ADDITIVE-QUADRATIC FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND STABILITY

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Abstract. In this paper, we investigate the following functional inequality

$$N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)] - f(-x)-f(-y)-f(-z),t) \geq N(f(x+y+z),t)$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [11] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called *a fuzzy norm on X* if for all $x, y \in X$ and all $c, s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for all $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X, N) *and one denotes it by* $N - \lim_{n\to\infty} x_n = x$. *A sequence* $\{x_n\}$ *in* (X, N) *is said to be Cauchy if for any* $\epsilon > 0$, *there is an* $m \in N$ *such that for any* $n \geq m$ *and any positive integer* p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ *for all* $t > 0$. *A fuzzy normed space is said to be complete if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called a fuzzy Banach space.*

In 1940, Ulam proposed the following stability problem (cf.[21]*):*

"Let G_1 *be a group and* G_2 *a metric group with the metric* d. *Given a constant* $\delta > 0$, *does there exist a constant* $c > 0$ *such that if a mapping* $f : G_1 \longrightarrow G_2$ *satisfies* $d(f(xy), f(x)f(y)) < c$ *for all* $x, y \in G_1$, *then there exists an unique homomorphism* $h : G_1 \longrightarrow G_2$ *with* $d(f(x), h(x)) < \delta$ *for all* $x \in G_1$*?"*

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In the next year, Hyers [10] *gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki* [1] *for additive mappings, and by Rassias* [17] *for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias' theorem was obtained by Găvruta* [7] *by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. In 2008, for the first time, Mirmostafaee and Moslehian* [14], [15] *used the definition of a fuzzy norm in* [2] *to obtain a fuzzy version of stability for the Cauchy functional equation*

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

Glányi [8] *and Rätz* [18] *showed that if a mapping* $f : X \longrightarrow Y$ *satisfies the following functional inequality*

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

then f *satisfies the following Jordan-Von Neumann functional equation*

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

for an abelian group X *divisible by* 2 *into an inner product space* Y. *Glányi* [9] *and Fechner* [6] *proved the Hyers-Ulam stability of (1.3). The stability problems of several functional equations and inequalities have been extensively investigated by a number of authors and there are many interesting results concerning the stability of various functional equations and inequalities.*

Now, we consider the following fixed point theorem on generalized metric spaces.

Definition 1.2. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions:
(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.
In case, (X, d) is called *a generalized metric space.*

Theorem 1.3. [4] *Let* (X, d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *a strictly contractive mapping with some Lipschitz constant* L *with* $0 < L < 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers* n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

The following function equation $f : X \longrightarrow Y$ *is called the Drygas functional equation :*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

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for all $x, y \in X$. The Drygas functional equation has been studied by Szabo [20] *and Ebanks, Făiziev and Sahoo* [5]*. The solutions of the Drygas functional equation in abelian group are obtained by H. Stetkær in* [19].

In this paper, we investigate the following functional inequality which is related with the Drygas type functional equation

$$\begin{aligned} &N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)] \\ &-f(-x)-f(-y)-f(-z),t) \geq N(f(x+y+z),t) \end{aligned} \tag{1.4}$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and $(\mathbb{R}, N')$ is a fuzzy normed space.

2. Solutions and the stability for (1.4)

In this section, we investigate the functional equation (1.4) and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$f_o(x) = \frac{f(x)-f(-x)}{2}, \quad f_e(x) = \frac{f(x)+f(-x)}{2}.$$

In [12]*, the authors proved the following theorem:*

Lemma 2.1. [12] *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Then f is quadratic if and only if f satisfies the following functional equation*

$$f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y) = k[f(x+y)+f(x-y)-2f(x)-2f(y)]$$

for all $x, y \in X$, a fixed nonzero rational number a and fixed real numbers b, k with $a^2 \neq b^2$.

