Examining the Distinct Characteristics of *f-alpha* Flexible Q-Fuzzy Subgroups and *f-alpha* Flexible Normal Q-Fuzzy Subgroups

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Abstract

This paper introduces the concepts Q-fuzzy subsets and Q-fuzzy subgroups and establishes essential properties related to these two notions. This study aims to present and investigate new ideas of fuzzy subsets and fuzzy subsets that are $(f-\alpha)$-flexible. We define$(f-\alpha)$-Q-fuzzy subgroups, $(f-\alpha)$-flexible Q-fuzzy subgroups and $(f-\alpha)$-flexible normal Q-fuzzy subgroups based on these definitions. We investigate and analyze fundamental characteristics while incorporating new discoveries into the research on these topics.

Keywords: Fuzzy subset, $(f-\alpha)$-flexible fuzzy subset, $(f-\alpha)$-flexible Q-fuzzy subset, $(f-\alpha)$-flexible Q-fuzzy subgroup, $(f-\alpha)$-flexible Q-fuzzy normal subgroups.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965. Following this, Rosenfeld in 1971 advanced the idea of fuzzy subgroups, marking the initial fuzzification of algebraic structures. This development inspired many mathematicians to explore various fuzzy algebraic structures, including fuzzy ideals in rings and semi-rings. Zadeh further expanded the theory in 1975 with the introduction of interval-valued fuzzy sets, where the values represent ranges rather than precise points. This concept was subsequently generalized by Anthony and Sherwood in 1979, leading to their definition of fuzzy normal subgroups. Additionally, Mukherjee and Bhattacharya (1986) investigated fuzzy cosets and normal fuzzy groupings.

Murali and Makamba (2006) introduced the concept of fuzzy subgroups in abelian groups by analyzing the number of fuzzy subgroups in an abelian group of order $p^n q$ where p, n, q are positive integers. In contrast, Solairaju and Nagarajan (2011) explored higher Q-fuzzy orders and Q-fuzzy subgroups. This study builds on findings from the works of Sarangapani and Muruganantham (2016) and Geethalakshmi Manickam and Solairaju. The aim of this work is to introduce the concepts of $(f-\alpha)$-Q-fuzzy subsets and $(f-\alpha)$-Q-fuzzy subgroups and to establish some basic algebraic properties associated with these concepts. We introduce the concepts of $(f-\alpha)$-flexible Q-fuzzy groups and $(f-\alpha)$-flexible normal Q-fuzzy subgroups and explore some of their essential properties through discussion and derivation.

2. Preliminaries and Definitions

The fundamental definitions and results are now outlined within the framework of a $(f-\alpha)$-flexible Q-fuzzy subgroups.

Definition 2.1: Let M_α be a set. A mapping μ_α: $M_\alpha \rightarrow [0,1]$ is referred to as a fuzzy subset of M_α.

Definition 2.2: Let M_α be any subgroup. A mapping $\Omega_\alpha^f : M_\alpha \rightarrow [0,1]$ is considered a $(f-\alpha)$-fuzzy subgroup on M_α if it satisfies the following conditions:

i) $\Omega_\alpha^f (m_\alpha n_\alpha) \leq \max\{\Omega_\alpha^f (m_\alpha), \Omega_\alpha^f (n_\alpha)\}$

ii) $\Omega_\alpha^f (m_\alpha^{-1}) \leq \Omega_\alpha^f (m_\alpha)$ for all $m_\alpha, n_\alpha \in M_\alpha$ and for any α in M_α .

Definition 2.3: For any Q-fuzzy subset Z_α in M_α and k ∈ [0,1], the set

$U(Z_\alpha, \text{k}) = \{ m_\alpha \in M_\alpha \mid Z_\alpha(m_\alpha, q_{rl}) \geq \text{k}$ for all $q_{rl} \in$ Q} is called the cut-set of Z_α at k.

Note 2.4: We use Q = $\{q_{rl}\}$ throughout this article

Definition 2.5: A $(f-\alpha)$-Q-fuzzy subset Z_α^f is termed as $(f-\alpha)$-Q-fuzzy subgroup of M_α if it satisfies the following conditions:

($(f-\alpha)$- QFG1): $Z_\alpha^f (m_\alpha\ n_\alpha, q_{rl}) \geq \min\{Z_\alpha^f (m_\alpha, q_{rl}), Z_\alpha^f (n_\alpha, q_{rl})\}$

($(f-\alpha)$-QFG2): $Z_\alpha^f (m_\alpha^{-1}, q_{rl}) = Z_\alpha^f (m_\alpha, q_{rl})$

($(f-\alpha)$-QFG3): $Z_\alpha^f (m_\alpha, q_{rl}) = 1$ for all m_α , $n_\alpha \in M_\alpha$ and $q_{rl} \in$ Q.

Definition 2.6: If Ω_α^f is a $(f-\alpha)$-fuzzy subgroup of a subgroup M_α with identity e_α, then

i) $\Omega_\alpha^f (m_\alpha^{-1}, q_{rl}) = \Omega_\alpha^f (m_\alpha , q_{rl})$
ii) $\Omega_\alpha^f (e_\alpha, q_{rl}) \leq \Omega_\alpha^f(m_\alpha, q_{rl})$ for all $m_\alpha \in M_\alpha$.

