TOEPLITZ DUALS OF FIBONACCI SEQUENCE SPACES

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Abstract. In this paper we introduce and study some classes of almost strongly convergent difference sequences of Fibonacci numbers defined by a sequence of modulus functions. We also make an effort to study some topological properties and inclusion relations between these classes of sequences. Further, we compute toeplitz duals of theses classes and study matrix transformations on these classes of sequences.

1. Introduction and Preliminaries

Let w be the vector space of all real sequences. We shall write c, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Also, we use the conventions that $e = (1, 1, 1, ...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix transformation from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is a sequence space. By using the matrix domain of a triangle infinite matrix, so many sequence spaces have recently been defined by several authors, (see [1], [2], [15], [25]). In the literature, the matrix domain X_Δ is called the difference sequence space whenever X is a normed or paranormed sequence space, where Δ denotes the backward difference matrix $\Delta = (\Delta_{nk})$ and $\Delta' = (\Delta'_{nk})$ denotes the forward difference matrix (the transpose of the matrix Δ), which are defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n, \\ 0 \quad , & 0 \le k < n-1 \text{ or } k > n \end{cases}$$

$$\Delta'_{nk} = \begin{cases} (-1)^{n-k}, & n \le k \le n+1, \\ 0 \quad , & 0 \le k < n \text{ or } k > n+1 \end{cases}$$

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for all $k, n \in \mathbb{N}$ respectively. The notion of difference sequence spaces was introduced by Kızmaz [16], who defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

for $X = l_\infty, c$ and c_0. The difference space $b\nu_p$, consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space l_p, was studied in the case $0 < p < 1$ by Altay and Başar [3] and in the case $1 \leq p \leq \infty$ by Başar and Altay [7] and Çolak et al. [9]. Kirişçi and Başar [15] have been introduced and studied the generalized difference sequence spaces

$$\hat{X} = \{x = (x_k) \in w : B(r,s)x \in X\}$$

where X denotes any of the spaces l_∞, l_p, c and c_0, $(1 \leq p < \infty)$ and $B(r,s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R}\backslash\{0\}$. Following Kirişçi and Başar [15], Sönmez [31] have been examined the sequence space $X(B)$ as the set of all sequences whose $B(r,s,t)$-transforms are in the space $X \in \{l_\infty, l_p, c, c_0\}$, where $B(r,s,t)$ denotes the triple band matrix $B(r,s,t) = \{b_{nk}(r,s,t)\}$ defined by

$$b_{nk}(r,s,t) = \begin{cases} r, & n = k \\ s, & n = k+1 \\ t, & n = k+2 \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R}\backslash\{0\}$. Also in ([10-13], [26]) authors studied certain difference sequence spaces.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is a BK-space with $\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$. The sequence space X is said to be solid (see [17, p. 48]) if and only if

$$\widetilde{X} = \{(v_k) \in w : \exists (x_k) \in X \text{ such that } |v_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset X.$$

A sequence (b_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

The following lemma (known as the Toeplitz Theorem) contains necessary and sufficient condition for regularity of a matrix.

Lemma 1.1. *(Wilansky, 1984): Matrix* $A = (a_{nk})_{n,k=1}^{\infty}$ *is regular if and only if the following three conditions hold:*

(1) There exists $M > 0$ *such that for every* $n = 1, 2, ...$ *the following inequality holds:*

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M;$$

(2) $\lim_{n\to\infty} a_{nk} = 0$ *for every* $k = 1, 2, ...$

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$

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The sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers is given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2},\ n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, in [18] some basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi \text{ (golden ratio)},$$
$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ (n \in \mathbb{N}),$$
$$\sum_{k} \frac{1}{f_k} \text{ converges},$$
$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula)}.$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Now, let $A = (a_{nk})$ be an infinite matrix and list the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk}\right| < \infty \tag{1.3}$$

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N} \tag{1.4}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{1.5}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{1.6}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{1.7}$$

$$\sup_{k\in\mathcal{H}} \sum_n \Big| \sum_{k\in K} a_{nk} \Big| < \infty \tag{1.8}$$

where $\mathbb{C}$ and $\mathcal{H}$ denote the set of all complex numbers and the collection of all finite subsets of $\mathbb{N}$, respectively.
Now, we may give the following lemma on the characterization of the matrix transformations between some classical sequence spaces.

