On Pentapartitioned Neutrosophic Binary Γ- Subsemirings of a crisp Γ- Semiring

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ABSTRACT

In this paper, we introduce and examine the concept of pentapartitioned neutrosophic binary Γ- subsemiring of a crisp Γ-semiring, and exploring some of the properties associated to it. Additionally some examples are provided to illustrate the definitions. Moreover, we discuss pentapartitioned neutrosophic binary cut of pentapartitioned neutrosophic binary Γ-subsemirings in a crisp Γ-semiring.

KEYWORDS

Γ –semiring, pentapartitioned neutrosophic binary set, Γ- subsemiring, pentapartitioned neutrosophic binary cut.

MATHEMATICS SUBJECT CLASSIFICATION CODE

03E72, 16Y60, 03B52

1. INTRODUCTION

In 1965, Zadeh L.A. [11] proposed the concept of fuzzy sets. As an extension of fuzzy set theory, intuitionistic fuzzy set theory was developed by K. Atanassov [2]. In 1998, Smarandache F. [9] introduced neutrosophic sets, a concept that generalizes both fuzzy sets and intuitionistic sets. Rama Malik & Surapati Pramanik [6] expanded on neutrosophic sets by introducing pentapartitioned neutrosophic sets, which assign five membership values to each element: true, contradictory, ignorant, unknown and false. Researchers have continued to expand on neutrosophic sets. In 2019, Francina Shalini and Remya [4] introduced neutrosophic vague binary sets and explored its properties, as well as neutrosophic vague binary BCK/BCI algebra. More recently, we [1] developed the Pentapartitioned Neutrosophic Binary Set (PNBS) and identified its key characteristics.

In 1996, M.K. Rao [7,8] extended the concept of semirings by introducing Γ-semirings, which also generalize Γ-rings. Y. Bhargavi & Eswarlal [3] investigated the properties of fuzzy Γ-semirings, a concept that expands on ternary semirings. Also, Yella Bhargavi & Akbar Rezaei [10] introduced the concept of neutrosophic Γ- Semirings. In this paper, an attempt has been made to study some algebraic nature of pentapartitioned neutrosophic binary set under

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Γ-semirings. Based on this, the concept of pentapartitioned neutrosophic binary Γ- subsemiring of a crisp Γ- semiring is introduced and some of its properties are investigated. We also include some examples to clarify the definitions. Additionally, pentapartitioned neutrosophic binary cut of pentapartitioned neutrosophic binary Γ-subsemirings is formulated.

2. PRELIMINARIES

Definition 2.1.

Let $\mathbb{E}$ and Γ be two additive commutative semigroups. Then $\mathbb{E}$ is called Γ-*semiring* if there exists a mapping $\mathbb{E} \times \Gamma \times \mathbb{E} \rightarrow \mathbb{E}$ where images of (u,ζ,v) to be denoted by $u\zeta v$ if it satisfies the following conditions: for all $u, v, w \in \mathbb{E}$; $\zeta, \lambda \in \Gamma$.

(i) $w\zeta(u+v) = w\zeta u + w\zeta v$,
(ii) $(u+v)\zeta w = u\zeta w + v\zeta w$,
(iii) $u(\zeta+\lambda)v = u\zeta v + u\lambda v$,
(iv) $u\zeta(v\lambda w) = (u\zeta v)\lambda w$.

Definition 2.2.

A nonempty subset H of a Γ-semiring $\mathbb{E}$ is said to be a *Γ-subsemiring* of $\mathbb{E}$ if $(H,+)$ is a sub semigroup of $(\mathbb{E},+)$ and $u\zeta w \in H$, for all $u, w \in H$; $\zeta \in \Gamma$.

Definition 2.3.

A *Neutrosophic set (NS)* $\tilde{A}$ over $\mathbb{E}$ is defined as follows:
$\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \nu_{\tilde{A}}(u), \gamma_{\tilde{A}}(u)\rangle : u \in \mathbb{E}\}$, where $\mu_{\tilde{A}}(u), \nu_{\tilde{A}}(u), \gamma_{\tilde{A}}(u)$ are the truth, indeterminant, and falsity membership values of each $u \in \mathbb{E}$. So, $0 \leq \mu_{\tilde{A}}(u) + \nu_{\tilde{A}}(u) + \gamma_{\tilde{A}}(u) \leq 3$.

Definition 2.4.