Definition 2.7: Let Ω_α^f be a $(f-\alpha)$-Q-fuzzy subgroup of M_α . Then Ω_α^f is called a $(f-\alpha)$-

normal Q-fuzzy subgroup of M_α if Ω_α^f (m_α n_α , q_{rl}) = Ω_α^f (n_α m_α , q_{rl}) for all $m_\alpha , n_\alpha \in M_\alpha$.

Definition 2.8: A $T - nrm$ is a function that maps pairs of numbers from the unit interval to the unit interval, defined as $T - nrm$:[0,1] × [0,1] → [0,1]. The following four conditions are satisfied for all $\gamma_{1,}\gamma_{2,}\gamma_{3,}$and γ_4∈[0,1].

i. $T - nrm$ ($\gamma_{1,}$, γ_2) = $T - nrm$ (γ_2, $\gamma_{1,}$)
ii. $T - nrm$ ($\gamma_{1,}$, $T - nrm$ (γ_2, γ_3)) = $T - nrm$ ($T - nrm$ ($\gamma_{1,}$, γ_2), γ_3)
iii. $T - nrm$ ($\gamma_{1,}$, 1) = $T - nrm$ (1, γ_1) = 1
iv. If $\gamma_{1,} \le \gamma_3$and $\gamma_2 \le \gamma_4$,then $T - nrm$ ($\gamma_{1,}$, γ_2) ≤ $T - nrm$ (γ_3, γ_4).

Note: The $T - nrm$ is a minimum-based norm.

Definition 2.9: Let M_α be a set. A mapping Ω_α^f: M_α×Q→Q^{sub*}([0,1]) is referred to as a $(f - \alpha)$ - flexible Q-fuzzy subset of M_α, where Q^{sub*}([0,1]) denotes the collection of all non-empty subsets of the interval [0,1].

Definition 2.10: Let M_α be a non-empty set and let Ω_α^f and δ_α^f be two $(f - \alpha)$ flexible Q-fuzzy subsets of M_α. The intersection of Ω_α^f and δ_α^f denoted by $\Omega_\alpha^f \cap \delta_\alpha^f$ and is defined by
$\Omega_\alpha^f \cap \delta_\alpha^f$ = {min {m_α, n_α }/m_α □ Ω_α^f (M_α), n_α □□δ_α^f□(M_α)} for all $m_\alpha \in M_\alpha$.
Similarly, the union of Ω_α^f and δ_α^f is denoted by (Ω_α^f □δ_α^f) is defined by

$\Omega_\alpha^f \cup \delta_\alpha^f$= {max {$m_\alpha$, n_α }/m_α □ Ω_α^f (M_α), n_α □□δ_α^f□(M_α)} for all $m_\alpha \in M_\alpha$.

Definition 2.10: Let M_α be a non-empty set. Let Ω_α^f and δ_α^f be two Q-fuzzy subsets ofM_α.Then for all $m_\alpha \in M_\alpha$ and q_{rl}□ Q, the following statements hold true:

i) $\Omega_\alpha^f \subseteq \delta_\alpha^f \Leftrightarrow \Omega_\alpha^f (m_\alpha, q_{rl}) \le \delta_\alpha^f (m_\alpha, q_{rl})$
ii) $\Omega_\alpha^f = \delta_\alpha^f \Leftrightarrow \Omega_\alpha^f (m_\alpha, q_{rl}) = \delta_\alpha^f (m_\alpha, q_{rl})$

Definition 2.11:
If Ω_α is a $(f - \alpha)$- Q fuzzy subgroup of G_r, then Com(Ω_α^f) represents the complement of a $(f - \alpha)$-Q-fuzzy subgroup of Ω_α^f and is defined as Com{$\Omega_\alpha^f(m_\alpha, q_{rl})$}=1-{$\Omega_\alpha^f(m_\alpha, q_{rl})$}, for all $m_\alpha \in M_\alpha$ and q_{rl}□ Q.

Definition 2.12: Let M_α be a groupoid, meaning it is a set closed under a binary operation (multiplication). A mapping is termed a $(f - \alpha)$-Q-fuzzy groupoid if it satisfies the following conditions:

(i) inf Ω_α^f (m_α n_α , q_{rl}) ≥ $T - nrm$□{(inf Ω^f (m_α , q_{rl}), inf Ω^f (n_α , q_{rl})),α}

(ii) sup Ω_α^f (m_α n_α , q_{rl}) ≥ $T - nrm$□□{(sup Ω^f (m_α , q_{rl}), sup Ω^f (n_α , q_{rl})), α}

for all $m_\alpha , n_\alpha \in M_\alpha$ and q_{rl}□ Q.