Lemma 1.2. *The following statements hold:*
(a) $A = (a_{nk}) \in (c_0, c_0)$ *if and only if (1.3) and (1.4) hold.*
(b) $A = (a_{nk}) \in (c_0, c)$ *if and only if (1.3) and (1.5) hold.*
(c) $A = (a_{nk}) \in (c, c_0)$ *if and only if (1.3), (1.4) and (1.6) hold.*
(d) $A = (a_{nk}) \in (c, c)$ *if and only if (1.3), (1.5) and (1.7) hold.*
(e) $A = (a_{nk}) \in (c_0, l_\infty) = (c, l_\infty)$ *if and only if condition (1.3) holds.*
(f) $A = (a_{nk}) \in (c_0, l_1) = (c, l_1)$ *if and only if condition (1.8) holds.*

Recently, Kara [19] has defined the sequence spaces $l_p(\hat{F})$ as follows:

$$l_p(\hat{F}) = \{x \in w : \hat{F}x \in l_p\},\ (1 \leq p \leq \infty)$$

where $\hat{F} = (\hat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & 0 \le k < n-1 \text{ or } k > n \end{cases} \qquad (k, n \in \mathbb{N}).$$

Also, in [20] Kara et al. have characterized some classes of compact operators on the spaces $l_p(\hat{F})$ and $l_\infty(\hat{F})$, where $1 \le p < \infty$.
The inverse $\hat{F}^{-1} = (g_{nk})$ of the Fibonacci matrix $\hat{F}$ is given by

$$g_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \le k \le n, \\ 0, & k > n \end{cases} \quad (k, n \in \mathbb{N}).$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{4}{1} & \frac{4}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{9}{1} & \frac{9}{2} & \frac{9}{6} & 0 & 0 & 0 & \dots \\ \frac{25}{1} & \frac{25}{2} & \frac{25}{6} & \frac{25}{15} & 0 & 0 & \dots \\ \frac{64}{1} & \frac{64}{2} & \frac{64}{6} & \frac{64}{15} & \frac{64}{40} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is obvious that the matrix $\hat{F}$ is a triangular matrix, that is, $f_{nn} \neq 0$ and $f_{nk} = 0$ for $k > n$ $(n = 1, 2, 3...)$. Also, it follows by Lemma 1.1 that the method $\hat{F}$ is regular.
In [8] Başarir et al. introduce the Fibonacci difference sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ as the set of all sequences whose $\hat{F}$-transforms are in the spaces c_0 and c, respectively, i.e.,

$$c_0(\hat{F}) = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},$$

and

$$c(\hat{F}) = \left\{ x = (x_n) \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}.$$

Define the sequence $y = (y_n)$ by the $\hat{F}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \hat{F}_n(x) = \begin{cases} x_0, & n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \ge 1 \end{cases} \quad (n \in \mathbb{N}). \tag{1.9}$$

A linear functional L on l_∞ is said to be a Banach limit if it has the following properties:
(1) $L(x) \ge 0$ if $n \ge 0$ (i.e. $x_n \ge 0$ for all n),
(2) $L(e) = 1$, where $e = (1, 1, ...)$,
(3) $L(Dx) = L(x)$,
where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$ (see [6]).
Let B be the set of all Banach limits on l_∞. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [22], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exits, uniformly in s} \right\}$$

In ([23], [24]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number l such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \text{ uniformly in s.}$$

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Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183).
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([4], [27], [28], [29], [30]) and references therein.
Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = 0 \right\},$$

and

$$c(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \exists l \in \mathbb{C} \ni \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = l \right\}.$$

