Approximation by modified Lupaş operators based on (p, q)–integers

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Abstract

The purpose of this paper is to introduce a new modification of Lupaş operators in the frame of post quantum setting and to investigate their approximation properties. First using the relations between q-calculus and post quantum calculus, the post quantum analogue of operators constructed will be linear and positive but will not follow Korovkin's theorem. Hence a new modification of q-Lupaş operators is constructed which will preserve test functions. Based on these modification of operators, approximation properties have been investigated. Further, the rate of convergence of operators by mean of modulus of continuity and functions belonging to the Lipschitz class as well as Peetre's K-functional are studied.

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1. Introduction and preliminaries

A. Lupaş [17] introduced the linear positive operators at the International Dortmund Meeting held in Witten (Germany, March, 1995) as follows:

$$L_m(f;u) = (1-a)^{-mu} \sum_{l=0}^{\infty} \frac{(mu)_l a^l}{l!} f\left(\frac{l}{m}\right), \; u \geq 0, \tag{1.1}$$

with $f : [0, \infty) \to \mathbb{R}$. If we impose $L_m(u) = u$, we get $a = \frac{1}{2}$. Thus operators (1.1) becomes

$$L_m(f;u) = 2^{-mu} \sum_{l=0}^{\infty} \frac{(mu)_l}{l! 2^l} f\left(\frac{l}{m}\right), \; u \geq 0, \tag{1.2}$$

where $(mu)_l$ is the rising factorial defined as:

$$(mu)_0 = 1, \; (mu)_l = mu(mu+1)(mu+2)\cdots(mu+l-1), \; l \geq 0.$$

The q-analogue of Lupaş operators (1.2) is defined in [26] as:

$$L_m^{p,q}(f;u) = 2^{-[m]_q u} \sum_{l=0}^{\infty} \frac{([m]_q u)_l}{[l]! 2^l} f\left(\frac{[l]_q}{[m]_q}\right), \; u \geq 0. \tag{1.3}$$

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2. Construction of new operators and auxiliary results

Let us recall certain notations and definitions of (p,q)-calculus. Let $p>0,\ q>0$. For each non negative integer $l,\ m,\ m\geq l\geq 0$, the (p,q)-integer, (p,q)-binomial are defined, as

$$[j]_{p,q}=p^{j-1}+p^{j-2}q+p^{j-3}q^2+...+pq^{j-2}+q^{j-1}=\begin{cases}\frac{p^j-q^j}{p-q}, & \text{when } p\neq q\neq 1,\\ j\ p^{j-1}, & \text{when } p=q\neq 1,\\ [j]_q, & \text{when } p=1,\\ j, & \text{when } p=q=1.\end{cases}$$

where $[j]_q$ denotes the q-integers and $m=0,1,2,\cdots$.

The formula for (p,q)-binomial expansion is as follows:

$$(au+bv)^m_{p,q}:=\sum_{l=0}^{m}p^{\frac{(m-l)(m-l-1)}{2}}q^{\frac{l(l-1)}{2}}\left[\begin{array}{c} m\\ l\end{array}\right]_{p,q}a^{m-l}b^lu^{m-l}v^l,$$

$$(u+v)^m_{p,q}=(u+v)(pu+qv)(p^2u+q^2v)\cdots(p^{m-1}u+q^{m-1}v),$$

$$(1-u)^m_{p,q}=(1-u)(p-qu)(p^2-q^2u)\cdots(p^{m-1}-q^{m-1}u),$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} m\\ l\end{array}\right]_{p,q}=\frac{[m]_{p,q}!}{[l]_{p,q}![m-l]_{p,q}!}.$$

Details on (p,q)-calculus can be found in [9, 11, 21].

