [21] deduced the following asymptotic formula for $x \to \infty$

$$G(x) - \frac{1}{x} \sim \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{k} x^{-2k}, \tag{9}$$

where $B_m{}'s$ are the Bernoulli numbers [17] and they also presented the following inequality

$$\frac{1}{2x^2 + \frac{3}{2}} < G(x) - x^{-1} < \frac{1}{2x^2}, \qquad x > 0 \tag{10}$$

which improves the lower bound of the inequality (7) for $x > \left(\frac{9-12\ln 2}{16\ln 2 - 11}\right)^{1/2}$. In [22] Mahmoud and Almuashi proved the following inequality

$$\sum_{n=1}^{2m} \frac{(2^{2n}-1)}{n} B_{2n} x^{-2n} < G(x) - x^{-1} < \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)}{n} B_{2n} x^{-2n}, \qquad m \in \mathbb{N} \tag{11}$$

where $\frac{(2^{2n}-1)}{n} B_{2n}$ are the best possible constants. Also, Mahmoud, Talat and Moustafa [23] studied the following family of approximations of Bateman's $G-$function

$$\chi(\rho, x) = \ln\left(1 + \frac{1}{x+\rho}\right) + \frac{2}{x(x+1)}, \qquad 1 \leq \rho \leq 2;\ x > 0$$

which is asymptotically equivalent to the function $G(x)$ for $x \to \infty$.

Recently, Mahmoud and Almuashi [24] presented some identities, functional equations and an asymptotic expansion of the generalized Bateman's $G-$function $G_\sigma(x)$ defined by

$$G_\sigma(x) = \psi\left(\frac{x+\sigma}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq -2r, -2r-\sigma;\ \sigma \in (0,2);\ \text{for}\ \ r = 0, 1, 2, \ldots .$$

Also, they presented the double inequality

$$\ln\left(1 + \frac{\sigma}{x+\phi}\right) < G_\sigma(x) - \frac{2\sigma}{x(x+\sigma)} < \ln\left(1 + \frac{\sigma}{x+\theta}\right), \quad x > 0;\ \sigma \in (0,2)$$

where $\phi = \frac{\sigma}{e^{\gamma + \frac{2}{\sigma} + \psi(\frac{\sigma}{2})} - 1}$ and $\theta = 1$ are the best possible constants.

In this paper, we will study the complete monotonicity of some functions involving the function $G(x)$ and as a consequence, we will deduce a double inequality of it. Also, we will prove that the function

$$q(x) = \frac{1}{G(x) - \frac{1}{x}} - 2x^2, \qquad x > 0$$

is strictly increasing and present a refinement of the lower bound of the inequality (10). We will apply our results to present a new proof of Slavić inequality about Wallis ratio $W_m = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}$ for $m \in N$. We will also present a new inequality of W_m, which improves some recent results. Further, we will present two inequalities involving the hyperbolic tangent function.

2 Main Results

We begin by proving some auxiliary results involving Bernoulli numbers.

Lemma 2.1. *For any positive integer $s \geq 1$, we have*

$$B_{2s} = \frac{1}{2(2^{2s}-1)}\left[1 - \frac{1}{2s+1}\sum_{k=1}^{s-1} 2(2^{2k}-1)\binom{2s+1}{2k} B_{2k}\right] \tag{12}$$

and

$$B_{2s} = \frac{1}{2(2^{2s}-1)}\left[s - \sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k} B_{2k}\right]. \tag{13}$$

Proof. The identity [30]

$$B_m = \frac{1}{2(1-2^m)}\sum_{j=0}^{m-1} 2^j \binom{m}{j} B_j, \qquad m \in \mathbb{N} \tag{14}$$

can be rewritten as

$$B_m = \frac{1}{2(1-2^m)}[1 - m + \sum_{j=1}^{[\frac{m-1}{2}]} 2^{2j}\binom{m}{2j} B_{2j}], \qquad m \geq 1$$

where $B_{2r+1} = 0$ for $r \in \mathbb{N}$ and hence

$$\sum_{k=1}^{s} 2^{2k}\binom{2s+1}{2k} B_{2k} = 2s, \quad s \geq 1 \tag{15}$$

$$\sum_{k=1}^{s-1} 2^{2k}\binom{2s}{2k} B_{2k} = (2s-1) + 2(1-2^{2s})B_{2s}, \quad s \geq 2. \tag{16}$$