If $F_k(x) = x$, for all $k \in \mathbb{N}$. Then above sequence spaces reduces to $c_0(\hat{F}, u, p)$ and $c(\hat{F}, u, p)$.
By taking $p_k = 1$ and $u_k = 1$, for all $k \in \mathbb{N}$, then we get the sequence spaces $c_0(\hat{F}, \mathcal{F})$ and $c(\hat{F}, \mathcal{F})$.
With the notation of (1.2), the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ can be redefined as follows:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \{c_0(\mathcal{F}, u, p)\}_{\hat{F}} \text{ and } c(\hat{F}, \mathcal{F}, u, p) = \{c(\mathcal{F}, u, p)\}_{\hat{F}}. \tag{1.10}$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.11}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper, we introduce the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$. We investigate some topological properties of these new sequence spaces and establish some inclusion relations between these spaces. Also we determine the $\alpha-, \beta-$ and $\gamma-$ duals of these spaces and construct the matrix transformation of the spaces $(c_0(\hat{F}, \mathcal{F}, u, p), X)$ and $(c(\hat{F}, \mathcal{F}, u, p), X)$, where X denote the spaces $l_\infty, f, c, f_0, c_0, bs, fs$ and l_1.

2. Some topological properties of the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$

Theorem 2.1. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are linear spaces over the field $\mathbb{R}$ of real numbers.*

Proof. Let $x=(x_k),\ y=(y_k)\in c_0(\hat{F},\mathcal{F},u,p)$ and $\lambda,\ \mu\in\mathbb{C}$. Then there exist integers M_λ and N_μ such that $|\lambda|\leq M_\lambda$ and $|\mu|\leq N_\mu$. Using inequality (1.11) and definition of modulus function, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\lambda\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\mu\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\right]^{p_k}$$

$$\leq\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k|\lambda|\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k|\mu|\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$$\leq K\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_kM_\lambda\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+K\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_kN_\mu\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$$\leq KM_\lambda^H\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}+KN_\mu^H\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}$$

$\to 0$ as $n\to\infty$.

Thus $\lambda x+\mu y\in c_0(\hat{F},\mathcal{F},u,p)$. This proves that $c_0(\hat{F},\mathcal{F},u,p)$ is a linear space. Similarly we can prove that $c(\hat{F},\mathcal{F},u,p)$ is a linear space over the real field $\mathbb{R}$. □

Theorem 2.2. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are paranormed space with the paranorm defined by*

$$g(x)=\sup\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

where $0\leq p_k\leq\sup p_k=H,\ \ M=\max(1,H)$.

Proof. Since the proof is similar for the space $c(\hat{F},\mathcal{F},u,p)$, we consider only the space $c_0(\hat{F},\mathcal{F},u,p)$. Clearly $g(-x)=g(x)$, for all $x\in c_0(\hat{F},\mathcal{F},u,p)$. It is trivial that $\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}=0$, for $x=0$. Hence we get $g(0)=0$. Since $\frac{p_k}{M}\leq 1$, using Minkowski's inequality, we have

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|+u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}$$

$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}+\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\right]^{p_k}\right)^{\frac{1}{M}}.$$

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Now it follows that $g(x)$ is subadditive. Finally to check the continuity of scalar multiplication let us take any real number ρ. By definition of modulus function F_k, we have

$$\begin{aligned} g(\rho x) &= \sup_k \left(\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \rho \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \right)^{\frac{1}{M}} \\ &\leq C_\rho^{\frac{H}{M}} g(x). \end{aligned}$$

where C_ρ is a positive integer such that $|\rho| \leq C_\rho$. Now, Let $\rho \to 0$ for any fixed x with $g(x) = 0$. By definition for $|\rho| < 1$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon \ \text{ for } n > N(\epsilon). \tag{2.1}$$

Also for $1 \leq n < N$, taking ρ small enough. Since F_k is continuous, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon. \tag{2.2}$$

Now from equation (2.1) and (2.2), we have

$$g(\rho x) \to 0 \ \text{ as } \rho \to 0.$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $u = (u_k)$ be a sequence of strictly positive real numbers. If $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers with $0 \leq p_k \leq q_k < \infty$ for each k, then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c(\hat{F}, \mathcal{F}, u, q)$.*

Proof. Let $x \in c_0(\hat{F}, \mathcal{F}, u, p)$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

This implies that

$$\left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq 1,$$

for sufficiently large values of k. Since F_k is increasing and $p_k \leq q_k$ we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{q_k} &\leq \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $x \in c(\hat{F}, \mathcal{F}, u, q)$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions and $\varrho = \lim_{t \to \infty} \frac{F_k(t)}{t} > 0$. Then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$.*