Using this, we have the following theorem:

Theorem 2.2. *If a mapping $f : X \longrightarrow Y$ saisfies (1.4), then f is an additive-quadratic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $x = y = z = 0$ in (1.4), by (N3), we have

$$N(f(0),t) \leq N(6f(0),t) = N\Big(f(0), \frac{t}{6}\Big)$$

for all $t > 0$ and by (N5), $N(f(0), \frac{t}{6}) \leq N(f(0), t)$ for all $t > 0$. Hence we have

$$N(f(0),t) = N(f(0),6t)$$

for all $t > 0$. By induction, we get

$$N(f(0),t) = N(f(0),6^n t)$$

for all $t > 0$ and all $n \in \mathbb{N}$. By (N5), we get

$$N(f(0),t) = \lim_{n\to\infty} N(f(0),6^n t) = 1$$

for all $t > 0$ and hence by (N2), $f(0) = 0$. Letting $z = -x-y$ in (1.4), we have

$$\begin{aligned} &N(f(x-y)+f(x+2y)+f(-2x-y)-2f(x)-2f(y)-2f(-x-y) \\ &-f(-x)-f(-y)-f(x+y),t) \geq N(0,t) = 1 \end{aligned}$$

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for all $x, y \in X$ and all $t > 0$ and so by (N2), we get

$$\begin{aligned} &f(x-y)+f(x+2y)+f(-2x-y) \\ &= 2f(x)+2f(y)+2f(-x-y)+f(x+y)+f(-x)+f(-y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$. By (2.1), we have

$$f_o(x-y)+f_o(x+2y)-f_o(2x+y) = -f_o(x+y)+f_o(x)+f_o(y) \tag{2.2}$$

for all $x, y \in X$ and interchanging x and y in (2.2), we have

$$-f_o(x-y)+f_o(2x+y)-f_o(x+2y) = -f_o(x+y)+f_o(x)+f_o(y) \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f_o(x+y) = f_o(x)+f_o(y)$$

for all $x, y \in X$ and hence f_o is an additive mapping. By (2.1), we have

$$f_e(x-y)+f_e(2x+y)+f_e(x+2y) = 3f_e(x+y)+3f_e(x)+3f_e(y) \tag{2.4}$$

for all $x, y \in X$ and letting $y = -y$ in (2.4), we have

$$f_e(x+y)+f_e(2x-y)+f_e(x-2y) = 3f_e(x-y)+3f_e(x)+3f_e(y) \tag{2.5}$$

for all $x, y \in X$. By (2.4) and (2.5), we have

$$\begin{aligned} &f_e(2x+y)+f_e(2x-y)+f_e(x+2y)+f_e(x-2y) \\ &= 2f_e(x+y)+2f_e(x-y)+6f_e(x)+6f_e(y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Letting $y = 0$ in (2.6), we get

$$f_e(2x) = 4f_e(x) \tag{2.7}$$

for all $x \in X$ and letting $y = 2y$ in (2.6), by (2.7), we have

$$\begin{aligned} &4f_e(x+y)+4f_e(x-y)+f_e(x+4y)+f_e(x-4y) \\ &= 2f_e(x+2y)+2f_e(x-2y)+6f_e(x)+24f_e(y) \end{aligned} \tag{2.8}$$

for all $x, y \in X$. By (2.6) and (2.8), we have

$$2f_e(2x+y)+2f_e(2x-y)+f_e(x+4y)+f_e(x-4y) = 18f_e(x)+36f_e(y) \tag{2.9}$$

for all $x, y \in X$. Letting $y = 2y$ in (2.9), by (2.7), we have

$$f_e(x+8y)+f_e(x-8y)+8f_e(x+y)+8f_e(x-y) = 18f_e(x)+144f_e(y)$$

for all $x, y \in X$. By Lemma 2.1, f_e is a quadratic mapping. Thus f is an additive-quadratic mapping. □

Now, we will prove the generalized Hyers-Ulam stability of (1.4) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, let

$$Df(x,y,z) = f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)]-f(-x)-f(-y)-f(-z).$$

Theorem 2.3. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2},\frac{y}{2},\frac{z}{2}\Big),t\Big) \geq N'\Big(\frac{L}{4}\phi(x,y,z),t\Big) \tag{2.10}$$

for all $x, y, z \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x,y,z),t) \geq \min\{N(f(x+y+z),t), N'(\phi(x,y,z),t)\} \tag{2.11}$$

for all $x, y, z \in X$ and all $t > 0$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{L}{4(1-L)}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \tag{2.12}$$

for all $x \in X$ and all $t > 0$. Further, we have

$$F(x) = N - \lim_{n\to\infty} \Big[\frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)\Big] \tag{2.13}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g, h) = \inf\{c \in [0, \infty)|\ N(g(x) - h(x), ct) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}, \\ \forall x \in X, \forall t > 0\}.$$