Also, Bernoulli numbers satisfy [4]

$$s - \frac{1}{2} = \sum_{k=1}^{s}\binom{2s+1}{2k} B_{2k}, \quad s \geq 1 \tag{17}$$

$$s - 1 = \sum_{k=1}^{s-1}\binom{2s}{2k} B_{2k}, \quad s \geq 2. \tag{18}$$

From the two identities (15) and (17), we get

$$\sum_{k=1}^{s} 2(2^{2k}-1)\binom{2s+1}{2k} B_{2k} = 2s+1 \qquad s \geq 1 \tag{19}$$

and the two identities (16) and (18) give us

$$2(2^{2s}-1)B_{2s} + \sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k} B_{2k} = s \qquad s \geq 1. \tag{20}$$

□

Lemma 2.2. *For $v = 2, 3, 4, \cdots$, Bernoulli numbers satisfy*

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\pi^2 < \frac{|B_{2v}|}{|B_{2v+2}|}(2v+1)(2v+2) < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1). \tag{21}$$

Proof. The function

$$f(x) = x(8x-(9+3\pi^2))+1, \qquad x \geq \frac{9+3\pi^2+\sqrt{49+54\pi^2+9\pi^4}}{16} \approx 4.80006...$$

is increasing and positive, and hence

$$2^{2v-1}(2^{2v+2}-(9+3\pi^2))+1 > 0, \qquad v \geq 2.$$

Then

$$(\pi^2+1)(2^{2v+2}-1)(2^{2v-1}-1)-\pi^2(2^{2v+1}-1)(2^{2v}-1) > 0, \qquad v \geq 2$$

or

$$\frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\pi^2 < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1), \qquad v \geq 2. \tag{22}$$

From the Qi's result [36]

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\frac{\pi^2}{(2v+1)(2v+2)} < \frac{|B_{2v}|}{|B_{2v+2}|} < \frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\frac{\pi^2}{(2v+1)(2v+2)}, \qquad v \geq 1 \tag{23}$$

and the inequality (22), we complete the proof. □

Now we will prove the complete monotonicity of some functions involving the function $G(x)$.

Lemma 2.3. *For a positive integer m, the function*

$$F(x) = G(x) - \frac{1}{x} - \sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{24}$$

is strictly completely monotonic.

Proof. Using the formula [1]

$$\frac{1}{x^k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-xt}dt, \qquad k \in \mathbb{N} \tag{25}$$

and the integral representation of $G(x)$, we get

$$\begin{aligned} F(x) &= \int_0^\infty \left[e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}\right]\frac{e^{-xt}}{1+e^t}dt \\ &= \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt, \end{aligned}$$

where

$$\varphi(t) = e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}. \tag{26}$$

Now

$$\begin{aligned}
\varphi(t) &= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!}\\
&= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^s}{(s-2k+1)!}\\
&= \sum_{r=1}^{4m}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m}\frac{t^s}{(s-2k+1)!}\\
&\quad+\sum_{r=4m+1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=4m+1}^{\infty}\frac{t^s}{(s-2k+1)!}.
\end{aligned}$$

Rewrite infinite summations from 0 and split finite summations by even and odd power of t we obtain

$$\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s-1}}{(2s-2k)!}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s}}{(2s-2k+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k+2)!}t^{s+4m+1},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{s=1}^{2m}\frac{1}{(2s)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{s=1}^{2m}\frac{1}{(2s+1)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)((2s+1)!)B_{2k}}{(2k)!(2s-2k+1)!}t^{2s}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k)+2)!}+\frac{2(2^{4k-2}-1)B_{4k-2}}{(4k-2)!(s+4(m-k)+4)!}\right)t^{s+4m+1}\\
&= \sum_{s=1}^{2m}\left[2s-4(2^{2s}-1)B_{2s}-\sum_{k=1}^{s-1}2(2^{2k}-1)\binom{2s}{2k}B_{2k}\right]\frac{t^{2s-1}}{(2s)!}\\
&\quad+\sum_{s=1}^{2m}\left[2s+1-\sum_{k=1}^{s}2(2^{2k}-1)\binom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1+\frac{(2^{4k-2}-1)(4k)(4k-1)B_{4k-2}}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)B_{4k}}\right)\right.\\
&\qquad\left.\frac{2(2^{4k}-1)t^{s+4m+1}B_{4k}}{(4k)!(s+4(m-k)+2)!}\right].
\end{aligned}$$