Proof. In order to prove that $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$. Let $\varrho > 0$. By definition of ϱ, we have $F_k(t) \geq \varrho(t)$, for all $t > 0$. Since $\varrho > 0$, we have $t \leq \frac{1}{\varrho} F_k(t)$ for all $t > 0$.
Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}, u, p)$. Thus, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq \frac{1}{\varrho n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k}$$

which implies that $x = (x_k) \in c_0(\hat{F}, u, p)$. This completes the proof. □

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Theorem 2.5. *Let $\mathcal{F}' = (F'_k)$ and $\mathcal{F}'' = (F''_k)$ are sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p) \subseteq c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p)$. Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F''_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

Then we have

$$\begin{aligned}
&\tfrac{1}{n}\sum_{k=1}^{n}\left[u_k (F'_k + F''_k)\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\\
&\qquad \leq \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\}\\
&\qquad + \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F''_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\}\\
&\qquad \longrightarrow \quad 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\frac{1}{n}\sum_{k=1}^{n}\left[u_k (F'_k + F''_k)\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0$ as $n \to \infty$.

Therefore $x = (x_k) \in c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p)$ and this completes the proof. □

Theorem 2.6. *Let $\mathcal{F} = (F_k)$ and $\mathcal{F}' = (F'_k)$ be two sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F} o \mathcal{F}', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p)$. Then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} = 0.$$

Let $\epsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $F_k(t) < \epsilon$ for $0 \leq t \leq \delta$.

Write $y_k = \left[u_k F'_k\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]$ and consider

$$\frac{1}{n}\sum_{k=1}^{n}[F_k(y_k)]^{p_k} = \frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} + \frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and second summation is over $y_k \geq \delta$. Since F_k is continuous, we have

$$\frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} < \epsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

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By the definition, we have for $y_k > \delta$

$$F_k(y_k) < 2F_k(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k} \le \max\Big(1, (2F_k(1)\delta^{-1})^H\Big)\frac{1}{n}\sum_{k}[y_k]^{p_k}. \tag{2.4}$$

From equation (2.3) and (2.4), we have

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F}o\mathcal{F}', u, p).$$

This completes the proof. □

Theorem 2.7. *The sets $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are BK-spaces with the norm $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} = \|x\|_{c(\hat{F},\mathcal{F},u,p)} = \|\hat{F}x\|_\infty$.*

Proof. Since (1.10) holds, c_0 and c are the BK-spaces with respect to their natural norms and the matrix $\hat{F}$ is a triangle; Theorem 4.3.12 of Wilansky [33, p.63] gives the fact that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are BK-spaces with the given norms. This completes the proof. □

Remark 2.8. One can easily check that the absolute property does not hold on the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, that is, $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c_0(\hat{F},\mathcal{F},u,p)}$ and $\|x\|_{c(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c(\hat{F},\mathcal{F},u,p)}$ for at least one sequence in the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, and this shows that $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$.

Theorem 2.9. *The Fibonacci difference sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c respectively, i.e., $c_0(\hat{F}, \mathcal{F}, p, u) \cong c_0$ and $c(\hat{F}, \mathcal{F}, p, u) \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0. Consider the transformation T defined with the notation of (1.9), from $c_0(\hat{F}, \mathcal{F}, u, p)$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.
We assume that $y = (y_k) \in c_0$, for $1 \le p \le \infty$ and defined the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j, \quad \text{for all} \quad k \in \mathbb{N}.$$