Then (S, d) is a complete metric space([16]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 3g(\frac{x}{2}) + g(-\frac{x}{2})$ for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (2.10), we have

$$\begin{aligned}
&N(Jg(x) - Jh(x), cLt) \\
&= N\Big(3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big) - 3h\Big(\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), cLt\Big) \\
&\geq \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g\Big(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\
&\geq \min\Big\{N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big), N'\Big(\phi\Big(-\frac{x}{2}, \frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big)\Big\} \\
&\geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = -x$ and $z = 0$ in (2.11), we get

$$N(f(2x) - 3f(x) - f(-x), t) \geq N'(\phi(x, -x, 0), t) \tag{2.14}$$

for all $x \in X$, $t > 0$ and hence

$$\begin{aligned}
&N\Big(f(x) - Jf(x), \frac{L}{4}t\Big) = N\Big(f(x) - 3f\Big(\frac{x}{2}\Big) - f\Big(-\frac{x}{2}\Big), \frac{L}{4}t\Big) \\
&\geq N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{L}{4}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}
\end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{L}{4} < \infty$. By Theorem 1.3, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, we have

$$J^n f(x) = \frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence we have (2.13).

Replacing x, y, z, and t by $\frac{x}{2^n}$, $\frac{y}{2^n}$, $\frac{z}{2^n}$, and $\frac{t}{2^{2n}}$ in (2.11), respectively, by (2.11), we have

$$\begin{aligned} &N\Big(Df_e\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\\ &\geq \min\Big\{N\Big(Df\Big(\frac{x}{2^n},\frac{y}{2^n},-\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big),N\Big(Df\Big(-\frac{x}{2^n},-\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\Big\}\\ &\geq \min\{N'(L^n\phi(x,y,z),t),N'(L^n\phi(-x,-y,z),t)\} \end{aligned} \tag{2.15}$$

for all $x, y, z \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.15), we have

$$DF_e(x,y,z) = 0$$

for all $x, y, z \in X$ and by Lemma 2.1, F_e is an quadratic mapping. Similarly, F_o is an additive mapping and thus F is an additive-quadratic mapping. Since $d(f, Jf) \leq \frac{L}{4}$, by Theorem 1.3, we have (2.12).

Now, we show the uniqueness of F. Let G be another additive-quadratic mapping with (2.12). Then clearly, G is a fixed point of J and

$$d(Jf,G) = d(Jf,JG) \leq Ld(f,G) \leq \frac{L^2}{4(1-L)} < \infty \tag{2.16}$$

and hence by (3) in Theorem 1.3, $F = G$. □

Similar to Theorem 2.3, we have the following threoem:

Theorem 2.4. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'(\phi(2x,2y,2z),t) \geq N'(2L\phi(x,y,z),t) \tag{2.17}$$

for all $x, y, z \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.11). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x)-F(x),\frac{1}{2(1-L)}t\Big) \geq \min\{N'(\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\} \tag{2.18}$$

for all $x \in X$ and all $t > 0$. Further, we have

$$F(x) = \lim_{n\to\infty}\Big[\frac{2^n+1}{2^{2n+1}}f(2^nx) - \frac{2^n-1}{2^{2n+1}}f(-2^nx)\Big]$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g,h) = \inf\{c\in[0,\infty)|\ &N(g(x)-h(x),ct) \geq \min\{N'(\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\},\\ &\forall x\in X,\forall t>0\}. \end{aligned}$$