A neutrosophic set $\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \nu_{\tilde{A}}(u), \gamma_{\tilde{A}}(u)\rangle : u \in \mathbb{E}\}$ in a Γ-semiring $\mathbb{E}$ is called a *Neutrosophic Γ-subsemiring* if it satisfies the following conditions: for any $u,v \in \mathbb{E}$; $\zeta \in \Gamma$

(i) $\mu_{\tilde{A}}(u+v) \geq \mu_{\tilde{A}}(u) \wedge \mu_{\tilde{A}}(v)$
(ii) $\nu_{\tilde{A}}(u+v) \geq \nu_{\tilde{A}}(u) \wedge \nu_{\tilde{A}}(v)$
(iii) $\gamma_{\tilde{A}}(u+v) \leq \gamma_{\tilde{A}}(u) \vee \gamma_{\tilde{A}}(v)$
(iv) $\mu_{\tilde{A}}(u\zeta v) \geq \mu_{\tilde{A}}(u) \wedge \mu_{\tilde{A}}(v)$
(v) $\nu_{\tilde{A}}(u\zeta v) \geq \nu_{\tilde{A}}(u) \wedge \nu_{\tilde{A}}(v)$
(vi) $\gamma_{\tilde{A}}(u\zeta v) \leq \gamma_{\tilde{A}}(u) \vee \gamma_{\tilde{A}}(v)$

Definition 2.5.

Let U and V be two universes of discourse. The *Pentapartitioned neutrosophic binary set* (PNBS) $(\tilde{A}_1, \tilde{A}_2) \subseteq (\widetilde{\mathbb{E}}_1, \widetilde{\mathbb{E}}_2)$ is given by

$$(\tilde{A}_1, \tilde{A}_2) = \left\{ \begin{array}{c} \langle u, \mu_{\tilde{A}_1}(u), \sigma_{\tilde{A}_1}(u), \vartheta_{\tilde{A}_1}(u), \phi_{\tilde{A}_1}(u), \gamma_{\tilde{A}_1}(u)\rangle, \\ \langle v, \mu_{\tilde{A}_2}(v), \sigma_{\tilde{A}_2}(v), \vartheta_{\tilde{A}_2}(v), \phi_{\tilde{A}_2}(v), \gamma_{\tilde{A}_2}(v)\rangle : u \in \widetilde{\mathbb{E}}_1, v \in \widetilde{\mathbb{E}}_2 \end{array} \right\}$$

where $\mu_{\tilde{A}_1}(u), \sigma_{\tilde{A}_1}(u), \vartheta_{\tilde{A}_1}(u), \phi_{\tilde{A}_1}(u), \gamma_{\tilde{A}_1}(u) : \widetilde{\mathbb{E}}_1 \rightarrow [0,1]$ are the degrees of the membership of truth, contradiction, ignorance, unknown, and falsity membership values of $u \in \widetilde{\mathbb{E}}_1$ and $\mu_{\tilde{A}_2}(v), \sigma_{\tilde{A}_2}(v), \vartheta_{\tilde{A}_2}(v), \phi_{\tilde{A}_2}(v), \gamma_{\tilde{A}_2}(v) : \widetilde{\mathbb{E}}_2 \rightarrow [0,1]$ are the degrees of the membership of truth, contradiction, ignorance, unknown, and falsity membership values of $v \in \widetilde{\mathbb{E}}_2$ such that $0 \leq \mu_{\tilde{A}_1}(u) + \sigma_{\tilde{A}_1}(u) + \vartheta_{\tilde{A}_1}(u) + \phi_{\tilde{A}_1}(u) + \gamma_{\tilde{A}_1}(u) \leq 5$ and $0 \leq \mu_{\tilde{A}_2}(v) + \sigma_{\tilde{A}_2}(v) + \vartheta_{\tilde{A}_2}(v) + \phi_{\tilde{A}_2}(v) + \gamma_{\tilde{A}_2}(v) \leq 5$.

3. MAIN RESULTS

Definition 3.1.