Using the identities (12) and (13) with the relation

$$(-1)^{r+1}B_{2r} > 0, \qquad r \in \mathbb{N} \tag{27}$$

we obtain

$$\begin{aligned}\varphi(t) &= \sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)|B_{4k}|}\right)\right.\\ &\quad \left.\frac{2(2^{4k}-1)|B_{4k}|t^{s+4m+1}}{(4k)!(s+4(m-k)+2)!}\right]+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}.\end{aligned}$$

For $s \geq 0$ and $m \geq k \geq 1$, we have

$$(s+4(m-k)+3)(s+4(m-k)+4) \geq (s+3)(s+4) \geq 12$$

and then

$$\begin{aligned}\varphi(t) &\geq \sum_{s=0}^{\infty}\sum_{k=1}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1}\\ &\quad +\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\ &\geq \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1}\\ &\quad +\sum_{s=0}^{\infty}\frac{30|B_4|}{(4!)(s+4m-2)!)}\left(1-\frac{|B_2|}{5|B_4|}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}.\end{aligned}$$

Using inequality (21) with $v = 2k-1$ for $k \in \mathbb{N}$, we get

$$\varphi(t) > \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{\pi^2+1}{12}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!} > 0,$$

which complete the proof. □

Lemma 2.4. *For a positive integer m, the function*

$$M(x) = \frac{1}{x} - G(x) + \sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{28}$$

is strictly completely monotonic.

Proof. Using the formula (25) and the integral representation of $G(x)$, we have

$$\begin{aligned}M(x) &= \int_0^{\infty}\left[(1+e^t)\sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}-(e^t-1)\right]\frac{e^{-xt}}{1+e^t}dt\\ &= \int_0^{\infty}\mu(t)\frac{e^{-xt}}{1+e^t}dt,\end{aligned}$$

where
$$\mu(t) = (1+e^t)\sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} - (e^t-1).$$
Now
$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!} - \sum_{r=1}^{\infty}\frac{t^r}{r!}\mu(t)\\
&= \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^{s}}{(s-2k+1)!} - \sum_{r=1}^{\infty}\frac{t^r}{r!}\\
&= \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m-2}\frac{t^{s}}{(s-2k+1)!} - \sum_{r=1}^{4m-2}\frac{t^r}{r!}\\
&\quad + \sum_{s=4m-1}^{\infty}\sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s}}{(s-2k+1)!} - \sum_{r=4m-1}^{\infty}\frac{t^r}{r!}.
\end{aligned}$$
Rewrite infinite summations from 0 and split finite summations by even and odd power of t, we obtain
$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s-1}}{(2s-2k)!} - \sum_{r=1}^{2m-1}\frac{t^{2r-1}}{(2r-1)!}\\
&\quad + \sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s}}{(2s-2k+1)!} - \sum_{r=1}^{2m-1}\frac{t^{2r}}{(2r)!}\\
&\quad + \sum_{s=0}^{\infty}\sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!} - \sum_{r=0}^{\infty}\frac{t^{r+4m-1}}{(r+4m-1)!},
\end{aligned}$$
which can be rewritten as
$$\begin{aligned}
\mu(t) &= \sum_{s=1}^{2m-1} \frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1} + \sum_{s=1}^{2m-1}\frac{1}{(2s)!}\sum_{k=1}^{s} \frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1} - \sum_{s=1}^{2m-1}\frac{t^{2s-1}}{(2s-1)!}\\
&\quad + \sum_{s=1}^{2m-1}\frac{1}{(2s+1)!}\sum_{k=1}^{s} \frac{2(2^{2k}-1)(2s+1)!B_{2k}}{(2k)!(2s-2k+1)!}t^{2s} - \sum_{s=1}^{2m-1}\frac{t^{2s}}{(2s)!}\\
&\quad + \sum_{s=0}^{\infty}\sum_{k=1}^{2m-1} \frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!} - \sum_{s=0}^{\infty}\frac{t^{s+4m-1}}{(s+4m-1)!}\\
&= \sum_{s=1}^{2m-1}\left[2(2^{2s}-1)B_{2s} - s + \sum_{k=1}^{s-1}(2^{2k}-1)\binom{2s}{2k}B_{2k}\right]\frac{2t^{2s-1}}{(2s)!}\\
&\quad + \sum_{s=1}^{2m-1}\left[-(2s+1) + \sum_{k=1}^{s}2(2^{2k}-1)\binom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!}\\
&\quad + \sum_{s=0}^{\infty}\left[\frac{1}{2}\frac{s+4m-3}{(s+4m-1)!} + \sum_{k=2}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k)!}\right]t^{s+4m-1}.
\end{aligned}$$