In the case of $p=1$, the above notations reduce to q-analogues and one can easily see that $[m]_{p,q}=p^{m-1}[m]_{q/p}$. Mursaleen et al. [21] introduced (p,q)-calculus in approximation theory and constructed post quantum analogue of Bernstein operators. On the other hand Khalid and Lobiyal defined the (p,q)- analogue of Lupaş Bernstein operators in [12] and have shown its application in computer aided geometric design for construction of Beizer curves and surfaces. For another applications of extra parameters p in the field of approximation on compact disk, one can refer [4]. For related literature, one can refer [1, 2, 9, 3, 13, 14, 18, 19, 20, 22, 23, 25, 24] papers based on q and (p,q) integers in approximation theory and CAGD. Motivated by the above mentioned work, we introduce a new analogue of Lupaş operators. The post quantum analogue of (1.3) are as follows:

Definition 2.1. Let $f:[0,\infty)\to\mathbb{R}$, $0<q<p\leq 1$ and for any $m\in\mathbb{N}$. we define the (p,q)-analogue of Lupaş operators as

$$L^{p,q}_m(f;u)=2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}f\left(\frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}}\right),\ u\geq 0. \tag{2.1}$$

The operators (2.1) are linear and positive. For $p=1$, the operators (2.1) turn out to be q-Lupaş operators defined in (1.3). Next, we prove some auxiliary results for (2.1).

Lemma 2.2. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. We have*

(i) $L^{p,q}_m(1;u)=1$,

(ii) $L^{p,q}_m(t;u)=\frac{u}{p^{m-1}(2-p)^{[m]_{p,q}u+1}}$,

(iii) $L^{p,q}_m(t^2;u)=\frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}+\frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}}+\frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}$.

Proof. we have

(i)

$$L_m^{p,q}(1;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
L_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} \frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u} \sum_{l=1}^{\infty} \frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l} \frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}} p^{l+1-m} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{m-1}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l p^l}{[l]_{p,q}!2^l} \\
&= \frac{u}{(p^{m-1})(2-p)^{([m]_{p,q}u+1)}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
L_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} \frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u} \sum_{l=1}^{\infty} \frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!a^l} \frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}} \frac{p^{2l+2-2m}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}[p^l+q[l]_{p,q}]}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{3l}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}} \\
&= I_1 + I_2 (say),
\end{aligned}$$

□

we find that I_1 and I_2 are

$$\begin{aligned} I_1 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{3l}}{[m]_{p,q}} \\ &= \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}. \\ I_2 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}} \\ &= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{p^{2m-2}} qu \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+2)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l} \frac{p^{2l}[l]_{p,q}}{[m]_{p,q}} \\ &= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-2}} qu \sum_{l=1}^{\infty} \frac{([m]_{p,q}u+2)_{l-1}p^{2l}}{[l-1]_{p,q}!2^l} \\ &= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-4}} qu \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+2)_l p^{2l}}{[l]_{p,q}!2^l} \\ &= \frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}} + \frac{qu}{p^{2n-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}. \end{aligned}$$

On adding $I_1$and I_2, we get

$$L_m^{p,q}(t^2;u) = \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}} + \frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}} + \frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}.$$

The sequence of (p,q)-Lupaş operators constructed in (2.1) however do not preserve the test functions t and t^2. Hence one can not guarantee approximation via these operators. Therefore, we construct the modified (p,q)- Lupaş operators as follows:

Lemma 2.3. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. For $f:[0,\infty)\to\mathbb{R}$, we define the (p,q)-analogue of Lupaş operators as:*

$$\widetilde{L}_m^{p,q}(f;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!a^l} f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right), \ u \geq 0. \tag{2.2}$$

The operators (2.2) are linear and positive. For $p=1$, the operators (2.2) turn out to be q-Lupaş operator defined in (1.3).
We shall investigate approximation properties of the operators (2.2). We obtain rate of convergence of the operators via modulus of continuities. We also obtain approximation behaviors of the operators for functions belonging to Lipschitz spaces.

Lemma 2.4. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. We have*

(i) $\widetilde{L}_m^{p,q}(1;u)=1$,
(ii) $\widetilde{L}_m^{p,q}(t;u)=u$,
(iii) $\widetilde{L}_m^{p,q}(t^2;u)=\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2$.

