GENERALIZED ADDITIVE-CUBIC FUCNTIONAL EQUATION AND ITS STABILITY

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Abstract. In this paper, we establish some stability results for the following additive-cubic functional equation with an extra term G_f

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x).$$

in Banach spaces, where G_f is a functional operator of f. Using these, we give new additive-cubic functional equations and prove their stability.

1. Introduction

In 1940, Ulam [12] raised the following question concerning the stability of group homomorphisms: "Under what conditions does there is an additive mapping near an approximately additive mapping between a group and a metric group ? "
In the next year, Hyers [5] gave a partial solution of Ulam's problem for the case of additive mappings. Hyers 's result, using unbounded Cauchy different, was generalized for additive mappings in [1] and for a linera mapping in [11]. Some stability results for additive, quardartic and mixed additve-cubic functional equations were investigated ([2], [3], [4], [6], [7], [8], [9], [10]).

The generalized Hyers–Ulam stability for the mixed additive-cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x) \tag{1.1}$$

in quasi-Banach spaces has been investigated by Najati and Eskandani [8]. Functional equation (1.1) is called *an additive-cubic functional equation*, since the function $f(x)=ax^3+bx$ is its solution. Every solution of this mixed additive-cubic functional equation is said to be *an additive-cubic mapping*.

In this paper, we are interested in what kind of a term $G_f(x,y)$ can be added to (1.1) while the solution of the new functional equation is also an additive-cubic funtional equation and the generalized Hyers-Ulam stability for it still holds, where $G_f(x,y)$ is a functional operator depending on the variables x,y, and function f. The new functional equation can be written as

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.2}$$

We give some new functional equations in section 3 as examples of our results and prove the generalized Hyers-Ulam stability for these.

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2. The generalized Hyers-Ulam stability for (1.2)

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Throughout this section we always assume that G_f satisfies the following two conditions unless a specific expression for G_f is given.

Condition $\mathbf{P_1}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 2f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) \tag{2.1}$$

for all $x, y \in X$. Then f is an additive mapping.

Condition $\mathbf{P_2}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 8f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{2.2}$$

for all $x, y \in X$. Then f is a cubic mapping.

For any $f : X \longrightarrow Y$, let

$$f_a(x) = \frac{4}{3}f(x) - \frac{1}{6}f(2x), \ \ f_c(x) = -\frac{1}{3}f(x) + \frac{1}{6}f(2x)$$

Now, we prove the following main theorem.

Theorem 2.1. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$**. *Further, suppose that there is a real number $\lambda(\lambda \neq -1)$ such that*

$$G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) = \lambda[t(4x) - 10t(2x) + 16t(x)] \tag{2.3}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{2.4}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned} &\|f(2x + y) + f(2x - y) + G_f(x, y) \\ &- 2f(x + y) - 2f(x - y) - 2f(2x) + 4f(x)\| \le \phi(x, y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} &\|F_a(x) - f_a(x)\| \\ &\le \frac{1}{12|\lambda + 1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.6}$$

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and

$$\begin{aligned}&\|F_c(x) - f_c(x)\| \\ &\leq \frac{1}{48|\lambda+1|}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.7}$$

for all $x \in X$*.*

Proof. By (2.5), we have

$$\|f(x) + f(-x) - G_f(0, x)\| \leq \phi(0, x), \tag{2.8}$$

$$\|f(3x) - 4f(2x) + 5f(x) + G_f(x, x)\| \leq \phi(x, x), \tag{2.9}$$

and

$$\|f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x) + G_f(x, 2x)\| \leq \phi(x, 2x) \tag{2.10}$$

for all $x \in X$. By (2.3), (2.8), (2.9), and (2.10), we have

$$\|2^{-1}f_a(2x) - f_a(x)\| \leq \frac{1}{12|\lambda+1|}[\phi(x, 2x) + 2\phi(x, x) + 2\phi(0, x)] \tag{2.11}$$