Definition 2.12: Let M_α be a group. A mapping Ω_α^f: $M_\alpha \times Q \to Q^{sub*}$([0,1]) is called a $(f-\alpha)$-flexible Q-fuzzy subgroup on M_α if it satisfies the following conditions:

(i) inf Ω_α^f $(m_\alpha n_\alpha, q_{rl}) \geq T-nrm$ { (inf Ω^f (m_α, q_{rl}), inf $\Omega^f(n_\alpha, q_{rl})$), α }

(ii) sup Ω_α^f $(m_\alpha n_\alpha, q_{rl}) \geq T-nrm$ { (sup Ω^f (m_α, q_{rl}), sup Ω_α^f (n_α, q_{rl})), α}

(iii) inf Ω_α^f $(m_\alpha^{-1}, q_{rl}) \geq$ inf Ω_α^f (m_α, q_{rl})

(iv) sup Ω_α^f $(m_\alpha^{-1}, q_{rl}) \geq$ sup Ω_α^f (m_α, q_{rl}) for all $m_\alpha, n_\alpha \in M_\alpha$ and $q_{rl} \in$ Q.

Example 2.13: Let G_r = {e_α,p_α,z_α,r_α} be a Klein's four-group. The multiplication operation in the group G_r is defined as follows:

•	e_α	p_α	z_α	r_α
e_α	e_α	e_α	e_α	e_α
p_α	p_α	p_α	p_α	p_α
z_α	e_α	e_α	e_α	z_α
r_α	p_α	p_α	p_α	e_α

Then (G_r ,·) is a group. Define a flexible fuzzy subset μ_α: $G_r \to Q^{sub*}$([0,1]) as follows:

μ_α(e) = 0.75, μ_α (p) = 0.25, μ_α (z) = 0.0.25, μ_α (r) = 0.75. With these assignments, μ_α qualifies as a flexible fuzzy subgroup of G_r.

Note 2.14: In definition *, if Ω_α^f: $M_\alpha \times Q \to Q^{sub*}$([0,1]) , then Ω_α^f (m_α, q_{rl}) for all $m \in M_\alpha$ are real values in [0,1] and it holds that inf (Ω_α^f (m_α, q_{rl})) = sup(Ω_α^f (m_α, q_{rl}))= Ω_α^f (m_α, q_{rl}) for all $m \in M_\alpha$ and $q_{rl}\in$Q. Consequently, definition * reduces to Rosenfeld's definition of a fuzzy subgroup. Therefore, $(f-\alpha)$-flexible Q-fuzzy subgroup is a generalization of Rosenfeld's fuzzy group.

3. Properties of $(f-\alpha)$-Flexible Q-Fuzzy Subgroups

Definition 3.1: Let G_r and Q be two non-empty subsets and for any $\alpha \in [0,1]$.Then , a mapping

Ω_α^f: $G_r \times Q \rightarrow [0,1]$ is called a $(f-\alpha)$-flexible Q-fuzzy set of G_r , with respect to the fuzzy subset Ω_α^f, if Ω_α^f (m_α, q_{rl}) = $T-nrm$ { Ω^f (m_α, q_{rl}),α } for $m_\alpha \in G_r$ and $q_{rl} \in$ Q.

Remark 3.2: Obviously if $\alpha = 1$, then Ω_α^f $(m_\alpha, q_{rl}) = (\Omega^f)^\alpha(m_\alpha, q_{rl}) = (\Omega^f)^1(m_\alpha, q_{rl}) = \Omega^f$ (m_α, q_{rl}) and if $\alpha = 0$, then Ω_α^f $(m_\alpha, q_{rl}) = (\Omega^f)^\alpha(m_\alpha, q_{rl}) = (\Omega^f)^0(m_\alpha, q_{rl}) = 0$.

Theorem 3.3: Every $(f-\alpha)$-Q- fuzzy subgroup of a group M_α is $(f-\alpha)$-Q fuzzy subgroup of M_α.

Proof: Suppose Ω_α^f be a $(f-\alpha)$- Q- fuzzy subgroup of a group M_α.

Let m_α, n_α be any two elements in a subgroup M_α.

Then, Ω_α^f (m_α, q_{rl}) = $T-nrm$ { Ω^f $(m_\alpha n_\alpha, q_{rl})$,$\alpha$ }

$\geq T-nrm$ { $T-nrm\{\Omega^f(m_\alpha, q_{rl}), \Omega^f(n_\alpha, q_{rl})\}$, α }

= $T-nrm$ { $T-nrm\{\Omega^f(m_\alpha, q_{rl})$,$\alpha$ }, $T-nrm$ $\{\Omega^f(m_\alpha, q_{rl})\}$, α }}

= $T-nrm\{(\Omega_\alpha^f(m_\alpha, q_{rl}), \Omega_\alpha^f(m_\alpha, q_{rl}))^\alpha\}$, α }

$\Omega_\alpha^f(m_\alpha, q_{rl}) \geq T-nrm\{(\Omega_\alpha^f(m_\alpha, q_{rl}),\Omega_\alpha^f(n_\alpha, q_{rl}))$, α }and

Ω_α^f (m_α^{-1}, q_{rl})= $T-nrm$\{ Ω^f (m_α^{-1}, q_{rl}),α }

$\geq T-nrm$ { Ω^f (m_α, q_{rl}),α }

=Ω_α^f (m_α, q_{rl})

Therefore $\Omega_\alpha^f(m_\alpha^{-1}, q_{rl}) \geq \Omega_\alpha^f$ (m_α, q_{rl}). Thus, Ω_α^f is a $(f-\alpha)$- Q fuzzy subgroup of M_α.