Proof. We have

(i)

$$\widetilde{L}_m^{p,q}(1;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\\
&= u.
\end{aligned}$$

(iii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}}\frac{[l+1]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{[p^l+q[l]_{p,q}]}{[m]_{p,q}}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&+ \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= I_1+I_2(Say).
\end{aligned}$$

After solving I_1 and I_2, we get

$$\begin{aligned}
I_1 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&= \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}.\\
I_2 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=1}^{\infty}\frac{([m]_{p,q}u+1)([m]_{p,q}u+2)_{l-1}[l]_{p,q}}{[l]_{p,q}[l-1]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)qu}{[n]_{p,q}}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+2)_l}{[l]_{p,q}!2^l}\\
&= \frac{qu}{[m]_{p,q}}+qu^2.
\end{aligned}$$

On adding $I_1$and I_2, we get

$$\widetilde{L}_m^{p,q}(t^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2.$$

□

Corollary 2.5. *Using Lemma 2.4, we get the following central moments.*

$\widetilde{L}_n^{p,q}(t-u;u) = 0$
$\widetilde{L}_n^{p,q}((t-u)^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2 = \delta_m(u)$ *(say).*

Remark 2.6. One can observe that

$$\lim_{m\to\infty}[m]_{p,q} = \begin{cases} 0, & \text{p,q} \in (0,1), \\ \frac{1}{1-q}, & p=1 \text{ and } q \in (0,1). \end{cases}$$

Thus for approximation processes, one need to choose convergent sequences (p_m) and (q_m) such that for each n, $0 < q_m < p_m \leq 1$ and $p_m, q_m \to 1$ so that $[m]_{p_m,q_m} \to \infty$ as $m \to \infty$.

Theorem 2.7. *Let $f \in C_B[0,\infty)$ and $q_m \in (0,1)$, $p_m \in (q_m, 1]$ such that $q_m \to 1$, $p_m \to 1$, as $m \to \infty$. Then for each $u \in [0,\infty)$ we have*

$$\lim_{n\to\infty} \widetilde{L}_m^{p_m,q_m}(f;u) = f(u).$$

Proof. By Korovkin's theorem it is enough to show that

$$\lim_{m\to\infty} \widetilde{L}_m^{p_m,q_m}(t^m;u) = u^m, \ m = 0,1,2.$$

By Lemma 2.4, it is clear that

$$\lim_{m\to\infty} \widetilde{L}_m^{p_m,q_m}(1;u) = 1$$

$$\lim_{m\to\infty} \widetilde{L}_m^{p_m,q_m}(1;u) = u$$

and

$$\lim_{m\to\infty} \widetilde{L}_m^{p_m,q_m}(t^2;u) = \lim_{m\to\infty}\left[\frac{u}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}} + \frac{q_m u}{[m]_{p_m,q_m}} + q_m u^2\right]$$

$$= u^2.$$

This completes the proof. ∎

3. Direct results

Let $C_B[0,\infty)$ be the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\leq x<\infty} | f(x) | .$$

Let us consider the K-functional as:

$$K_2(f,\delta) = \inf_{s\in W^2}\{\| f-s \| + \delta \| s^{''} \|\},$$

where $\delta > 0$ and $W^2 = \{s \in C_B[0,\infty) : s^{'}, s^{''} \in C_B[0,\infty)\}$.
Then as in ([4], p. 177, Theorem 2.4), there euists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}). \tag{3.1}$$

Second order modulus of smoothness of $f \in C_B[0,\infty)$ is as follows

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} \mid f(u+2h)-2f(u+h)+f(u) \mid .$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} \mid f(u+h)-f(u) \mid .$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$, $p,q \in (0,1)$ such that $0<q<p\leq 1$. Then for every $u\in[0,\infty)$ we have*

$$\mid \widetilde{L}_m^{p,q}(f;u) - f(u) \mid \leq C\omega_2(f;\delta_m(u)),$$

where

$$\delta_m^2(u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$$

Proof. Let $s \in \mathcal{W}^2$. Then from Taylor's expansion, we get

$$s(t) = s(u) + s'(u)(t-u) + \int_u^t (t-u)s''(u)\mathrm{d}u,\ t\in[0,\mathcal{A}],\ \mathcal{A}>0.$$