Then we have

$$\lim_{k\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_jf_{j+1}}y_j - \frac{f_{k+1}}{f_k}\sum_{j=0}^{k-1}\frac{f_k^2}{f_jf_{j+1}}y_j\Big|\right]^{p_k}\right\} = \lim_{k\to\infty}y_k = 0$$

which shows that $x \in c_0(\hat{F}, \mathcal{F}, p, u)$. Additionally, we have for every $x \in c_0(\hat{F}, \mathcal{F}, p, u)$ that

$$\begin{aligned}
\|x\|_{c_0(\hat{F},\mathcal{F},p,u)} &= \sup_{k\in\mathbb{N}} \left| \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k \Big| \frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right| \\
&= \sup_{k\in\mathbb{N}} \left| \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k \Big| \frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j - \frac{f_{k+1}}{f_k}\sum_{j=0}^{k-1}\frac{f_k^2}{f_j f_{j+1}}y_j\Big|\right]^{p_k}\right| \\
&= \sup_{k\in\mathbb{N}} \Big(|y_k|^{p_k}\Big) \\
&= \|y\|_\infty < \infty.
\end{aligned}$$

Consequently, we see from here that T is surjective and norm preserving. Hence, T is a linear bijection which shows that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are linearly isomorphic. It is clear here that if the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are respectively replaced by the spaces $c(\hat{F}, \mathcal{F}, u, p)$ and c, then we obtain the fact that $c(\hat{F}, \mathcal{F}, p, u) \cong c$. This concludes the proof. □

Now, we give some inclusion relations concerning with the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.

Theorem 2.10. *The inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. It is clear that the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ holds. Further, to show that this inclusion is strict, consider the sequence $x = (x_k) = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2}$. Then, we obtain (1.9) for all $k \in \mathbb{N}$ that

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\Big|\frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j^2} - \frac{f_{k+1}}{f_k}\sum_{j=0}^{k-1}\frac{f_{k+1}^2}{f_j^2}\Big|\right]^{p_k} = \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\Big(\frac{f_{k+1}}{f_k}\Big)\right]^{p_k}$$

which shows that $\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\Big(\frac{f_{k+1}}{f_k}\Big)\right]^{p_k} \to \varphi$, as $k \to \infty$. This is to say that $\hat{F}(x) \in c \backslash c_0$. Thus, the sequence x is in the $c(\hat{F}, \mathcal{F}, u, p)$ but not in $c_0(\hat{F}, \mathcal{F}, u, p)$. Hence, the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ is strict. □

Theorem 2.11. *The space l_∞ does not include the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.*

Proof. Let us consider the sequence $x = (x_k) = (f_{k+1}^2)$. Since $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\hat{F}(x) = e^{(0)} = (1, 0, 0, ...)$, the sequence x is in the space $c_0(\hat{F}, \mathcal{F}, u, p)$ but is not in the space l_∞. This shows that the space l_∞ does not include the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and the space $c(\hat{F}, \mathcal{F}, u, p)$, as desired. □

Theorem 2.12. *The inclusions $c_0 \subset c_0(\hat{F}, \mathcal{F}, u, p)$ and $c \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. Let $X = c_0$ or c. Since the matrix $\hat{F} = (f_{nk})$ satisfies the conditions

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\sum_k |f_{nk}| = \sup_{n\in\mathbb{N}}\Big(\frac{f_n}{f_{n+1}} + \frac{f_{n+1}}{f_n}\Big) = 2 + \frac{1}{2} = \frac{5}{2}, \\
&\lim_{n\to\infty} f_{nk} = 0, \\
&\lim_{n\to\infty}\sum_k f_{nk} = \lim_{n\to\infty}\Big(\frac{f_n}{f_{n+1}} - \frac{f_{n+1}}{f_n}\Big) = \frac{1}{\varphi} - \varphi
\end{aligned}$$

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we conclude by parts (a) and (c) of Lemma 1.2 that $(\hat{F},\mathcal{F},u,p)\in(X,X)$. This leads that $(\hat{F},\mathcal{F},u,p)x\in X$ for any $x\in X$. Thus, $x\in X_{(\hat{F},\mathcal{F},u,p)}$. This shows that $X\subset X_{(\hat{F},\mathcal{F},u,p)}$. Now, let $x=(x_k)=(f_{k+1}^2)$. Then, it is clear that $x\in X_{(\hat{F},\mathcal{F},u,p)}\backslash X$. This says that the inclusion $X\subset X_{(\hat{F},\mathcal{F},u,p)}$ is strict. □

Theorem 2.13. *The spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are not solid.*