Let $(\tilde{A}_1, \tilde{A}_2)$ be a Pentapartitioned Neutrosophic Binary set (PNBS) over $\widetilde{\mathbb{E}}_1$ and $\widetilde{\mathbb{E}}_2$ and Γ be a non empty set. A *Pentapartitioned Neutrosophic Binary Γ- subsemiring* (*PNB Γ- subsemiring*) is a structure $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)} = (\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}, \Gamma, +, \cdot)$ where $\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)} = \mathbb{E} = \{\widetilde{\mathbb{E}}_1 \cup \widetilde{\mathbb{E}}_2\}$ forms a crisp Γ- semiring to the pentapartitioned neutrosophic binary set $(\tilde{A}_1, \tilde{A}_2)$ under the usual binary operation $'+'$ and $'\cdot'$ which satisfies the following $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ inequality: for any $u, v \in \mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}$ and $\zeta \in \Gamma$,

(i) $\mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \succcurlyeq \mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(v)$

(ii) $\mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \succcurlyeq \mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \mathrm{PNB}_{(\tilde{A}_1,\tilde{A}_2)}(v)$ for all $u, v \in \mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}$ and $\zeta \in \Gamma$.

That is for every $u, v \in \mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}$ and $\zeta \in \Gamma$,

(i)
$$\mu_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \geq \mu_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \mu_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\sigma_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \geq \sigma_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \sigma_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\vartheta_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \leq \vartheta_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \vartheta_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\phi_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \leq \phi_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \phi_{(\tilde{A}_1,\tilde{A}_2)}(v) \text{ and,}$$
$$\gamma_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \leq \gamma_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \gamma_{(\tilde{A}_1,\tilde{A}_2)}(v).$$

(ii)
$$\mu_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \geq \mu_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \mu_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\sigma_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \geq \sigma_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge \sigma_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\vartheta_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \leq \vartheta_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \vartheta_{(\tilde{A}_1,\tilde{A}_2)}(v),$$
$$\phi_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \leq \phi_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \phi_{(\tilde{A}_1,\tilde{A}_2)}(v) \text{ and}$$
$$\gamma_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \leq \gamma_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee \gamma_{(\tilde{A}_1,\tilde{A}_2)}(v).$$

Throughout this paper, $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)} = (\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}, \Gamma, +, \cdot)$ always means an algebraic structure of PNB Γ- subsemiring $(\tilde{A}_1, \tilde{A}_2)$ of a Γ-Semiring $\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}$.

Remark 3.2

(i) In PNB Γ- subsemiring, $\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)} = \widetilde{\mathbb{E}}$ is taken as the "union" of elements of $\widetilde{\mathbb{E}}_1$ and $\widetilde{\mathbb{E}}_2$.

(ii) Before implementing the condition $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$, it is necessary to perform a operation (Here max, max, min, min, min) on each membership grades of the overlapping elements in the universes $\widetilde{\mathbb{E}}_1$ and $\widetilde{\mathbb{E}}_2$. This operation will determine the membership grades of the combined pentapartitioned neutrosophic binary set, ensuring that the binary effect in pentapartitioned neutrosophic sets is effectively represented from a practical perspective.

Example 3.3

Let $\widetilde{\mathbb{E}}_1 = \{z \in \mathbb{C} \,/\, Im(z) \geq 0\}$ and $\widetilde{\mathbb{E}}_2 = \{z \in \mathbb{C} \,/\, Im(z) \leq 0\}$ be two universes under consideration where $Im(z)$ represents the imaginary part of z and let $\Gamma = \{x \in \mathbb{Z}/x \geq 0\}$. Then the combined universes $\widetilde{\mathbb{E}} = \{\widetilde{\mathbb{E}}_1 \cup \widetilde{\mathbb{E}}_2\}$ be the set of complex numbers. Clearly $\widetilde{\mathbb{E}}$ and Γ are additive commutative semigroup. Define $\widetilde{\mathbb{E}} \times \Gamma \times \widetilde{\mathbb{E}} \to \widetilde{\mathbb{E}}$ by $u\zeta v$ usual product of u, ζ, v for all $u, v \in \widetilde{\mathbb{E}}, \zeta \in \Gamma$. Then $\widetilde{\mathbb{E}}$ is a Γ- semiring.

Let $(\tilde{A}_1, \tilde{A}_2)$ be a PNBS defined over $\widetilde{\mathbb{E}}$ as follows : for any $u \in \widetilde{\mathbb{E}}_1$ and $v \in \widetilde{\mathbb{E}}_2$,

For	$Im(u) = 0$	$Im(u) \neq 0$
$\mu_{\tilde{A}_1}(u)$	0.9	0.8
$\sigma_{\tilde{A}_1}(u)$	0.5	0.2
$\vartheta_{\tilde{A}_1}(u)$	0.4	0.6
$\phi_{\tilde{A}_1}(u)$	0.7	0.9
$\gamma_{\tilde{A}_1}(u)$	0.3	0.4