Using the identities (12) and (13) with the relation (27), $\mu(t)$ satisfies

$$\begin{aligned}\mu(t) &> \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k))!}+\frac{2(2^{4k+2}-1)B_{4k+2}}{(4k+2)!(s+4(m-k)-2)!}\right)t^{s+4m-1}\\ &> \left[\sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{(2^{4k}-1)(4k+1)(4k+2)|B_{4k}|}{(2^{4k+2}-1)(s+4(m-k)-1)(s+4(m-k))|B_{4k+2}|}\right)\right.\\ &\qquad\left.\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!}\right].\end{aligned}$$

For $s \geq 0$ and $m-k \geq 1$, we have

$$(s+4(m-k)-1)(s+4(m-k)) \geq (s+3)(s+4) \geq 12$$

and then μ satisfies

$$\mu(t) > \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{\pi^2+1}{12}\right)\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!} > 0,$$

which complete the proof. □

From the complete monotonicity of the two functions $F(x)$ and $M(x)$ with the asymptotic expansion (9), we get the following double inequality which posed as a conjecture in [21].

Lemma 2.5. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k} < G(x)-x^{-1} < \sum_{k=1}^{2l-1}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k}, \qquad l,m\in N;\ x>0. \tag{29}$$

From the positivity of the two functions $\varphi(t)$ and $\mu(t)$ in the proofs of Lemmas 2.3 and 2.4, we obtain the following result:

Lemma 2.6. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1} \leq \tanh(x) \leq \sum_{k=1}^{2l-1}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad l,m\in N;\ x\geq 0 \tag{30}$$

and the inequality is reversed if $x \leq 0$. *Equality holds if* $x = 0$.

Remark 1. In the case $|x| < \frac{\pi}{2}$ and l or $m =$ tends to ∞, in the inequality (30) in fact equality holds, since

$$\tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad |x| < \frac{\pi}{2}.$$

Elbert and Laforgia established the following lemma to study the monotonicity of some functions involving gamma function [9] (see also [48]).

Lemma 2.7. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then $K(x) > 0$, if $K(x) > K(x+1)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+1)$ for all $x > a$.*

To present our next result, we can easily prove the following simple modification on Lemma 2.7:

Corollary 2.8. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then for $m \in \mathbb{N}$, $K(x) > 0$, if $K(x) > K(x+m)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+m)$ for all $x > a$.*

Proof. For $m \in \mathbb{N}$, if we have $K(x) > K(x+m)$ and $\lim_{x\to\infty} K(x) = 0$, then

$$K(x) > K(x+m) > ... > K(x+rm) > ... > \lim_{r\to\infty} K(x+rm) = \lim_{y\to\infty} K(y) = 0.$$

The other case is similarly treated. □

Lemma 2.9. *The function*

$$q(x) = \frac{1}{G(x) - \frac{1}{x}} - 2x^2, \qquad x > 0 \tag{31}$$

is strictly increasing.

Proof. For $x > 0$, we have

$$q'(x) = \frac{L(x)}{[G(x) - \frac{1}{x}]^2},$$

where

$$L(x) = -G'(x) - 4xG^2(x) + 8G(x) - \frac{(4x+1)}{x^2}.$$

Now,

$$\begin{aligned} L(x+1) - L(x) &= G'(x) - G'(x+1) + 4x\left[G^2(x) - G^2(x+1)\right] - 4G^2(x+1) \\ &- 8\left[G(x) - G(x+1)\right] + \frac{4x^2+6x+1}{x^2(x+1)^2} \end{aligned}$$

and using equation (4) and its derivative, we get

$$L(x+1) - L(x) = 2G'(x) - 4G^2(x+1) + \frac{6x^2+10x+3}{x^2(x+1)^2} \triangleq L_1(x).$$