Now by Corollary 2.5, we have

$$\widetilde{L}_m^{p,q}(s;u) = s(u) + \widetilde{L}_m^{p,q}\left(\int_u^t (t-u)s''(u)\mathrm{d}u;u\right).$$

$$\begin{aligned}|\widetilde{L}_m^{p,q}(s;u)(s;u)-s(u)| &\leq \widetilde{L}_m^{p,q}\left(\left|\int_u^t \mid (t-u)\mid \mid s''(u)\mid \mathrm{d}u;u\right|\right)\\ &\leq \widetilde{L}_m^{p,q}\left((t-u)^2;u\right) \parallel s'' \parallel,\end{aligned}$$

hence we get

$$|\widetilde{L}_m^{p,q}(s;u(s;u)-s(u)| \leq \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By Lemma 2.3, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \leq 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\left|f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right)\right| \leq \parallel f \parallel .$$

Thus, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \ \leq \mid \widetilde{L}_m^{p,q}\left((f-s);u\right) - (f-s)(u) \mid + |\widetilde{L}_m^{p,q}(s;u)-s(u)|.$$

After substituting all values, we get

$$|\widetilde{L}_m^{p,q}(f;u)-f(u)| \ \leq \parallel f-s \parallel + \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By taking the infimum on the right hand side over all $s\in\mathcal{W}^2$, we get

$$|\widetilde{L}_m^{p,q}(f;u)-f(u)| \ \leq \ \mathcal{C}K_2\left(f,\delta_m^2(u)\right).$$

By using the property of K-functional, we have

$$|\widetilde{L}_m^{p,q}(f;u)-f(u)| \ \leq \ \mathcal{C}\omega_2\left(f,\delta_m(u)\right).$$

This completes the proof. □

4. Pointwise estimates

Theorem 4.1. *Let $0 < \alpha \leq 1$ and Ȩ be any bounded subset of the interval $[0, \infty)$. If $f \in C_B[0, \infty)$, is locally Lip(α), i.e., the condition*

$$|f(v) - f(u)| \ \leq Ł|v - u|^{\alpha}, \ v \in Ȩ \text{ and } u \in [0, \infty) \tag{4.1}$$

holds, then, for each $u \in [0, \infty)$, we have

$$|\widetilde{L}_m^{p,q}(f; u) - f(u)| \ \leq Ł\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, Ȩ))^{\alpha}\right\}, \ u \in [0, \infty)$$

where L is a constant depending on α and f and $d(u; Ȩ)$ is the distance between u and Ȩ defined by

$$d(u, Ȩ) = inf\left\{|t - u|; t \in E\right\} \text{ and } \delta_m(u) = \widetilde{L}_m^{p,q}((t - u)^2; u).$$

Proof. Let $\overline{Ȩ}$ be the closure of Ȩ in $[0, 1)$. Then, there exists a point $t_0 \in \overline{Ȩ}$ such that $d(u, Ȩ) = |u - t_0|$.
Using the triangle inequality, we have

$$|f(t) - f(u)| \ \leq |f(t) - f(t_0)| + |f(t_0) - f(u)|.$$

By using (4.1) we get,

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq \widetilde{L}_m^{p,q}\left(|f(t) - f(t_0)|; u\right) + \widetilde{L}_m^{p,q}\left(|f(u) - f(t_0)|; u\right) \\
&\leq Ł\left\{\widetilde{L}_m^{p,q}\left(|t - t_0|^{\alpha}; u\right) + (|u - t_0|^{\alpha}; u) + |u - t_0|^{\alpha}\right\} \\
&\leq Ł\left\{\widetilde{L}_m^{p,q}\left(|t - u|^{\alpha}; u\right) + 2|u - t_0|^{\alpha}\right\}.
\end{aligned}$$

By choosing $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get $\frac{1}{p} + \frac{1}{q} = 1$. Then by using Hölder's inequality we get