Proof. Consider the sequences $r=(r_k)$ and $s=(s_k)$ defined by $r_k=f_{k+1}^2$ and $s_k=(-1)^{k+1}$ for all $k\in\mathbb{N}$. Then, it is clear that $r\in c_0(\hat{F},\mathcal{F},u,p)$ and $s\in l_\infty$. Nevertheless $rs=\{(-1)^{k+1}f_{k+1}^2\}$ is not in the space $c_0(\hat{F},\mathcal{F},u,p)$, since

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\Big|\frac{f_k}{f_{k+1}}(-1)^{k+1}f_{k+1}^2-\frac{f_{k+1}}{f_k}(-1)^kf_k^2\Big|\right]^{p_k}\\&=\frac{1}{n}\sum_{k=1}^{n}\left[u_kF_k\big(2(-1)^{k+1}f_kf_{k+1}\big)\right]^{p_k}\quad\text{for all }\ k\in\mathbb{N}.\end{aligned}$$

This shows that the multiplication $l_\infty c_0(\hat{F},\mathcal{F},u,p)$ of the spaces l_∞ and $c_0(\hat{F},\mathcal{F},u,p)$ is not a subset of $c_0(\hat{F},\mathcal{F},u,p)$. Hence, the space $c_0(\hat{F},\mathcal{F},u,p)$ is not solid.
It is clear here that if the spaces $c_0(\hat{F},\mathcal{F},u,p)$ is replaced by the space $c(\hat{F},\mathcal{F},u,p)$, then we obtain the fact $c(\hat{F},\mathcal{F},u,p)$ is not solid. This completes the proof. □

It is known from Theorem 2.3 of Jarrah and Malkowsky [14] that the domain X_T of an infinite matrix $T=(t_{nk})$ in a normed sequence space X has a basis if and only if X has a basis, if T is a triangle. As a direct consequence of this fact, we have

Corollary 2.14. *Define the sequences $c^{(-1)}=\{c_k^{(-1)}\}_{k\in\mathbb{N}}$ and $c^{(n)}=\{c_k^{(n)}\}_{k\in\mathbb{N}}$ for every fixed $n\in\mathbb{N}$ by*

$$c_k^{(-1)}=\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_jf_{j+1}}\quad\text{and}\quad c_k^{(n)}=\begin{cases}0 & ,\ 0\le k\le n-1\\ \frac{f_{k+1}^2}{f_nf_{n+1}}, & k\ge n\end{cases}$$

Then, the following statements hold:
(a) The sequence $\{c^{(n)}\}_{n=0}^{\infty}$ is a basis for the space $c_0(\hat{F},\mathcal{F},u,p)$ and every sequence $x\in c_0(\hat{F},\mathcal{F},u,p)$ has a unique representation $x=\sum_n\hat{F}_n(x)c^{(n)}$.
(b) The sequence $\{c^{(n)}\}_{n=-1}^{\infty}$ is a basis for the space $c(\hat{F},\mathcal{F},u,p)$ and every sequence $z=(z_n)\in c(\hat{F},\mathcal{F},u,p)$ has a unique representation $z=lc^{(-1)}+\sum_n[\hat{F}_n(z)-l]c^{(n)}$, where $l=\lim_{n\to\infty}\hat{F}_n(z)$.

3. The $\alpha-,\beta-$ and $\gamma-$ duals of the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ and some matrix transformations

The $\alpha-,\beta-$ and $\gamma-$ duals of the sequence space X are respectively defined by

$$X^\alpha=\{a=(a_k)\in w: ax=(a_kx_k)\in l_1\ \text{ for all } x=(x_k)\in X\},$$
$$X^\beta=\{a=(a_k)\in w: ax=(a_kx_k)\in cs\ \text{ for all } x=(x_k)\in X\}$$

and

$$X^\gamma=\{a=(a_k)\in w: ax=(a_kx_k)\in bs\ \text{ for all } x=(x_k)\in X\}$$

In this section, we determine $\alpha-,\beta-$ and $\gamma-$ duals of the sequence spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$, and characterize the classes of infinite matrices from the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ to the spaces $c_0,c,l_\infty,f,f_0,bs,fs,cs$ and l_1, and from the space f to the

spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.
The following two lemmas are essential for our results.