Table (a): Membership values of $u \in \widetilde{\mathbb{E}}_1$

For	$Im(v) = 0$	$Im(v) \neq 0$
$\mu_{\tilde{A}_2}(v)$	0.7	0.5
$\sigma_{\tilde{A}_2}(v)$	0.4	0.15
$\vartheta_{\tilde{A}_2}(v)$	0.5	0.9
$\phi_{\tilde{A}_2}(v)$	0.1	0.91
$\gamma_{\tilde{A}_2}(v)$	0.3	0.6

Table (b): Membership values of $v \in \widetilde{\mathbb{E}}_2$

The membership grade of the combined PNBS from Table (a) and (b) is given in Table (c)

For	$Im(w) = 0$	$Im(w) > 0$	$Im(w) < 0$
$\mu_{(\tilde{A}_1,\tilde{A}_2)}(w)$	0.9	0.8	0.5
$\sigma_{(\tilde{A}_1,\tilde{A}_2)}(w)$	0.5	0.2	0.15
$\vartheta_{(\tilde{A}_1,\tilde{A}_2)}(w)$	0.4	0.6	0.9
$\phi_{(\tilde{A}_1,\tilde{A}_2)}(w)$	0.1	0.9	0.91
$\gamma_{(\tilde{A}_1,\tilde{A}_2)}(w)$	0.3	0.4	0.6

Table (c): Combined membership values of $w \in \widetilde{\mathbb{E}}$

From the values of the Table (c), we get $(\tilde{A}_1, \tilde{A}_2)$ is a PNB Γ- subsemiring.

Example 3.4

Let $\widetilde{\mathbb{E}}_1 = \{f_1 : f_1$ is an even function with $f_1(0) = 0\}$ and
$\widetilde{\mathbb{E}}_2 = \{f_2 : f_2$ is an even function with $f_2(0) \neq 0\}$. Then the combined universe $\widetilde{\mathbb{E}} = \{\widetilde{\mathbb{E}}_1 \cup \widetilde{\mathbb{E}}_2\}$ be the set of all even functions. And let Γ be the identity function on $\widetilde{\mathbb{E}}$. Clearly, $\widetilde{\mathbb{E}}$ and Γ are additive commutative semigroups. Define the mapping $\widetilde{\mathbb{E}} \times \Gamma \times \widetilde{\mathbb{E}} \to \widetilde{\mathbb{E}}$ by $f \circ g \circ h$ composition of three functions f, g, h for all $f, h \in \widetilde{\mathbb{E}}, g \in \Gamma$. Then $\widetilde{\mathbb{E}}$ is a Γ- semiring.

Let $(\tilde{A}_1, \tilde{A}_2)$ be a $PNBS$ defined over $\widetilde{\mathbb{E}}$ as follows:

For any $f_1 \in \widetilde{\mathbb{E}}_1$: $\begin{cases} \mu_{\tilde{A}_1}(f_1) = .8 \\ \sigma_{\tilde{A}_1}(f_1) = .6 \\ \vartheta_{\tilde{A}_1}(f_1) = .2 \\ \phi_{\tilde{A}_1}(f_1) = .4 \\ \gamma_{\tilde{A}_1}(f_1) = .3 \end{cases}$ and for any $f_2 \in \widetilde{\mathbb{E}}_2$: $\begin{cases} \mu_{\tilde{A}_2}(f_2) = .6 \\ \sigma_{\tilde{A}_2}(f_2) = .5 \\ \vartheta_{\tilde{A}_2}(f_2) = .5 \\ \phi_{\tilde{A}_2}(f_2) = .7 \\ \gamma_{\tilde{A}_2}(f_2) = .6 \end{cases}$

From this $PNBS\,(\tilde{A}_1, \tilde{A}_2)$, the combined membership grades of $PNBS$ in $\widetilde{\mathbb{E}} = \{\widetilde{\mathbb{E}}_1 \cup \widetilde{\mathbb{E}}_2\}$ is given by

$$\text{For any } f \in \widetilde{\mathbb{E}}: \begin{cases} \mu_{(\tilde{A}_1,\tilde{A}_2)}(f) = \begin{cases} .8 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .6 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \sigma_{(\tilde{A}_1,\tilde{A}_2)}(f) = \begin{cases} .6 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .5 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \vartheta_{(\tilde{A}_1,\tilde{A}_2)}(f) = \begin{cases} .2 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .5 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \phi_{(\tilde{A}_1,\tilde{A}_2)}(f) = \begin{cases} .4 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .7 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \gamma_{(\tilde{A}_1,\tilde{A}_2)}(f) = \begin{cases} .3 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .6 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \end{cases}$$

Therefore, we get $(\tilde{A}_1, \tilde{A}_2)$ is a PNB Γ- subsemiring of a crisp Γ-semiring.