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq Ł\left\{\widetilde{L}_m^{p,q}\left(|t - u|^{\alpha p}; u\right)^{\frac{1}{p}} [\widetilde{L}_m^{p,q}(1^q; u)]^{\frac{1}{q}} + 2(d(u, Ȩ))^{\alpha}\right\} \\
&\leq Ł\left\{\widetilde{L}_m^{p,q}\left(((t - u)^2; u)\right)^{\frac{\alpha}{2}} + 2(d(u, Ȩ))^{\alpha}\right\} \\
&\leq Ł\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, Ȩ))^{\alpha}\right\}.
\end{aligned}$$

Hence the proof is completed. □

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given by Lenze [16] as

$$\widetilde{\omega}_{\alpha}(f; u) = \sup_{t \neq u, t \in (0, \infty)} \frac{|f(t) - f(u)|}{|t - u|^{\alpha}}, \ u \in [0, \infty) \text{ and } \alpha \in (0, 1]. \tag{4.2}$$

Then we get the next result

Theorem 4.2. *Let $f \in C_B[0, \infty)$ and $\alpha \in (0, 1]$. Then, for all $u \in [0, \infty)$, we have*

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\Big(\delta_m(u)\Big)^{\frac{\alpha}{2}},$$

where $\delta_m(u)$ is defined in Corollary 2.5.

Proof. We know that

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{L}_m^{p_m, q_m}(|f(t) - f(u)|; u).$$

From equation (4.2), we have

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\widetilde{L}_m^{p_m, q_m}(|t - u|^{\alpha}; u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \quad \leq \quad \widetilde{\omega}_{\alpha}(f; u)\big(\widetilde{L}_m^{p_m, q_m}(|t - u|^2; u)\big)^{\frac{\alpha}{2}},$$

which proves the desired result. □

5. Weighted approximation by $\widetilde{L}_m^{p,q}$

In this section we shall discuss weighted approximation theorems for the operators $\widetilde{L}_m^{p,q}$ on the interval $[0,\infty)$.

Theorem 5.1 (cf. [5, 15]). *Let (T_m) be the sequence of linear positive operators from $C_{u^2}[0,\infty)$ to $B_{u^2}[0,\infty)$ satisfy*

$$\lim_{m\to\infty} \parallel T_m\kappa_i - \kappa_i \parallel_{u^2} = 0, \qquad i = 0,1,2.$$

Then for any function $f \in C^*_{u^2}[0,\infty)$

$$\lim_{m\to\infty} \parallel T_m f - f \parallel_{u^2} = 0.$$

Theorem 5.2 (cf. [6, 7]). *Let (q_m) and (p_m) be two sequences such that $0 < q_m < p_m \leq 1$, for all n and both converge to 1. Then for each function $f \in C^*_{u^2}[0,\infty)$, we get*

$$\lim_{m\to\infty} \parallel L_m^{p_m,q_m} f - f \parallel_{u^2} = 0.$$

Proof. By Theorem 5.1, it is enough to show

$$\lim_{m\to\infty} \parallel \widetilde{L}_m^{p_m,q_m}\kappa_i - \kappa_i \parallel_{u^2} = 0, \qquad i = 0,1,2. \tag{5.1}$$

By Lemma 2.4 (i) and (ii), it is clear that

$$\begin{aligned} \parallel L_m^{p_m,q_m}(1;u) - 1 \parallel_{u^2} &= 0 \\ \parallel \widetilde{L}_m^{p_m,q_m}(t;u) - u \parallel_{u^2} &= 0 \end{aligned}$$

and by Lemma 2.4 (iii), we have

$$\begin{aligned} \parallel \widetilde{L}_m^{p_m,q_m}(t^2;u) - u^2 \parallel_2 &= \sup_{u\in[0,\infty)} \frac{\left(\frac{1}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right)u + (q_m-1)u^2}{1+u^2} \\ &\leq \left(\frac{1}{(2-p_m)[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right) + (q_m - 1). \end{aligned}$$

Last inequality means that (5.1) holds for $i = 2$. By Theorem 5.1, the proof is completed. □