Lemma 3.1. [8] *Let X be any of the spaces c_0 or c and $a = (a_n) \in w$, and the matrix $B = (b_{nk})$ be defined by $B_n = a_n \hat{F}_n^{-1}$, that is ,*

$$b_{nk} = \begin{cases} a_n g_{nk}, & 0 \leq k \leq n, \\ 0 \quad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then $a \in X_{\hat{F}}^{\beta}$ if and only if $B \in (X, l_1)$.

Lemma 3.2 (5, Theorem 3.1)**.** *Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $Z = (z_{nk})$ by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \leq k \leq n, \\ \quad 0 \quad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then for any sequence space X,

$$X_Z^{\gamma} = \{a = (a_k) \in w : C \in (X, l_\infty)\},$$

$$X_Z^{\beta} = \{a = (a_k) \in w : C \in (X, c)\}.$$

Combining Lemmas (1.2), (3.1), and (3.2), we have

Corollary 3.3. *Consider the sets d_1, d_2, d_3 and d_4 defined as follows:*

$$d_1 = \left\{ a = (a_k) \in w : \sup_{k \in \mathcal{H}} \sum_n \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{k \in K} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right| \right]^{p_k} < \infty \right\},$$

$$d_2 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} < \infty \right\},$$

$$d_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \text{ exists for each } k \in \mathbb{N} \right\},$$

$$d_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \text{ exists} \right\}.$$

Then the following statements hold:
(a) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = d_1$.
(b) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3$ and $\{c(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3 \cap d_4$.
(c) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = d_2$.

Theorem 3.4. *Let $X = c_0$ or c and Y be an arbitrary subset of w. Then, we have $A = (a_{nk}) \in (X_{\hat{F}}, Y)$ if and only if*

$$D^{(m)} = (d_{nk}^{(m)}) \in (X, c) \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

$$D = (d_{nk}) \in (X, Y), \tag{3.2}$$

where

$$d_{nk}^{(m)} = \begin{cases} \frac{1}{n} \sum_{k=1}^{n} \left(u_k F_k \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| \right)^{p_k}, & 0 \leq k \leq m \\ 0 & , \quad k > m \end{cases}$$

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and

$$d_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left(u_k F_k\left|\sum_{j=k}^{\infty}\frac{f_{j+1}^2}{f_k f_{k+1}}a_{nj}\right|\right)^{p_k} \text{ for all } k, m, n \in \mathbb{N}.$$

By changing the roles of the spaces $X_{\hat{F}}$ and X with Y in Theorem 3.4, we have

Theorem 3.5. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\left| -\frac{f_{n+1}}{f_n}a_{n-1,k} + \frac{f_n}{f_{n+1}}a_{nk}\right|\right]^{p_k} \tag{3.3}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y, X_{\hat{F}})$ if and only if $B \in (Y, X)$.

Proof. Let $z = (z_k) \in Y$. Then, by taking into account the relation (3.3) one can easily derive the following equality

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{k=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\left| -\frac{f_{n+1}}{f_n}a_{n-1,k} + \frac{f_n}{f_{n+1}}a_{nk}\right|\right]^{p_k}\right)z_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = [\hat{F}(Az)]_n$. Therefore, we conclude that $Az \in X_{\hat{F}}$ whenever $z \in Y$ if and only if $Bz \in X$ whenever $z \in Y$. This completes the proof. □

By f_0, f and fs we denote the spaces of almost null and almost convergent sequences and series respectively. Now, the following two lemmas characterizing the strongly and almost conservative matrices:

Lemma 3.6. *(see [32]) $A = (a_{nk}) \in (f, c)$ if and only if (1.3), (1.5), and (1.7) hold, and*

$$\lim_{n\to\infty}\sum_k \Delta(a_{nk} - \alpha_k) = 0 \tag{3.4}$$

also holds, where $\Delta(a_{nk} - \alpha_k) = a_{nk} - \alpha_k - (a_{n,k+1} - \alpha_{k+1})$ for all $k, n \in \mathbb{N}$.