Theorem 3.5

Intersection of two PNB Γ- subsemirings in a crisp Γ- semiring $\widetilde{\mathbb{E}}$ is also a PNB Γ- subsemiring of a crisp Γ-semiring.

Proof.

Let $(\tilde{A}_1, \tilde{A}_2)$ and $(\tilde{B}_1, \tilde{B}_2)$ be two PNB Γ- subsemiring of a Γ- semiring $\widetilde{\mathbb{E}}$.

Let $u, v \in \widetilde{\mathbb{E}}$ and $\zeta \in \Gamma$.

$$\begin{aligned} PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(u+v) &= PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(u+v) \\ &\succcurlyeq \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v)\right) \wedge \left(PNB_{(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(v)\right) \\ &= \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(u)\right) \wedge \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(v)\right) \\ &= PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(v) \end{aligned}$$

Also, $PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(u\cdot\zeta\cdot v) = PNB_{(\tilde{A}_1,\tilde{A}_2)}(u\cdot\zeta\cdot v) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(u\cdot\zeta\cdot v)$

$$\begin{aligned} &\succcurlyeq \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v)\right) \wedge \left(PNB_{(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(v)\right) \\ &= \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(u)\right) \wedge \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(v)\right) \\ &= PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)\cap(\tilde{B}_1,\tilde{B}_2)}(v) \qquad \text{for all } u, v \in \widetilde{\mathbb{E}}, \quad \zeta \in \Gamma \end{aligned}$$

Therefore, $(\tilde{A}_1, \tilde{A}_2) \cap (\tilde{B}_1, \tilde{B}_2)$ is also an PNB Γ- subsemiring of a crisp Γ- semiring $\widetilde{\mathbb{E}}$.

Corollary 3.6

Let $\left\{\left(\tilde{A}_1, \tilde{A}_2\right)_i / i \in I\right\}$ be a family of PNB Γ- subsemirings of a crisp Γ- semiring $\widetilde{\mathbb{E}}$, then $\bigcap_{i\in I}\left(\tilde{A}_1, \tilde{A}_2\right)$ is also a PNB Γ- subsemiring.

Theorem 3.7

Union of two PNB Γ- subsemirings of a crisp Γ-semiring $\widetilde{\mathbb{E}}$ is also a PNB Γ- subsemiring if one is contained in the other.

Proof.

Let $\left(\tilde{A}_1, \tilde{A}_2\right)$ and $\left(\tilde{B}_1, \tilde{B}_2\right)$ be two PNB Γ- subsemirings of a crisp Γ-semiring $\widetilde{\mathbb{E}}$ over $\widetilde{\mathbb{E}}_1$ and $\widetilde{\mathbb{E}}_2$ and let $\left(\tilde{A}_1, \tilde{A}_2\right) \subseteq \left(\tilde{B}_1, \tilde{B}_2\right)$. Let $u, v \in \widetilde{\mathbb{E}}$ and $\zeta \in \Gamma$.

Now,
$$\begin{aligned} PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(u+v) &= PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \vee PNB_{(\tilde{B}_1,\tilde{B}_2)}(u+v) \\ &= PNB_{(\tilde{B}_1,\tilde{B}_2)}(u+v) \\ &\succcurlyeq PNB_{(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{B}_1,\tilde{B}_2)}(v) \\ &= \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \vee PNB_{(\tilde{B}_1,\tilde{B}_2)}(u)\right) \wedge \left(PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \vee PNB_{(\tilde{B}_1,\tilde{B}_2)}(v)\right) \\ &= PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(v) \end{aligned}$$

Also, by the above similar arguments, it can be easily proved that $PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(u \cdot \zeta . v) = PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)\cup(\tilde{B}_1,\tilde{B}_2)}(v)\ \ \forall u, v \in \widetilde{\mathbb{E}}, \zeta \in \Gamma$

Therefore, the union of the two PNB Γ- subsemirings of a crisp Γ- semiring is also a

PNB Γ- subsemiring.

Note:

The condition mentioned above is sufficient for the union of two PNB Γ- subsemirings to be considered a PNB Γ- subsemiring. However, the example provided below demonstrates that this condition is not necessarily required.