Remark 3.4: A $(f-\alpha)$- Q- fuzzy subgroup of M_α is not necessarily a Q-fuzzy subgroup of a group M_α.

Example 3.5: Consider the four Klein's group $G_r = \{p_\alpha, k_\alpha, r_\alpha, e_\alpha\}$ where $p_\alpha k_\alpha = q_\alpha k_\alpha = r_\alpha$ and $p_\alpha^2 = k_\alpha^2 = r_\alpha^2 = e_\alpha$ and a non-empty set $Q = \{q_{rl}\}$.

A Q-fuzzy subset Ω_α^f of a group G_r is defined as

$$(m_\alpha, q_{rl}) = \begin{cases} 0.05 & if\ m_\alpha = e_\alpha \\ 0.08 & if\ m_\alpha = p_\alpha\ or\ m_\alpha = k_\alpha \\ 0.06 & if\ m_\alpha = r_\alpha \end{cases}$$

It is clear that the Q- fuzzy subset Ω_α^f is not a Q-fuzzy subgroup of a group G_r since first part of definition fails to hold as Ω_α^f $(r_\alpha, q_{rl}) < T-nrm\{(\Omega^f(p_\alpha, q_{rl}), \Omega^f(k_\alpha, q_{rl}))$, α }

However, it can be demonstrated that $\Omega_\alpha^f$is a $(f-\alpha)$ - Q-fuzzy subgroups of a group G_r. If we set $\alpha = 0.03$, we find that Ω_α^f $(r_\alpha, q_{rl}) > \alpha$ for all elements in the group G_r and for all q_{rl} in Q. This indicates that , Ω_α^f $(m_\alpha, q_{rl}) = T-nrm$ { Ω^f $(m_\alpha,$

$q_{rl}),\alpha\}=\alpha$, for all $m_\alpha \in G_r$ and q_{rl} □ Q. Which fulfills the first condition of the definition of $(f-\alpha)$-Q-fuzzy subgroups:

$\Omega^f_\alpha\ (m_\alpha n_\alpha, q_{rl}) \geq T-nrm\ \{(\Omega^f\ (m_\alpha, q_{rl}),\ \Omega^f(n_\alpha, q_{rl})), \alpha\}$

Furthermore, for the second condition of the definition, since $p^{-1}_\alpha = p_\alpha, k^{-1}_\alpha = k_\alpha\ and\ r^{-1}_\alpha = r_\alpha$ it follows that $\Omega^f_\alpha(m^{-1}_\alpha, q_{rl}) \geq\ \Omega^f_\alpha\ (m_\alpha, q_{rl})$.

Therefore , Ω^f_α is an $(f-\alpha)$-Q-fuzzy subgroup of the group G_r, denoted as Ω^f_α.

Theorem 3.6: A $(f-\alpha)$-flexible Q-fuzzy subset Ω^f_α of a group M_α is considered a $(f-\alpha)$-flexible Q-fuzzy subgroup if and only if the following conditions are met.

(i) $\inf \Omega^f_\alpha\ (m_\alpha n^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\inf \Omega^f\ (m_\alpha, q_{rl}),\ \inf \Omega^f\ (n_\alpha, q_{rl})), \alpha\ \}$

(ii) $\sup \Omega^f_\alpha\ (m_\alpha n^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\sup \Omega^f\ (m_\alpha, q_{rl}), \sup \Omega^f(n_\alpha, q_{rl})), \alpha\ \}$
for all $m_\alpha, n_\alpha \in M_\alpha$ and q_{rl} □ Q.

Proof: Initially, let Ω^f_α be a $(f-\alpha)$-flexible Q-fuzzy subgroup of M_α, and let m_α and n_α be elements of M_α. Then

$\inf \Omega^f_\alpha(m_\alpha n^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\inf\ \Omega^f\ (m_\alpha, q_{rl}), \inf\ \Omega^f\ (n^{-1}_\alpha, q_{rl})), \alpha\ \}$
$= T-nrm\ \{(\inf\ \Omega^f\ (m_\alpha, q_{rl}), \inf \Omega^f(n_\alpha, q_{rl})), \alpha\}$ and

$\sup \Omega^f_\alpha\ (m_\alpha n^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\sup\ \Omega^f\ (m_\alpha, q_{rl}), \sup \Omega^f(n_\alpha, q_{rl})), \alpha\}$
$= T-nrm\ \{(\sup\ \Omega^f\ (m_\alpha, q_{rl}) \sup\ \Omega^f(n_\alpha, q_{rl})), \alpha\}$.

Conversely, if Ω^f_α is a $(f-\alpha)$-flexible Q-fuzzy subset of M_α and the given conditions are satisfied, then it follows that

$\inf \Omega^f_\alpha\ (e_\alpha, q_{rl}) = \inf \Omega^f_\alpha\ (e_\alpha m^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\ \inf \Omega^f\ (e_\alpha, q_{rl}), \inf \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$
$= \inf \Omega^f\ (m_\alpha, q_{rl})$---------- (1)

$\sup \Omega^f_\alpha\ (e_\alpha, q_{rl}) = \sup \Omega^f_\alpha\ (e_\alpha m^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{\ (\ \sup \Omega^f\ (e_\alpha, q_{rl}), \sup \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$
$= \sup \Omega^f\ (m_\alpha, q_{rl})$----------- (2) for all $m_\alpha \in M_\alpha$.