Consider the difference

$$\begin{aligned} L_1(x+2) - L_1(x) &= 2\left[G'(x+2) - G'(x)\right] - 4\left[G^2(x+3) - G^2(x+1)\right] \\ &- \frac{4\left(27 + 135x + 220x^2 + 158x^3 + 51x^4 + 6x^5\right)}{x^2(x+1)^2(x+2)^2(x+3)^2} \end{aligned}$$

and using equation (4) and its derivative, we obtain

$$\begin{aligned} L_1(x+2)-L_1(x) &= \frac{16}{(x+1)(x+2)}\left\{G(x+1)-\frac{4x^5+34x^4+98x^3+99x^2+3x-9}{4x^2(x+1)(x+2)(x+3)^2}\right\} \\ &\triangleq \frac{16}{(x+1)(x+2)}L_2(x). \end{aligned}$$

Using equation (4), the function $L_2(x)$ satisfies

$$L_2(x+2)-L_2(x) = -\frac{3(7x+15)(7x+20)}{2x^2(x+1)(x+2)^2(x+3)^2(x+4)(x+5)^2} < 0.$$

From the asymptotic formula (9) and its derivative

$$G'(x) \sim -\frac{1}{x^2} - \sum_{k=1}^{\infty}\frac{2(2^{2k}-1)B_{2k}}{x^{2k+1}}, \qquad x\to\infty \tag{32}$$

we have

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} L_1(x) = \lim_{x\to\infty} L_2(x) = 0.$$

Hence, using Corollary 2.8, we get that $L(x) > 0$ for all $x > 0$ which completes the proof. □

As a consequence of the monotonicity of the function $q(x)$ with the asymptotic expansion (9), we obtain the following inequality:

Lemma 2.10. *The following double inequality holds*

$$\frac{1}{2x^2+\alpha} < G(x) - \frac{1}{x} < \frac{1}{2x^2+\beta}, \qquad x > 0 \tag{33}$$

where $\alpha = 1$ and $\beta = 0$ are the best possible constants.

Remark 2. The double inequality (33) is a refinement of the double inequality (10).

Lemma 2.11. *The function*

$$U(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2+1}, \qquad x > 0 \tag{34}$$

is strictly completely monotonic.

Proof. Using the formula (25), the integral representation of $G(x)$ and the Laplace transform of sine function, we have

$$U(x) = \int_0^\infty \lambda(t)e^{-xt}dt,$$

where

$$\lambda(t) = \frac{e^t-1}{e^t+1} - \frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right).$$

Since $\sin z < 1$, we get

$$\lambda(t) > \frac{e^t-1}{e^t+1} - \frac{1}{\sqrt{2}} > 0, \quad t > \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 1.76275\ .$$

Also, from the generalization of Redheffer-Williams's inequality [40], [41], [42], [46]

$$\frac{\pi^2-x^2}{\pi^2+x^2} \leq \frac{\sin x}{x} \leq \frac{12-x^2}{12+x^2}, \qquad 0 < x \leq \pi$$

and the inequality (30) for $m = 4$, we obtain $\lambda(t) > \frac{t^5(2352-240t^2-17t^4)}{40320(24+t^2)} > 0$ for $0 < t < \sqrt{\frac{4\sqrt{3399}-120}{17}} \approx 2.58051.$ □

As a consequence of the Lemma 2.11, we get

Lemma 2.12.

1. *For odd positive integer r , we have*

$$G^{(r)}(x) < -\frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r+1}{2}} (-1)^l \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{35}$$

2. *For even positive integer r , we have*

$$G^{(r)}(x) > \frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r}{2}+1} (-1)^{l+1} \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{36}$$

Also, as a consequence of the proof of Lemma 2.11, we obtain the following inequality:

Lemma 2.13. *The following double inequality holds*

$$\tanh(x) \geq \frac{1}{\sqrt{2}} \sin(\sqrt{2}x), \qquad x \geq 0. \tag{37}$$

Equality holds iff $x = 0$.

3 Applications: Some inequalities of Wallis ratio

The Wallis ratio

$$W_m = \frac{1.3.5...(2m-1)}{2.4.6...(2m)} = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}, \quad m \in N \tag{38}$$

plays an important role in mathematics especially in special functions, combinatorics, graph theory and many other branches. For further details about its history and applications, we refer to [7], [16], [18], [20], [26]-[29].