for all $x \in X$. By (2.11), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n x) - 2^{-m}f_a(2^m x)\| \\ &= 2^{-m}\|2^{-(n-m)}f_a(2^{n-m} \cdot 2^m x) - f_a(2^m x)\| \\ &\leq \frac{1}{12|\lambda+1|}\sum_{k=m}^{n-1} 2^{-k}[\phi(2^k x, 2^{k+1}x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)]\end{aligned} \tag{2.12}$$

for all $x \in X$. By (2.12), $\{2^{-n}f_a(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty} 2^{-n}f_a(2^n x)$$

for all $x \in X$. Moreover, by (2.12), we have

$$\begin{aligned}&\|A(x) - f_a(x)\| \\ &\leq \frac{1}{12|\lambda+1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.13}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned}&\|f_a(2x + y) + f_a(2x - y) + G_{f_a}(x, y) - 2f_a(x + y) - 2f_a(x - y) \\ &- 2f_a(2x) + 4f_a(x)\| \leq \frac{4}{3}\phi(x, y) + \frac{1}{6}\phi(2x, 2y)\end{aligned} \tag{2.14}$$

for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.14), respectively and deviding (2.5) by 2^n, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n(2x + y)) + 2^{-n}f_a(2^n(2x - y)) + 2^{-n}G_{f_a}(2^n x, 2^n y) \\ &- 2 \cdot 2^{-n}f_a(2^n(x + y)) - 2 \cdot 2^{-n}f_a(2^n(x - y)) - 2 \cdot 2^{-n}f_a(2^{n+1}x) \\ &+ 4 \cdot 2^{-n}f_a(2^n x)\| \leq \frac{4}{3} \cdot 2^{-n}\phi(2^n x, 2^n y) + \frac{1}{6} \cdot 2^{-n}\phi(2^{n+1}x, 2^{n+1}y)\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned} &A(2x+y) + A(2x-y) + \lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ &- 2A(x+y) - 2A(x-y) - 2A(2x) + 4A(x) = 0 \end{aligned} \tag{2.15}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned} &\lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ =& \lim_{n\to\infty} F(2^{-n} f_a(2^n \sigma_1(x,y)), 2^{-n} f_a(2^n \sigma_2(x,y)), \cdots, 2^{-n} f_a(2^n \sigma_l(x,y))) \\ =& G_A(x,y) \end{aligned}$$

for all $x, y \in X$. Hence by (2.15), we have

$$A(2x+y) + A(2x-y) + G_A(x,y) = 2A(x+y) + 2A(x-y) + 2A(2x) - 4A(x) \tag{2.16}$$

for all $x, y \in X$. Relpacing x by $2^n x$ in (2.11) and deviding (2.11) by 2^n, we have

$$\begin{aligned} &\|2^{-n-1} f_a(2^n \cdot 2x) - 2^{-n} f_a(2^n x)\| \\ \le& \frac{2^{-n}}{12|\lambda+1|}[\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$A(2x) = 2A(x) \tag{2.17}$$

for all $x, y \in X$. By (2.16) and (2.17), A satisfies (2.1). By **Condition $\mathbf{P_1}$**, A is an additive mapping .

By (2.3), (2.8), (2.9), and (2.10), we have

$$\|8^{-1} f_c(2x) - f_c(x)\| \le \frac{1}{48|\lambda+1|}[\phi(x, 2x) + 2\phi(x,x) + 2\phi(0,x)] \tag{2.18}$$

for all $x \in X$. By (2.18), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$, we have

$$\begin{aligned} &\|2^{-3n} f_c(2^n x) - 2^{-3m} f_c(2^m x)\| \\ =& 2^{-3m} \|2^{-3(n-m)} f_c(2^{n-m} \cdot 2^m x) - f_c(2^m x)\| \\ \le& \frac{1}{48|\lambda+1|} \sum_{k=m}^{n-1} 2^{-3k}[\phi(2^k x, 2^{k+1} x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)] \end{aligned} \tag{2.19}$$