This implies that

$\inf \Omega^f_\alpha\ (m^{-1}_\alpha, q_{rl}) = \inf \Omega^f_\alpha\ (e_\alpha m^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\ \inf \Omega^f\ (e_\alpha, q_{rl}), \inf \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$ by using (1)

and $\sup \Omega^f_\alpha\ (m^{-1}_\alpha, q_{rl}) = \sup \Omega^f_\alpha\ (e_\alpha m^{-1}_\alpha, q_{rl}) \geq T-nrm\ \{(\ \sup \Omega^f\ (e_\alpha, q_{rl}), \sup \Omega^f\ (m_\alpha, q_{rl})), \alpha\}$ by using (2).

Once again, we even prove that

$\inf \Omega^f_\alpha\ (m_\alpha n_\alpha, q_{rl}) \geq T-nrm\ \{(\inf \Omega^f\ (m_\alpha, q_{rl}), \inf \Omega^f\ (n^{-1}_\alpha, q_{rl})), \alpha\}$

$\geq T-nrm\ \{(\inf \Omega^f\ (m_\alpha, q_{rl}), \inf \Omega^f\ (n_\alpha, q_{rl})), \alpha\}$.

and $\sup \Omega^f_\alpha\ (m_\alpha n_\alpha, q_{rl}) \geq T-nrm\ \{(\sup \Omega^f\ (m_\alpha, q_{rl}), \sup \Omega^f\ (n^{-1}_\alpha, q_{rl})), \alpha\}$

$$\geq T - nrm\ \{(\sup \Omega^{f}\ (m_{\alpha}, q_{rl}), \sup \Omega^{f}\ (n_{\alpha}, q_{rl})), \alpha\}.$$

Therefore, Ω_{α}^{f} is a $(f - \alpha)$-flexible Q-fuzzy group of M_{α} .

Theorem 3.8: If Ω_{α}^{f} is a $(f - \alpha)$-flexible Q-fuzzy groupoid of an infinite group M_{α}, then Ω_{α}^{f} is a $(f - \alpha)$-flexible Q-fuzzy group of M_{α}.

Proof: Let $m \in M_{\alpha}$. Since M_{α} is finite, m_{α} has a finite order, say p. Therefore,$m_{\alpha}^{p} = e_{\alpha}$, where e_{α} is the identity element of M_{α}.

Thus, $m_{\alpha}^{-1}$is equivalent to m_{α}^{p-1} in the context of $(f - \alpha)$-flexible Q-fuzzy groupoids.

Thus, it follows that inf Ω_{α}^{f} $(m_{\alpha}^{-1}, q_{rl})$ = inf Ω_{α}^{f} $(m_{\alpha}^{p-1}, q_{rl})$ = inf Ω_{α}^{f} $(m_{\alpha}^{p-2}, q_{rl})$

$$\geq T - nrm\ \{(\inf \Omega^{f}\ (m_{\alpha}^{p-2}, q_{rl}), \Omega^{f}\ (m_{\alpha}, q_{rl})), \alpha\}.$$

Once again, we even prove that

$$\inf \Omega_{\alpha}^{f}\ (m_{\alpha}^{p-2}, q_{rl}) = \inf \Omega_{\alpha}^{f}\ (m_{\alpha}^{p-3}, q_{rl})$$

$$\geq T - nrm\ \{(\inf \Omega^{f}\ (m_{\alpha}^{p-3}, q_{rl}), \Omega^{f}\ (m_{\alpha}, q_{rl})), \alpha\}.$$

Consequently, it can be deduced that

$$\inf \Omega_{\alpha}^{f}\ (m_{\alpha}^{-1}, q_{rl}) \geq T - nrm\ \{(\inf \Omega^{f}\ (m_{\alpha}^{p-3}, q_{rl}), \Omega^{f}\ (m_{\alpha}, q_{rl})), \alpha\}.$$

By repeatedly applying the definition of a $(f - \alpha)$-flexible Q-fuzzy groupoid, we obtain:

$$\inf \Omega_{\alpha}^{f}\ (m_{\alpha}^{-1}, q_{rl})\ \square\ \inf \Omega_{\alpha}^{f}\ (m_{\alpha}, q_{rl}).$$

Similarly, we have, sup Ω_{α}^{f} $(m_{\alpha}^{-1}, q_{rl})$ □ sup Ω_{α}^{f} (m_{α}, q_{rl}).

Thus, Ω_{α}^{f} is a $(f - \alpha)$-flexible Q-fuzzy subgroup of M_{α}.

Theorem 3.8: The intersection of any two$(f - \alpha)$-flexible Q-fuzzy subgroups is also a $(f - \alpha)$-flexible Q-fuzzy subgroup of M_{α}.