Example 3.8

Let us consider a PNB Γ- subsemiring $\left(\tilde{A}_1, \tilde{A}_2\right)$ of a crisp Γ- semiring $\widetilde{\mathbb{E}}$ as specified in example: 3.4. Now let us assume another PNB Γ- subsemiring $\left(\tilde{B}_1, \tilde{B}_2\right)$ of the same crisp Γ-semiring $\widetilde{\mathbb{E}}$ as follows:

$$f \in \widetilde{\mathbb{E}}: \begin{cases} \mu_{(\tilde{B}_1,\tilde{B}_2)}(f) = \begin{cases} .9 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .6 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \sigma_{(\tilde{B}_1,\tilde{B}_2)}(f) = \begin{cases} .6 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .4 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \vartheta_{(\tilde{B}_1,\tilde{B}_2)}(f) = \begin{cases} .1 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .6 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \phi_{(\tilde{B}_1,\tilde{B}_2)}(f) = \begin{cases} .3 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .6 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \\ \gamma_{(\tilde{B}_1,\tilde{B}_2)}(f) = \begin{cases} .1 \text{ if } f \in \widetilde{\mathbb{E}}_1 \\ .7 \text{ if } f \in \widetilde{\mathbb{E}}_2 \end{cases} \end{cases}$$

It is straightforward to demonstrate that $(\tilde{A}_1, \tilde{A}_2) \cup (\tilde{B}_1, \tilde{B}_2)$ forms a PNB Γ- subsemiring, even though neither $(\tilde{A}_1, \tilde{A}_2)$ nor $(\tilde{B}_1, \tilde{B}_2)$ is a subset of the other.

Definition 3.9

Let the PNBS $(\tilde{A}_1, \tilde{A}_2)$ be a PNB Γ-subsemiring with structure $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)} = \left(\mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)} = \left(\widetilde{\mathbb{E}} = \{\widetilde{\mathbb{E}}_1 \cup \widetilde{\mathbb{E}}_2\}\right), \Gamma, +, \cdot\right)$. The membership functions of truth, contradiction, ignorance, unknown and falsity of $(\tilde{A}_1, \tilde{A}_2)$ are $\mu_{(\tilde{A}_1,\tilde{A}_2)}, \sigma_{(\tilde{A}_1,\tilde{A}_2)}, \vartheta_{(\tilde{A}_1,\tilde{A}_2)}, \phi_{(\tilde{A}_1,\tilde{A}_2)}$ and $\gamma_{(\tilde{A}_1,\tilde{A}_2)}$ respectively. A pentapartitioned Neutrosophic binary $\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle$-cut or PNB $\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle$-cut of $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ is a crisp subset $(\tilde{A}_1, \tilde{A}_2)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ of the $PNBS$ $(\tilde{A}_1, \tilde{A}_2)$ is given by

$$(\tilde{A}_1, \tilde{A}_2)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle} = \left\{ u \in \mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)} : PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \succcurlyeq \begin{cases} [\alpha_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \\ [\beta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_2 \\ [\delta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \cap \widetilde{\mathbb{E}}_2 \end{cases} \right\}$$

Here $\delta_i = \alpha_i \vee \beta_i$ for $i \leq 2$, and $\delta_i = \alpha_i \wedge \beta_i$ for $3 \leq i \leq 5$ with $\alpha_i, \beta_i, \delta_i \in [0,1]\ \forall i$

i.e, if $u \in \widetilde{\mathbb{E}}_1$, then

$$\mu_{(\tilde{A}_1,\tilde{A}_2)} \geq \alpha_1, \sigma_{(\tilde{A}_1,\tilde{A}_2)} \geq \alpha_2, \vartheta_{(\tilde{A}_1,\tilde{A}_2)} \geq \alpha_3\ , \phi_{(\tilde{A}_1,\tilde{A}_2)} \geq \alpha_4, \gamma_{(\tilde{A}_1,\tilde{A}_2)} \geq \alpha_5$$

if $u \in \widetilde{\mathbb{E}}_2$, then

$$\mu_{(\tilde{A}_1,\tilde{A}_2)} \geq \beta_1, \sigma_{(\tilde{A}_1,\tilde{A}_2)} \geq \beta_2, \vartheta_{(\tilde{A}_1,\tilde{A}_2)} \geq \beta_3\ , \phi_{(\tilde{A}_1,\tilde{A}_2)} \geq \beta_4, \gamma_{(\tilde{A}_1,\tilde{A}_2)} \geq \beta_5$$