for all $x \in X$. By (2.19), $\{2^{-3n} f_c(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $C : X \longrightarrow Y$ such that

$$C(x) = \lim_{n\to\infty} 2^{-3n} h(2^n x)$$

for all $x \in X$. Moreover, by (2.19), we have

$$\begin{aligned} &\|C(x) - f_c(x)\| \\ \le& \frac{1}{48|\lambda+1|} \sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.20}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned} &\|f_c(2x+y) + f_c(2x-y) + G_{f_c}(x,y) - 2f_c(x+y) - 2f_c(x-y) \\ &- 2f_c(2x) + 4f_c(x)\| \le \frac{1}{3}\phi(x,y) + \frac{1}{6}\phi(2x, 2y) \end{aligned} \tag{2.21}$$

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for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.21), respectively and deviding (2.21) by 2^{3n}, we have

$$\begin{aligned}
&\|2^{-3n} f_c(2^n(2x+y)) + 2^{-3n} f_c(2^n(2x-y)) + 2^{-3n} G_{f_c}(2^n x, 2^n y) \\
&- 2 \cdot 2^{-3n} f_c(2^n(x+y)) - 2 \cdot 2^{-3n} f_c(2^n(x-y)) - 2 \cdot 2^{-n} f_c(2^{n+1}x) \\
&+ 4 \cdot 2^{-3n} f_c(2^n x)\| \leq \frac{1}{3} \cdot 2^{-3n} \phi(2^n x, 2^n y) + \frac{1}{6} \cdot 2^{-3n} \phi(2^{n+1}x, 2^{n+1}y)
\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned}
&C(2x+y) + C(2x-y) + \lim_{n\to\infty} 2^{-3n} G_{f_c}(2^n x, 2^n y) \\
&- 2C(x+y) - 2C(x-y) - 2C(2x) + 4C(x) = 0
\end{aligned} \tag{2.22}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned}
&\lim_{n\to\infty} 2^{-3n} G_{f_c}(2^n x, 2^n y) \\
&= \lim_{n\to\infty} F(2^{-3n} h(2^n \sigma_1(x,y)), 2^{-3n} h(2^n \sigma_2(x,y)), \cdots, 2^{-3n} h(2^n \sigma_l(x,y))) \\
&= G_C(x,y)
\end{aligned}$$

for all $x, y \in X$. Hence by (2.22), we have

$$C(2x+y) + C(2x-y) + G_C(x,y) = 2C(x+y) + 2C(x-y) + 2C(2x) - 4C(x) \tag{2.23}$$

for all $x, y \in X$. Relpacing x by $2^n x$ in (2.18) and deviding (2.18) by 2^{3n}, we have

$$\begin{aligned}
&\|2^{-3} \cdot 2^{-3n} f_c(2^n \cdot 2x) - 2^{-3n} f_c(2^n x)\| \\
&\leq \frac{2^{-3n}}{48|\lambda+1|} [\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]
\end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$C(2x) = 8C(x) \tag{2.24}$$

for all $x, y \in X$. By (2.23) and (2.24), C satisifes (2.2). By **Condition $\mathbf{P_2}$**, C is a cubic mapping.

Let $F = A + C$. Then F is an additive-cubic mapping, $F_a = A$, and $F_c = C$. By (2.13) and (2.20), we have (2.6) and (2.7).

For the uniqueness of F, let H be another additive-cubic mapping with (2.6) and (2.7). Then F_a and H_a are additive mappings and hence

$$\begin{aligned}
&\|F_a(x) - H_a(x)\| = 2^{-k} \|F_a(2^k x) - H_a(2^k x)\| \\
&\leq \frac{1}{6|\lambda+1|} \sum_{n=k}^{\infty} 2^{-n} [\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]
\end{aligned}$$

for all $x \in X$. Hence, letting $k \to \infty$ in the above inequality, we have $F_a = H_a$ and similarly, we have $F_c = H_c$. Thus $F = H$. □