Proof: Let Z_{α}^{f} and Y_{α}^{f} be two $(f - \alpha)$-flexible Q-fuzzy subgroups of M_{α}. Consider the elements m_{α} and n_{α} are of M_{α}. Then

inf $(Z_{\alpha}^{f}\ \square\ Y_{\alpha}^{f})$ $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})$= $T - nrm$ {(inf Z^{f} $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})$, inf Y^{f} $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})), \alpha$ }

$\geq$ $T - nrm\{$ $T - nrm$ {(inf $Z^{f}(m_{\alpha}, q_{rl})$, inf $Z^{f}(m_{\alpha}, q_{rl})), \alpha\}$, {(inf Y^{f} (n_{α}, q_{rl}), inf $Y^{f}(n_{\alpha}, q_{rl})), \alpha\}\}\}$

= $T - nrm$ { $(T - nrm$ {inf Z^{f} (m_{α}, q_{rl}), inf Y^{f} $(n_{\alpha}, q_{rl})), \alpha\}$, $T - nrm$ {(inf Z^{f} (m_{α}, q_{rl}), inf Y^{f} $(n_{\alpha}, q_{rl})), \alpha\}\}\}$

= $T - nrm$ {(inf Z^{f} □ Y^{f} (m_{α}, q_{rl}), inf Z^{f} □ Y^{f} $(n_{\alpha}, q_{rl})), \alpha\}$ ….. (1).

We also establish that

sup $(Z_{\alpha}^{f}\ \square\ Y_{\alpha}^{f})$ $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})$ = $T - nrm$ {(sup Z^{f} $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})$, sup Y^{f} $(m_{\alpha} n_{\alpha}^{-1}, q_{rl})), \alpha\}$ by definition

$\geq T - nrm$ { $T - nrm$ {{(sup Z^{f} (m_{α}, q_{rl}) , sup Z^{f} (m_{α}, q_{rl})), α }, {(sup Y^{f} (m_{α}, q_{rl}),sup Y^{f} (n_{α}, q_{rl})), α }}}

= $T - nrm$ { $T - nrm$ {(sup Z^{f} (m_{α}, q_{rl}), sup Y^{f} (m_{α}, q_{rl})), α }, $T - nrm$ {(sup $Z^{f}(m_{\alpha}$,

q_{rl}), sup Y^f $(n_\alpha, q_{rl}), \alpha\}\}\}$

= $T - nrm$ {(sup Z^f □ Y^f (m_α, q_{rl}), sup Z^f □ Y^f (n_α, q_{rl})), α }…… (2).

Based on results (1) and (2), it follows that the intersection Z_α^f □ Y_α^f is a $(f - \alpha)$-flexible Q-fuzzy subgroup of M_α.

Remark 3.9: If Ω_α^f and δ_α^f be two $(f - \alpha)$-Q fuzzy subgroup of the group G_r, then $\Omega_\alpha^f \cup \delta_\alpha^f$ need not to be a $(f - \alpha)$-flexible Q-fuzzy subgroup of a group G_r.

Theorem 3.10: The intersection of any arbitrary collection of $(f - \alpha)$-flexible Q-fuzzy subgroups is also a $(f - \alpha)$-flexible Q-fuzzy subgroups of M_α.

Proof: This is straightforward.

Theorem 3.11: If Z_α^f is a $(f - \alpha)$-flexible Q-fuzzy subgroup of a group M_α with identity element e_α, then

(i) inf Z_α^f (m_α^{-1}, q_{rl}) = inf Z_α^f (m_α, q_{rl}), and sup Z_α^f (m_α^{-1}, q_{rl}) = sup Z_α^f (m_α, q_{rl})

(ii) inf Z_α^f (e_α, q_{rl})= inf $Z_\alpha^f(m_\alpha, q_{rl})$ and sup Z_α^f (e_α, q_{rl})= sup $Z_\alpha^f(m_\alpha, q_{rl})$

for all $m \in M_\alpha$.

Proof: (i) Since Z_α^f is a $(f - \alpha)$ −flexible Q-fuzzy subgroup of the group M_α,
it follows that inf Z_α^f (m_α^{-1}, q_{rl}) □ inf Z_α^f (m_α, q_{rl}).

In the same way, it can be established that

inf Z_α^f (m_α, q_{rl}) = inf Z_α^f $((m_\alpha^{-1})^{-1}, q_{rl})$

□ inf Z_α^f (m_α^{-1}, q_{rl}).

Thus, inf Z_α^f (m_α^{-1}, q_{rl}) = inf Z_α^f (m_α, q_{rl}) .

Similarly, it can be shown that sup Z_α^f (m_α^{-1}, q_{rl}) = sup Z_α^f (m_α, q_{rl})

(ii) Given that Z_α^f is a $(f - \alpha)$-flexible Q-fuzzy subgroup of the group M_α, it follows that

inf Z_α^f (e_α, q_{rl}) = inf Z_α^f $(m_\alpha n_\alpha^{-1}, q_{rl})$ $\geq T - nrm$ {(inf Z^f (m_α, q_{rl}), inf Z^f (m_α^{-1}, q_{rl})), α }and
sup Z_α^f (e_α, q_{rl}) = sup Z_α^f $(m_\alpha n_\alpha^{-1}, q_{rl})$ $\geq T - nrm$ {(sup Z^f (m_α, q_{rl}), sup Z^f (m_α^{-1}, q_{rl})), α }.