Then (S, d) is a complete metric space([16]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$ for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0,\infty)$. Then by (2.10), we have

$$\begin{aligned} &N(Jg(x)-Jh(x),cLt)\\ &= N\Big(\frac{3}{8}g(2x)-\frac{1}{8}g(-2x)-\frac{3}{8}h(2x)+\frac{1}{8}h(-2x),cLt\Big)\\ &\geq \min\{N(g(2x)-h(2x),2cLt),N(g(-2x)-h(-2x),2cLt)\}\\ &\geq \min\{N'(\phi(2x,-2x,0),2Lt),N'(\phi(-2x,2x,0),2Lt)\}\\ &\geq \min\{N'\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\} \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g, h)$ for any $g, h \in S$ and since $0 < L < 1$, J is a strictly contractive mapping. By (2.14), we get

$$\begin{aligned} & N\Big(f(x) - Jf(x), \frac{t}{2}\Big) \\ & = N\Big(\frac{3}{8}[f(2x) - 3f(x) - f(-x)] - \frac{1}{8}[f(-2x) - 3f(-x) - f(x)], \frac{t}{2}\Big) \\ & \ge \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Jf) \le \frac{1}{2} < \infty$. The rest of proof the proof is similar to Theorem 2.3. □

As examples of $\phi(x, y, z)$ and $N'(x, t)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and

$$N'(x, t) = \begin{cases} \frac{t}{t+k|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \le 0 \end{cases}$$

for all $x \in \mathbb{R}$, $t > 0$, and for some $\epsilon > 0$, where $k = 1, 2$. Then we can formulate the following corollary:

Corollary 2.5. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(Df(x, y, z), t) \ge \min\Big\{N(f(x + y + z), t), \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|z\|^p)}\Big\}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \ge \begin{cases} \dfrac{(2^p - 4)t}{(2^p - 4)t + 2k\epsilon\|x\|^p}, & \text{if } 2 < p \\ \\ \dfrac{(2 - 2^p)t}{(2 - 2^p)t + 2k\epsilon\|x\|^p}, & \text{if } 0 < p < 1 \end{cases} \tag{2.19}$$

for all $x \in X$ and all $t > 0$.

For any $f : X \longrightarrow Y$, let

$$\begin{aligned} D_1 f(x, y) = & f(x - y) + f(x + 2y) + f(-2x - y) \\ & - 2f(x) - 2f(y) - 2f(-x - y) - f(-x) - f(-y) - f(x + y) \end{aligned} \tag{2.20}$$

Using Corollary 2.5, we have the following corollary:

Corollary 2.6. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(D_1 f(x, y), t) \ge \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|x + y\|^p)} \tag{2.21}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ with (2.19).

We remark that the functional inequality (1.4) is not stable for $p = 1$ in Corollary 2.6. The following example shows that the inequality (2.21) is not stable for $p = 1$.

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Example 2.7. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \leq -1 \\ 1, & \text{if } 1 \leq x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \Big[\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n}\Big]$$

We will show that f satisfies the following inequality

$$|D_1 f(x, y)| \leq 112(|x| + |y| + |x + y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$ and so f satisfies (2.21). But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \leq K|x| \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $t_o(x) = t(x)$, $s_o(x) = 0$, and $|f_o(x)| \leq 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \leq |x| + |y| + |x + y|$. Then $|D_1 f_o(x, y)| \leq 48(|x| + |y| + |x + y|)$. Now suppose that $\frac{1}{4} > |x| + |y| + |x + y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+3}} \leq |x| + |y| + |x + y| < \frac{1}{2^{m+2}}$$

and so

$$2^m|x| < \frac{1}{4}, \quad 2^m|y| < \frac{1}{4}, \quad 2^m|x + y| < \frac{1}{4}.$$

Hence we have

$$\{2^m x, 2^m y, 2^m(x - y), 2^m(x + y),\ 2^m(x + 2y),\ 2^m(2x + y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, m$,

$$|D_1 t_0(2^n x, 2^n y)| = 0,$$

because $t(x) = t_o(x) = x$ on $(-1, 1)$. Thus

$$\begin{aligned} |D_1 f_o(x, y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| \\ &\leq \Big|\sum_{n=0}^{m} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| + \Big|\sum_{n=m+1}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y)\Big| \\ &\leq \frac{12}{2^{m+1}} \leq 48(|x| + |y| + |x + y|), \end{aligned}$$

because $|D_1 t_0(2^n x, 2^n y)| \leq 6$.