Theorem 5.3. *Let $q_m \in (0,1)$, $p_m \in (q,1]$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Let $f \in C^*_{u^2}[0,\infty)$, and its modulus of continuity $\omega_{d+1}(f;\delta)$ be defined on $[0,d+1] \subset [0,\infty)$. Then, we have*

$$|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)\|_{C[0,d]} \leq 6M_f(1+d^2)\delta_m(d) + 2\omega_{d+1}(f;\sqrt{\delta_m(d)}),$$

where $\delta_m(d) = \widetilde{L}_m^{p,q}((t-u)^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$

Proof. From ([10] p. 378), for $u \in [0,d]$ and $t \in [0,\infty)$, we have

$$|f(t) - f(u)| \leq 6M_f(1+d^2)(t-u)^2 + \left(1 + \frac{|t-u|}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying $\widetilde{L}_m^{p,q}$ both the sides, we have

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \leq 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) + \left(1 + \frac{\widetilde{L}_m^{p,q}(|t-u|;u)}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying Cauchy-Schwarz inequality,for $u \in [0,d]$ and $t \geq 0$, we get

$$\begin{aligned} |\widetilde{L}_m^{p,q}(f;u) - f(u)| &\leq \widetilde{L}_m^{p,q}\left(|(f;u) - f(u)|;u\right) \\ &\leq 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) \\ &+ \omega_{d+1}(f;\delta)\left(1 + \frac{1}{\delta}\widetilde{L}_m^{p,q}\left((t-u)^2;u\right)^{\frac{1}{2}}\right). \end{aligned}$$

Thus, from Lemma 2.4, for $u \in [0,d]$, we get

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \leq 6M_f(1+d^2)\delta_m(d) + \omega_{d+1}(f;\delta)\left(1 + \frac{\sqrt{\delta_m(d)}}{\delta}\right).$$

By Choosing $\delta = \sqrt{\delta_m(d)}$, we get the required result. ☐

Now, we prove a theorem to approximate all functions in $C^*_{u^2}$ Such type of results are given in [8] for locally integrable functions.

Theorem 5.4. *Let $0 < q_m < p_m \leq 1$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Then for each function $f \in C^*_{u^2}[0,\infty)$, and $\alpha > 1$*

$$\lim_{m\to\infty} \sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} = 0.$$

Proof. Let for any fixed $u_0 > 0$,

$$\begin{aligned}\sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} &\leq \sup_{u\leq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} + \sup_{u\geq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} \\ &\leq \| L_m^{p_m,q_m}(f) - f \|_{[c_0,u_0]} + \| f \|_{u^2} \sup_{u\leq u_0} \frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha} \\ &+ \sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^\alpha}. \qquad (5.2)\end{aligned}$$

Since, $|f(u)| \leq M_f(1+u^2)$ we have,

$$\sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^\alpha} \leq \sup_{u\geq u_0} \frac{M_f}{(1+u^2)^{\alpha-1}} \leq \frac{M_f}{(1+u^2)^{\alpha-1}}.$$

Let $\epsilon > 0$, and let us choose u_0 large then we have

$$\frac{M_f}{(1+{u_0}^2)^{\alpha-1}} < \frac{\epsilon}{3} \qquad (5.3)$$

and in view of (2.4), we get,

$$\begin{aligned}\| f \|_{u^2} \lim_{m\to\infty} \frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha} &= \| f \|_{u^2} \frac{1+u^2}{(1+u^2)^\alpha} \\ &\leq \frac{\| f \|_{u^2}}{(1+u^2)^{\alpha-1}} \\ &\leq \frac{\| f \|_{u^2}}{(1+{u_0}^2)^{\alpha-1}} \\ &\leq \frac{\epsilon}{3}. \qquad (5.4)\end{aligned}$$

By using Theorem 5.3, the first term of inequality (5.2) becomes

$$\| \widetilde{L}_m^{p_m,q_m}(f) - f \|_{[c_0,u_0]} < \frac{\epsilon}{3}, \text{as } m \to \infty. \qquad (5.5)$$

Hence we get the required proof by combining (5.3)-(5.5)

$$\sup_{u\in[0,\infty)} \frac{|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)|}{(1+u^2)^\alpha} < \epsilon.$$

☐

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