Theorem 3.12: Let Ω_α^f and δ_α^f be two $(f - \alpha)$-flexible Q-fuzzy subgroups of M_{α_1} and M_{α_2} respectively and let f be a homomorphism from M_{α_1} to M_{α_2} . Then:

- h$(\Omega_\alpha^f, q_{rl})$ is a $(f - \alpha)$-flexible Q-fuzzy group of M_{α_2}.
- h$(\delta_\alpha^f, q_{rl})$ is a $(f - \alpha)$-flexible Q-fuzzy group of M_{α_1}.

Proof: This is straightforward.

Remark 3.13: If Ω_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy group of M_{α} and S_{gr} is a subgroup of M_{α} , then the restriction of Ω_{α}^{f} to S_{gr} (denoted $\Omega_{\alpha}^{f}/S_{gr}$) is a$(f-\alpha)$-flexible Q-fuzzy subgroup of S_{gr}.

4. Normal $(f-\alpha)$-Flexible Q-Fuzzy Subgroups

Definition 4.1: If Ω_{α}^{f} is a $(f-\alpha)$-flexible Q-fuzzy group of a group M_{α}, then Ω_{α}^{f} is referred to as a normal $(f-\alpha)$-flexible Q-fuzzy subgroup of M_{α} if

$$\inf \Omega_{\alpha}^{f}\,(m_{\alpha}n_{\alpha}) = \inf \Omega_{\alpha}^{f}\,(n_{\alpha}m_{\alpha}) \text{ and}$$

$$\sup \Omega_{\alpha}^{f}\,(m_{\alpha}n_{\alpha}) = \sup \Omega_{\alpha}^{f}\,(n_{\alpha}m_{\alpha})\} \text{ for all } m_{\alpha}, n_{\alpha} \in M_{\alpha}.$$

Theorem 4.2: A normal $(f-\alpha)$-flexible Q-fuzzy subgroup of a group M_{α} can be expressed as the intersection of any two normal $(f-\alpha)$-flexible Q-fuzzy subgroups of M_{α}.

Proof: Let Z_{α}^{f} and Y_{α}^{f} be two normal $(f-\alpha)$-flexible Q-fuzzy subgroups of a group M_{α}. We need to show that $Z_{\alpha}^{f} \cap Y_{\alpha}^{f}$ is also a normal $(f-\alpha)$-flexible Q-fuzzy group of M_{α}.

To do this, consider any elements $m_{\alpha}, n_{\alpha} \in M_{\alpha}$. By definition,

$$\inf (Z_{\alpha}^{f} \cap Y_{\alpha}^{f})\,(m_{\alpha}n_{\alpha}, q_{rl}) = T-nrm\,\{(\inf Z^{f}\,(m_{\alpha}n_{\alpha}, q_{rl}), \inf Y^{f}\,(m_{\alpha}n_{\alpha}, q_{rl})), \alpha\}$$

$$= T-nrm\,\{(\inf Z^{f}\,(n_{\alpha}m_{\alpha}, q_{rl}), \inf Y^{f}\,(n_{\alpha}m_{\alpha}, q_{rl})), \alpha\}$$

$$= \inf Z_{\alpha}^{f} \cap Y_{\alpha}^{f}\,(m_{\alpha}n_{\alpha}, q_{rl}).$$

In a similar manner, $\sup (Z_{\alpha}^{f} \cap Y_{\alpha}^{f})\,(m_{\alpha}n_{\alpha}, q_{rl}) = \sup (Z_{\alpha}^{f} \cap Y_{\alpha}^{f})\,(n_{\alpha}m_{\alpha}, q_{rl})$.

This demonstrates that $Z_{\alpha}^{f} \cap Y_{\alpha}^{f}$ is a normal $(f-\alpha)$-flexible Q-fuzzy group of M_{α}

Theorem 4.3: A normal $(f-\alpha)$-flexible Q-fuzzy subgroup of a group M_{α} can be expressed as the intersection of any arbitrary collection of normal $(f-\alpha)$-flexible Q-fuzzy subgroups of M_{α}.

Proof: Let $m_\alpha, n_\alpha \in M_\alpha$ and $\alpha \in M_\alpha$. Now it finds that

$$\inf Z^f_\alpha (m_\alpha n_\alpha^{-1}, q_{rl}) = \inf Z^f_\alpha (\alpha^{-1} m_\alpha n_\alpha^{-1} \alpha, q_{rl}) \text{ by definition}$$

$$= \inf Z^f_\alpha (\alpha^{-1} m_\alpha \alpha\alpha^{-1} n_\alpha^{-1} \alpha, q_{rl})$$

$$= \inf (Z^f_\alpha (\alpha^{-1} m_\alpha \alpha, q_{rl}), Z^f_\alpha ((\alpha^{-1} n_\alpha \alpha)^{-1}, q_{rl}))$$

$$\geq T - nrm \{(\inf (Z^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), \inf Z^f (\alpha^{-1} n_\alpha \alpha, q_{rl})), \alpha\}$$

$$= T - nrm \{(\inf (Z^f (m_\alpha, q_{rl}), Z^f (n_\alpha, q_{rl})), \alpha\}.$$