Guo, Xu and Qi [14] deduced the inequality

$$\frac{C_1}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1} < W_m \le \frac{C_2}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1}, \quad m\ge 2 \tag{39}$$

with the best possible constants $C_1=\sqrt{\frac{e}{\pi}}$ and $C_2=\frac{4}{3}$.

Recently, Qi and Mortici [37] presented the following improvement of the double inequality (39)

$$\sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3} < W_m < \sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3} e^{\frac{1}{144m^3}}, \qquad m\in N. \tag{40}$$

Also, Zhang, Xu and Situ [47] presented the inequality

$$\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}} < W_m \le \frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m+16}}, \qquad m\in N. \tag{41}$$

Recently, Cristea [8] improved the upper bound of the inequality (41) by

$$W_m \le \frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}+\frac{1}{48m^2}-\frac{1}{2880m^3}}, \qquad m\in N \tag{42}$$

which is better than the upper bound of the inequality (40).

3.1 New proof of Slavić inequality

Slavić [43] presented the following double inequality

$$\frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right) < \frac{\Gamma(x+1/2)}{\Gamma(x+1)} < \frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right), \tag{43}$$

where $x>0$ and $l,m\in N$. In the following sequel, we will present a new proof of Slavić inequality (43). Consider the two functions

$$S_L(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right), \qquad l\in N$$

and

$$S_U(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right), \qquad m\in N.$$

Using Lemma 2.5, we obtain

$$\frac{S_L'(x)}{S_L(x)}=G(2x)-\frac{1}{2x}-\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{kx^{2k}}\right)<0, \qquad l\in N$$

and

$$\frac{S'_U(x)}{S_U(x)} = G(2x) - \frac{1}{2x} - \left(\sum_{k=1}^{2m} \frac{(1-2^{-2k})B_{2k}}{kx^{2k}} \right) > 0, \qquad m \in N.$$

Then the function $S_L(x)$ is decreasing and the function $S_U(x)$ is increasing and using the asymptotic expansion of the ratio of two gamma functions [19]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \left[1 + \frac{(a-b)(a+b-1)}{2x} + O(x^{-2}) \right], \qquad a, b \geq 0 \tag{44}$$

as $x \to \infty$, we have

$$\lim_{x\to\infty} S_L(x) = \lim_{x\to\infty} S_U(x) = 1.$$

Hence we get

$$S_L(x) > 1 \quad \text{and} \quad S_U(x) < 1,$$

which complete the proof of Slavić inequality (43).

Remark 3. In the case of $l = 1$, $m = 1$ and $x = m$, the inequality (43) will gives

$$\frac{e^{\frac{-1}{8m}}}{\sqrt{\pi m}} < W_m < \frac{e^{\frac{-1}{8m}+\frac{1}{192m^3}}}{\sqrt{\pi m}}, \qquad m \in N \tag{45}$$

which is better than inequality (40) of Qi and Mortici [37].

3.2 New upper bound of W_n

Consider the function

$$M_L(x) = \frac{\Gamma(x+1/2)}{\Gamma(x+1)} \sqrt{x} e^{\frac{-1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}x) - \frac{\pi}{2}\right]}, \qquad x > 0.$$

Using the inequality (33), we get

$$\frac{M'_L(x)}{M_L(x)} = G(2x) - \frac{1}{2x} - \frac{1}{8x^2+1} > 0$$

and using the expansion (44), we have $\lim_{x\to\infty} M_L(x) = 1$. Then

$$M_L(x) < 1$$

and we obtain the following result:

Lemma 3.1. *The following double inequality holds*

$$\frac{\Gamma(x+1/2)}{\Gamma(x+1)} < \frac{e^{\frac{1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}x) - \frac{\pi}{2}\right]}}{\sqrt{x}}, \qquad x > 0. \tag{46}$$

Remark 4. In the case of $x = m$ in the inequality (46), we have

$$W_m < \frac{e^{\frac{1}{2\sqrt{2}}\left[\tan^{-1}(2\sqrt{2}m) - \frac{\pi}{2}\right]}}{\sqrt{\pi m}}, \qquad m \in N \tag{47}$$

which is better than inequality (42) of Cristea [8].

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