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Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \leq \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \leq |x|+|y|+|x+y|$. Then $|D_1 f_e(x,y)| \leq 64(|x|+|y|+|x+y|)$. Now suppose that $\frac{1}{4} > |x|+|y|+|x+y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{2k+4}} \leq |x|+|y|+|x+y| < \frac{1}{2^{2k+2}}$$

and so

$$2^{2k}|x| < \frac{1}{4}, \quad 2^{2k}|y| < \frac{1}{4}, \quad 2^{2k}|x+y| < \frac{1}{4}.$$

Hence we have

$$\{2^k x, 2^k y, 2^k(x-y), 2^k(x+y),\ 2^k(x+2y),\ 2^k(2x+y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, k$,

$$|D_1 s_e(2^n x, 2^n y)| = 0.$$

Thus

$$\begin{aligned} |D_1 f_e(x,y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{2^n} D_1 s_e(2^n x, 2^n y)\Big| \\ &\leq \Big|\sum_{n=0}^{k} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| + \Big|\sum_{n=k+1}^{\infty} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| \\ &\leq \frac{16}{4^{k+1}} \leq 64(|x|+|y|+|x+y|), \end{aligned}$$

because $|D_1 s_e(2^n x, 2^n y)| \leq 12$. Hence we have

$$|D_1 f_o(x,y)| \leq 48(|x|+|y|+|x+y|),\ |D_1 f_e(x,y)| \leq 64(|x|+|y|+|x+y|)$$

for all $x, y \in X$ and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A+Q$ satisfies (2.23). Since $|f(x)| \leq \frac{10}{3}$, by (2.23), we have

$$-\frac{10}{3n^2} - K\frac{|x|}{n} \leq \frac{A(x)}{n} + Q(x) \leq \frac{10}{3n^2} + K\frac{|x|}{n}$$

for all $x \in X$ and all positive integers n and so $Q(x) = 0$ for all $x \in X$. Since A is additive,

$$-\frac{10}{3n} - K|x| \leq A(x) \leq \frac{10}{3n} + K|x|$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence $|A(x)| \leq K|x|$. By (2.23), we have

$$|f(x)| \leq 2K|x| \tag{2.24}$$

for all $x \in X$. Take a positive integer l such that $l > 2K$ and $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Since $x > 0$,

$$f(x) \geq \sum_{n=0}^{\infty} \frac{t(2^n x)}{2^n} \geq \sum_{n=0}^{l-1} \frac{t(2^n x)}{2^n} = lx > 2Kx$$

which contradicts to (2.24). □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach space*, J.Math.Soc.Japan **2** (1950), 64-66.

[2] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. **11**(2003), 687-705.

[3] S. C. Cheng and J. N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Cal. Math. Soc. **86**(1994), 429–436.

[4] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74**(1968), 305-309.

[5] B. R. Ebanks, PL. Kannappan, and P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner product spaces*, Canad. Math. Bull. **35(3)**(1992), 321-327.

[6] W. Fechner, *Stability of a functional inequality associated with the Jordan-Von Neumann functional equation*, Aequationes Mathematicae **71**(2006), 149-161.

[7] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436.

[8] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Mathematicae **62**(2001), 303-309.

[9] A. Gilányi, *On a problem by K. Nikoden*, Mathematical Inequalities and Applications **4**(2002), 707-710.

[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27**(1941), 222-224.

[11] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets Syst **12**(1984), 143-154.

[12] C. I. Kim, G. Han, and S. A. Shim, Hyers-Ulam Stability for a Class of Quadratic Functional Equations via a Typical Form, Abs. and Appl. Anal. **2013**(2013), 1-8.

[13] I. Kramosil and J. Michálek, Fuzzy metric and statistical metric spaces, Kybernetica **11**(1975), 336-344.

[14] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy almost quadratic functions, Results Math. **52**(2008), 161–177.

[15] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets Syst. **159**(2008), 720-729.

[16] M. S. Moslehian and Th. M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Anal. Discrete Math. **1**(2007), 325-334.

[17] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(1978), 297-300.

[18] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Mathematicae **66**(2003), 191-200.

[19] H. Stetkær, Functional equations on abelian groups with involution, II, Aequationes Math. **55**(1998), 227-240.

[20] G. Y. Szabó, *Some functional equations related to quadratic functions*, Glasnik Math. **38**(1983), 107-118.

[21] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960, Chapter VI.

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