Lemma 3.7. *(see [21]) $A = (a_{nk}) \in (c, f)$ if and only if (1.3) holds, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \tag{3.5}$$

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha. \tag{3.6}$$

Now, we list the following conditions:

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|d_{mk}^{(n)}\right| < \infty \tag{3.7}$$

$$\exists d_{nk} \in \mathbb{C} \ni \lim_{m\to\infty} d_{mk}^{(n)} = d_{nk} \text{ for each } k, n \in \mathbb{N} \tag{3.8}$$

$$\sup_{n\in\mathbb{N}}\sum_k |d_{nk}| < \infty \tag{3.9}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{3.10}$$

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$$\sup_{N,K\in\mathcal{H}}\Big|\sum_{n\in\mathbb{N}}\sum_{k\in K} d_{nk}\Big| < \infty \tag{3.11}$$

$$\exists\beta_n\in\mathbb{C}\ni\lim_{m\to\infty}\sum_{k=0}^{m} d^{(n)}_{mk}=\beta_n \text{ for each } n\in\mathbb{N} \tag{3.12}$$

$$\exists\alpha\in\mathbb{C}\ni\lim_{n\to\infty}\sum_{k} d_{nk}=\alpha \tag{3.13}$$

It is trivial that Theorem 3.4 and Theorem 3.5 have several consequences. Indeed, combining Theorem 3.4, 3.5 and Lemmas 1.1, 3.6 and 3.7 we derive the following results:

Corollary 3.8. *Let $A=(a_{nk})$ be an infinite matrix and $a(n,k)=\sum_{j=0}^{n} a_{jn}$ for all $k,n\in\mathbb{N}$. Then, the following statements hold:*
(a) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),c_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k=0$ for all $k\in\mathbb{N}$.
(b) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),cs_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k=0$ for all $k\in\mathbb{N}$ with $a(n,k)$ instead of a_{nk}.
(c) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),c)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold.
(d) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),cs)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $a(n,k)$ instead of a_{nk}.
(e) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),l_\infty)$ if and only if (3.7), (3.8) and (3.9) hold.
(f) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),bs)$ if and only if (3.7), (3.8) and (3.9) hold with $a(n,k)$ instead of a_{nk}.
(g) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),l_1)$ if and only if (3.7), (3.8) and (3.11) hold.
(h) $A=(a_{nk})\in(c_0(\hat{F},\mathcal{F},u,p),bv_1)$ if and only if (3.7), (3.8) and (3.11) hold with $a_{nk}-a_{n-1,k}$ instead of a_{nk}.

Corollary 3.9. *Let $A=(a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(a) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),l_\infty)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold.
(b) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),bs)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold with $a(n,k)$ instead of a_{nk}.
(c) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),c)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold.
(d) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),cs)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n,k)$ instead of a_{nk}.
(e) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),c_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k=0$ for all $k\in\mathbb{N}$, (3.12) and (3.13) also hold with $\alpha=0$.
(f) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),cs_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k=0$ for all $k\in\mathbb{N}$, (3.12) and (3.13) also hold with $\alpha=0$ with $a(n,k)$ instead of a_{nk}.
(g) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),l_1)$ if and only if (3.7), (3.8), (3.11) and (3.12)hold.
(h) $A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),bv_1)$ if and only if (3.7), (3.8), (3.11) and (3.12) hold with $a_{nk}-a_{n-1,k}$ instead of a_{nk}..

Corollary 3.10. *$A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),f)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk}.*

Corollary 3.11. *$A=(a_{nk})\in(c(\hat{F},\mathcal{F},u,p),f_0)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk} and $\alpha_k=0$ for all $k\in\mathbb{N}$.*

Corollary 3.12. *$A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), f_s)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n,k)$ instead of a_{nk} and (3.9), (3.10) also hold with $d(n,k)$ instead of d_{nk}.*

Corollary 3.13. *$A = (a_{nk}) \in (f, c(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3), (1.5), (1.7) and (3.8) hold with b_{nk} instead of a_{nk}, where $b(n,k)$ is defined by (3.3) .*

Corollary 3.14. *$A = (a_{nk}) \in (f, c_0(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3) and (1.7) hold, (1.5) and (3.8) also hold with b_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$, where $b(n,k)$ is defined by (3.3).*

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