Similarly, we have the following theorem:

Theorem 2.2. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$**, **Condition $\mathbf{P_2}$**, *and*

$$G_t(x, 0) = -G_t(0, x). \tag{2.25}$$

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for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Further, suppose that there are real numbers $\lambda, \delta (\lambda \neq -1)$ such that

$$\begin{aligned}(2.26)\qquad &G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) \\ &= \lambda[t(4x) - 10t(2x) + 16t(x)] + \delta[f(x) + f(-x)]\end{aligned}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.5). Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned}\|F_a(x) - f_a(x)\| \le \frac{1}{12|\lambda + 1|} \sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) \\ + 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2 + |\delta|)\phi(0, 2^n x)]\end{aligned}$$

and

$$\begin{aligned}\|F_c(x) - f_c(x)\| \le \frac{1}{48|\lambda + 1|} \sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) \\ + 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2 + |\delta|)\phi(0, 2^n x)]\end{aligned}$$

for all $x \in X$.

Proof. By (2.8) and (2.25), we have

$$\|f(x) + f(-x)\| \le \phi(x, 0) + \phi(0, x)$$

for all $x \in X$, because $\|G_f(x, 0)\| \le \phi(x, 0)$ and $G_f(x, 0) = -G_f(0, x)$. Similar to the proof of Theorem 2.1, we have

$$\begin{aligned}&\|(1 + \lambda)[f(4x) - 10f(2x) + 16f(x)]\| \\ &\le \phi(x, 2x) + 2\phi(x, x) + 2\phi(0, x) + |\delta|\|f(x) + f(-x)\| \\ &\le \phi(x, 2x) + 2\phi(x, x) + |\delta|\phi(x, 0) + (2 + |\delta|)\phi(0, x)\end{aligned}$$

for all $x \in X$ and so we get

$$\|2^{-1}f_a(2x) - f_a(x)\| \le \frac{1}{12|\lambda + 1|}[\phi(x, 2x) + 2\phi(x, x) + |\delta|\phi(x, 0) + (2 + |\delta|)\phi(0, x)]$$

for all $x \in X$. The rest of this proof is similar to the proof of Theorem 2.1. □

3. Applications

In this section, using Theorem 2.1 and Theorem 2.2, we will prove the generalized Hyers-Ulam stability for some additive-cubic functional equations.

First, we consider the following functional equation :

$$(3.1)\quad f(2x + y) + f(2x - y) - f(4x) = 2f(x + y) + 2f(x - y) - 8f(2x) + 12f(x).$$

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned}(3.2)\qquad &\|f(2x + y) + f(2x - y) - f(4x) - 2f(x + y) - 2f(x - y) \\ &+ 8f(2x) - 12f(x)\| \le \phi(x, y)\end{aligned}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$(3.3)\quad \|F_a(x) - f_a(x)\| \le \frac{1}{24} \sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]$$

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and

$$\|F_c(x) - f_c(x)\| \leq \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \tag{3.4}$$

for all $x \in X$.

Proof. Let $G_f(x,y) = -f(4x) + 10f(2x) - 16f(x)$. Then f satisfies (2.5) and

$$G_t(x, 2x) + 2G_t(x,x) - 2G_t(0,x) = -3[t(4x) - 10t(2x) + 16t(x)]$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. If $t : X \longrightarrow Y$ is a mapping with $t(2x) = 2t(x)$ for all $x \in X$ and (2.1), then $G_t(x,y) = 0$ for all $x, y \in X$ and so t is an additive mapping. Hence G_t satifies **Condition $\mathbf{P_1}$** and similarly G_t satifies **Condition $\mathbf{P_2}$**. By Theroem 2.1, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.3) and (3.4). □

Using the above theorem, we have the following corollaries:

Corollary 3.2. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (3.1) if and only if f is an additive-cubic mapping.*