And if $u \in \widetilde{\mathbb{E}}_1 \cap \widetilde{\mathbb{E}}_2$, then

$$\mu_{(\tilde{A}_1,\tilde{A}_2)} \geq \delta_1, \sigma_{(\tilde{A}_1,\tilde{A}_2)} \geq \delta_2, \vartheta_{(\tilde{A}_1,\tilde{A}_2)} \geq \delta_3\ , \phi_{(\tilde{A}_1,\tilde{A}_2)} \geq \delta_4, \gamma_{(\tilde{A}_1,\tilde{A}_2)} \geq \delta_5$$

Remark 3.10

(i) $(\tilde{A}_1, \tilde{A}_2)_{\langle [0,0,1,1,1],[0,0,1,1,1] \rangle} = \mathbb{E}_{(\tilde{A}_1,\tilde{A}_2)}$

(ii) $(\tilde{A}_1, \tilde{A}_2)_{\langle [\alpha_i^*]_{i=1}^5, [\beta_i^*]_{i=1}^5 \rangle} \supseteq (\tilde{A}_1, \tilde{A}_2)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ iff $\alpha_i^* \geq \alpha_i, \beta_i^* \geq \beta_i$ for $i = 1,2$ and $\alpha_i^* \leq \alpha_i, \beta_i^* \leq \beta_i$ for $i = 3,4,5$

Theorem 3.11

If $\left(\tilde{A}_1, \tilde{A}_2\right)$ represents a $PBNS$ over $\widetilde{\mathbb{E}}_1$ and $\widetilde{\mathbb{E}}_2$. Then $\left(\tilde{A}_1, \tilde{A}_2\right)$ is a PNB Γ- subsemiring $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ of a crisp Γ-semiring $\widetilde{\mathbb{E}}$ iff PNB $\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle$-cut of $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ is a crisp Γ- subsemiring of $\widetilde{\mathbb{E}}$ for all $\alpha_i, \beta_i \in [0,1]$.

Proof.

Let for any $\alpha_i, \beta_i \in [0,1]$, $\left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ be a PNB $\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle$-cut of $\left(\tilde{A}_1, \tilde{A}_2\right)$. For any $u, v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$,

$$PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \succcurlyeq \begin{cases} [\alpha_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \\ [\beta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_2 \\ [\delta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \cap \widetilde{\mathbb{E}}_2 \end{cases} \quad \text{and} \quad PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \succcurlyeq \begin{cases} [\alpha_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \\ [\beta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_2 \\ [\delta_i]_{i=1}^5 & if\ u \in \widetilde{\mathbb{E}}_1 \cap \widetilde{\mathbb{E}}_2 \end{cases}$$

Given that $\left(\tilde{A}_1, \tilde{A}_2\right)$ is a PNB Γ- subsemiring $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$. Then,

(i) $PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v)$

(ii) $PNB_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \quad \forall\, u, v \in \widetilde{\mathbb{E}}$ and $\zeta \in \Gamma$

$\Rightarrow$ Either $PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \succcurlyeq [\alpha_i]_{i=1}^5$ or $PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \succcurlyeq [\beta_i]_{i=1}^5$

$\Rightarrow u + v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle} \ \forall\, u, v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$

Similarly,

$u\zeta v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ for any $\zeta \in \Gamma$ and $u, v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$

$\Rightarrow \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ is a crisp Γ- subsemiring of $\widetilde{\mathbb{E}}$.

Conversely, let $\left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ is a crisp Γ- subsemiring of $\widetilde{\mathbb{E}}$ for each $\alpha_i, \beta_i \in [0,1]$.

Let $u, v \in \widetilde{\mathbb{E}}$ such that $[\alpha_i]_{i=1}^5 = PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v)$

$\Rightarrow PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \succcurlyeq [\alpha_i]_{i=1}^5$ and $PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \succcurlyeq [\alpha_i]_{i=1}^5$

$\Rightarrow u, v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$

$\Rightarrow u + v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$ and $u\zeta v \in \left(\tilde{A}_1, \tilde{A}_2\right)_{\langle [\alpha_i]_{i=1}^5, [\beta_i]_{i=1}^5 \rangle}$

$\Rightarrow PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+v) \succcurlyeq [\alpha_i]_{i=1}^5 = PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v)\ \forall\, u, v \in \widetilde{\mathbb{E}}$ and $\zeta \in \Gamma$

Similarly,

$PNB_{(\tilde{A}_1,\tilde{A}_2)}(u\zeta v) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(v) \quad \forall\, u, v \in \widetilde{\mathbb{E}}$ and $\zeta \in \Gamma$

Therefore, $\left(\tilde{A}_1, \tilde{A}_2\right)$ is a PNB Γ- subsemiring $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ of a crisp Γ-semiring $\widetilde{\mathbb{E}}$.