Again $\sup Z^f_\alpha (m_\alpha n_\alpha^{-1}, q_{rl}) = \sup Z^f_\alpha (\alpha^{-1} m_\alpha n_\alpha^{-1} \alpha, q_{rl})$ by definition

$$= \sup (Z^f_\alpha (\alpha^{-1} m_\alpha \alpha\alpha^{-1} n_\alpha^{-1} \alpha, q_{rl})$$

$$= \sup (Z^f_\alpha (\alpha^{-1} m_\alpha \alpha, q_{rl}), Z^f_\alpha ((\alpha^{-1} n_\alpha \alpha)^{-1}, q_{rl}))$$

$$\geq T - nrm \{(\sup (Z^f (\alpha^{-1} m_\alpha \alpha, q_{rl}), \sup (Z^f (\alpha^{-1} n_\alpha \alpha), q_{rl})), \alpha\}$$

$$= T - nrm \{(\sup (Z^f (m_\alpha, q_{rl}), Z^f (m_\alpha, q_{rl})), \alpha\}.$$

Consequently, it can be concluded that Z^f_α is a normal $(f - \alpha)$-flexible Q-fuzzy subgroup of the group M_α.

5. Conclusion

We proposed the notions of $(f - \alpha)$-flexible fuzzy sets and $(f - \alpha)$-flexible Q-fuzzy groups in this paper. Furthermore, we described and analyzed some of the fundamental features of $(f - \alpha)$-flexible Q-fuzzy normal groups. Many of these attributes have been demonstrated.

References

[1] A. Rosenfield. "Fuzzy groups" J. Math. Anal. Appl., Vol.35, (1965), 521-517

[2] J.H. Anthony and H. Sherwood, (1979) "Fuzzy groups redefined". J.Hath.Anal.Appl .69, 124-130. https://doi.org/10.1016/0022-247X(79)90182-3

[3] N.P. Mukherjee and P. Bhattacharya. "Fuzzy normal subgroups and fuzzy cosets", Information Sciences, Vol.34, (1984), 225-239. https://doi.org/10.1016/0020-0255(84)90050-1

[4] N.P. Mukherjee and P. Bhattacharya" Fuzzy groups: Some group theoretic analogs" Information Sciences, Vol.39. (1986), 247-269. https://doi.org/10.1016/0020-0255(86)90039-3

[5] Murali.V and Makamba B.B (2004) "Fuzzy subgroups of finite abelian groups "for East journal of Mathematical Science (EJMC), 14, 360-371.

[6] Murali.V and Makamba B.B (2006). "Counting the number of fuzzy subgroups of abelian group of order pn q" Fuzzy sets and systems, 44. 459-470. https://doi.org/10.1016/S0165-0114(03)00224-0

[7] A. Solairaju and R. Nagarajan "Q-fuzzy left R-subgroup of near rings with respect to T-norms" Antarctica Journal of Mathematics, 5 (1-2), (2008), 59-63**.**

[8] A. Solairaju and R. Nagarajan "A new structure and construction of Q-fuzzy groups" Advances in Fuzzy Mathematics 4(1), (2009), 23-29.

[9] A. Zadeh, "Fuzzy Sets", Information and Control Vol.8, (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[10] A. Solairaju and R. Nagarajan, Some structure properties of upper Q-fuzzy index order with upper Q-fuzzy subgroups Int. Journal of open Problem Math.Appl., 1, (2011), 21-29. https://doi.org/10.5120/1952-2116

[11] P. Sarangapani and P. Muruganantham "New structures on upper flexible Q-fuzzy groups " International Journal of Mathematics Research Vol8 (2), (2016), 107-112.

[12] W. H. Wu (1981), Normal fuzzy subgroups. Fuzzy Math, 1, 21-30.

[13] Massadeh M "Properties of fuzzy subgroups in particular the normal subgroups" Damacus University - Syrian Arab Republic, Doctorate thesis.

[14] V. Vanitha and G. Subbiah "Certain thresholds on flexible fuzzy subgroups with flexible fuzzy order" Int.Rearch journal of pure algebra -6(11), 2016, 436-442.

[15] Zimmerman H.J., "Fuzzy set theory and its applications" Kluwer Academic publishers London, third edition, (1997).

[16] Muhammad Gulzar and Ghazanfar Abbas, "Algebraic Properties of ω-Q-fuzzy subgroups" Int. Journal of Mathematics and Computer Science, 15(2020), no.1, 265-274.

[17] Mwafaq M. Salih and Delbrin H. Ahmed, α- Q- fuzzy subgroups. Academic Journal of Nawroz university (AJNU).

[18] Certain properties of ω- Q- fuzzy subrings, Indonesian Journal of Electrical Engineering and computer science vol.21, No.2 PP.822-828,2021. https://doi.org/10.11591/ijeecs.v21.i2.pp822-828

[19] Geethalakshmi Manickam and A. Solairaju, "New structure properties of flexible Q-fuzzy groups and flexible normal Q-fuzzy subgroups" Int. Research Journal of pure algebra-10, 7(2020), 14-18.

[20] Geethalakshmi Manickam, Kulandaivel Maruthamuthu Paramasivam, Gopal Thangavel, Sharmila Banu S and Kasper Sebastine, Development and Analysis of New Intuitionistic Q-Fuzzy Soft Rings and Ideals Using Triangular Norms. (Accepted for the Publication)