Ostadbashi and Kazemzadeh [9] investigated the following additive-cubic functinal equation :

$$\begin{aligned} &f(2x + y) + f(2x - y) - f(4x) \\ &= 2f(x + y) + 2f(x - y) - 8f(2x) + 10f(x) - 2f(-x). \end{aligned} \tag{3.5}$$

Corollary 3.3. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - f(4x) - 2f(x + y) - 2f(x - y) \\ &+ 8f(2x) - 10f(x) + 2f(-x)\| \leq \phi(x, y) \end{aligned} \tag{3.6}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\|F_a(x) - f_a(x)\| \leq \frac{1}{24}\sum_{n=0}^{\infty} 2^{-n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.7}$$

and

$$\|F_c(x) - f_c(x)\| \leq \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.8}$$

for all $x \in X$, where $\phi_1(x, y) = \phi(x, y) + \phi(0, x)$.

Proof. By (3.6), we have

$$\|f(x) + f(-x)\| \leq \phi(0, x)$$

for all $x \in X$ and hence we have

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - f(4x) - 2f(x + y) - 2f(x - y) \\ &+ 8f(2x) - 12f(x)\| \leq \phi(x, y) + \phi(0, x) = \phi_1(x, y) \end{aligned}$$

for all $x, y \in X$. By Theorem 3.3, we have the results. □

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Finally, we consider the following new functional equation :

$$\begin{aligned} &f(2x+y)+f(2x-y)-2f(x+y)+3f(x-y)-5f(y-x)\\ &-10(x)+14f(y)-2f(2y)=0. \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let G_f be a functional operator such that*

$$G_f(x,y)=-G_f(y,x) \tag{3.10}$$

for all mapping $f:X\longrightarrow Y$ and all $x,y\in X$. Then **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$** *hold.*

Proof. Suppose that $f:X\longrightarrow Y$ is a mapping with $f(2x)=2f(x)$ and (2.1). Letting $y=0$ in (2.1), we have

$$G_f(x,0)=0 \tag{3.11}$$

for all $x\in X$ and by (3.10) and (3.11), we get

$$G_f(x,0)=-G_f(0,x)=-[f(x)+f(-x)]=0$$

for all $x\in X$. Hence

$$f(-x)=-f(x) \tag{3.12}$$

for all $x\in X$. Interchaging x and y in (2.1), by (3.12), we have

$$f(x+2y)-f(x-2y)+G_f(y,x)=2f(x+y)-2f(x-y) \tag{3.13}$$

for all $x,y\in X$ and by (2.1), (3.10), and (3.13), we have

$$f(2x+y)+f(2x-y)+f(x+2y)-f(x-2y)=4f(x+y) \tag{3.14}$$

for all $x,y\in X$. Letting $y=-y$ in (3.14), we have

$$f(2x+y)+f(2x-y)+f(x-2y)-f(x+2y)=4f(x-y) \tag{3.15}$$

for all $x,y\in X$. By (3.14) and (3.15), we have

$$f(x+y)+f(x-y)=f(x+2y)+f(x-2y) \tag{3.16}$$

for all $x,y\in X$. Letting $x=x+y$ in (3.16), we get

$$f(x+2y)+f(x)=f(x+3y)+f(x-y) \tag{3.17}$$

for all $x,y\in X$ and letting $x=2x$ in (3.16), we get

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y) \tag{3.18}$$

for all $x,y\in X$. Letting $y=x+y$ in (3.18), we get

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y) \tag{3.19}$$

for all $x,y\in X$ and interchaging x and y in (3.19), we have

$$f(x+3y)-f(x-y)=2f(x+2y)-2f(x) \tag{3.20}$$

for all $x,y\in X$. By (3.17) and (3.20), we have

$$f(x+2y)-3f(x)+2f(x-y)=0 \tag{3.21}$$

for all $x,y\in X$. Letting $x=x-y$ in (3.21), we get

$$f(x+y)-3f(x-y)+2f(x-2y)=0 \tag{3.22}$$

for all $x,y\in X$ and letting $y=-y$ in (3.22), we get

$$f(x-y)-3f(x+y)+2f(x+2y)=0 \tag{3.23}$$

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for all $x, y \in X$. By (3.21) and (3.23), we have

$$f(x+y) + f(x-y) - 2f(x) = 0$$

for all $x, y \in X$ and hence f is an additive mapping. Thus **Condition $\mathbf{P_1}$** holds.

(2) Suppose that $f : X \longrightarrow Y$ is a mapping with $f(2x) = 8f(x)$ and (2.2). Similar to (1), we have

$$G_f(x, 0) = -G_f(0, x) = 0, \ f(-x) = -f(x)$$

for all $x, y \in X$. Interchaging x and y in (2.2), we have

$$f(x+2y) - f(x-2y) + G_f(y, x) = 2f(x+y) - 2f(x-y) + 12f(y) \tag{3.24}$$

for all $x, y \in X$ and by (2.2), (3.10), and (3.24), we have

$$f(2x+y) + f(2x-y) + f(x+2y) - f(x-2y) = 4f(x+y) + 12f(x) + 12f(y) \tag{3.25}$$

for all $x, y \in X$. Letting $y = -y$ in (3.25), we have

$$f(2x+y) + f(2x-y) + f(x-2y) - f(x+2y) = 4f(x-y) + 12f(x) - 12f(y) \tag{3.26}$$

for all $x, y \in X$. By (3.25) and (3.26), we have

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$ and hence f is a cubic mapping. Thus **Condition $\mathbf{P_2}$** holds. □

Using Lemma 3.4, we investigate solutions and the generalized Hyers-Ulam stability for (3.9).

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 2f(x+y) + 3f(x-y) - 5f(y-x) \\ &- 10(x) + 14f(y) - 2f(2y)\| \le \phi(x, y) \end{aligned} \tag{3.27}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} \|F_a(x) - f_a(x)\| \le \frac{1}{12} \sum_{n=0}^{\infty} 2^{-n} [&\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) \\ &+ 5\phi(2^n x, 0) + 7\phi(0, 2^n x)] \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \|F_c(x) - f_c(x)\| \le \frac{1}{48} \sum_{n=0}^{\infty} 2^{-3n} [&\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) \\ &+ 5\phi(2^n x, 0) + 7\phi(0, 2^n x)] \end{aligned} \tag{3.29}$$

for all $x \in X$.

Proof. Let $G_f(x, y) = 5[f(x-y) - f(y-x)] - 14[f(x) - f(y)] + 2[f(2x) - f(2y)]$. Then f satisfies (2.5) and

$$\begin{aligned} &G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) \\ &= -2[t(4x) - 10t(2x) + 16t(x)] - 5[f(x) + f(-x)] \end{aligned}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Since G_f satifies (3.10), by Lemma 3.4, **Condition $\mathbf{P_1}$** and **Condition $\mathbf{P_2}$** satisfy. By Theroem 2.2, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.28) and (3.29). □

Corollary 3.6. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (3.9) if and only if f is an additive-cubic mapping.*

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2**(1950), 64-66.

[2] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**(2003), 1-7.

[3] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74**(1968), 305-309.

[4] P. Găvruta, *A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184**(1994), 431-436.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27**(1941), 222-224.

[6] K. Jun, H. Kim and I. Chang, *On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation*, J. Comput. Anal. Appl., **7** (2005), 21-33 .

[7] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343**(2008), 567-572.

[8] A. Najati and G. Z. Eskandani, *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., **342**(2008), 1318-1331.

[9] S. Ostadbashi and J. Kazemzadeh, *Orthogonal stability of mixed type additive and cubic functional equation*, Int. J. Nonlinear. Anal. Appl., **6**(2015), 35-43

[10] C. Park, *Orthogonal Stability of an Additive-Quadratic Functional Equation*, Fixed Point Theory and Applications, **2011**(2011), 1-11.

[11] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[12] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Wiley, New York, 1960.

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