Corollary 3.12

Let the PNB set $\left(\tilde{A}_1, \tilde{A}_2\right)$ be a PNB Γ- subsemiring with structure $\mathfrak{E}_{(\tilde{A}_1,\tilde{A}_2)}$ of a crisp Γ-semiring $\widetilde{\mathbb{E}}$, then

$$PNB_{(\tilde{A}_1,\tilde{A}_2)}(nu) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \text{ for } n \geq 2, n \in N$$

Proof.

To prove: $PNB_{(\tilde{A}_1,\tilde{A}_2)}(nu) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u)$ -------(1)

We prove this theorem by applying the method of mathematical induction on n. When $n = 2$,

The equation (1) becomes

$$PNB_{(\tilde{A}_1,\tilde{A}_2)}(2u) = PNB_{(\tilde{A}_1,\tilde{A}_2)}(u+u)$$
$$\succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(u)$$
$$= PNB_{(\tilde{A}_1,\tilde{A}_2)}(u)$$

Therefore, the inequality holds true when $n = 2$.

Assume that the inequality holds true for $n = k-1$,

That is, $PNB_{(\tilde{A}_1,\tilde{A}_2)}\big((k-1)u\big) = PNB_{(\tilde{A}_1,\tilde{A}_2)}(u)$ -------(2)

Now, consider $PNB_{(\tilde{A}_1,\tilde{A}_2)}(ku) = PNB_{(\tilde{A}_1,\tilde{A}_2)}\Big(\big((k-1)+1\big)u\Big)$

$$= PNB_{(\tilde{A}_1,\tilde{A}_2)}\big((k-1)\,u+u\big)$$
$$= PNB_{(\tilde{A}_1,\tilde{A}_2)}\big((k-1)u\big) \wedge PNB_{(\tilde{A}_1,\tilde{A}_2)}(u)$$
$$\succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \qquad [by(2)]$$

$$PNB_{(\tilde{A}_1,\tilde{A}_2)}(nu) \succcurlyeq PNB_{(\tilde{A}_1,\tilde{A}_2)}(u) \text{ for } n \geq 2, n \in N$$

CONCLUSION

In this paper, we introduced the concept of pentapartitioned neutrosophic binary Γ- subsemiring in a crisp Γ-semiring, extending the pentapartitioned neutrosophic binary set to the algebraic structure of Γ-semiring. The key contributions of this paper include the formulation of pentapartitioned neutrosophic binary cut of the pentapartitioned neutrosophic binary Γ-semirings and some examples are included to clarify the definitions and theorems. Future work will focus on examining additional algebraic properties, like ideals, and homomorphisms in the context of pentapartitioned neutrosophic binary Γ- subsemirings and analyzing the computational complexity of operations to solve real-world problems.

REFERENCES

[1] Anit Yoha. A and Jaslin Melbha .M, *Pentapartitioned Neutrosophic Binary Set And Its Properties, "International Journal of Neutrosophic Science", vol. 24, No.02, pp. 237-245, 2024.*

[2] Atanassov, K. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, vol. 20, no. 1, 1986, pp. 87–96.

[3] Bhargavi, Y., and Eswarlal, T., *Fuzzy Γ-semirings*, International Journal of Pure and Applied Mathematics,vol. 98, no. 3, 2015, pp. 339–349.

[4] Francina Shalini.A and Remya P.B., *Neutrosophic Vague Binary BCK\BCI- algebra,* Neutrosophic Sets and Systems*, 2020 vol. 35, Issue 1, pp 45-67.*
[5] Muzaffar, A., Nafis, M.T., and Sohail, S.S., *Neutrosophy Logic and its Classification: An Overview,* Neutrosophic Sets and Systems, vol. 35, 2020, pp. 239–251.
[6] Rama Malik and Surapati Pramanik, *Pentapartitioned neutrosophic set and its properties*,,Neutrosophic Sets and Systems*, vol. 36, 2020.*
[7] Rao, M.K., *Γ-semiring II*, Southeast Asian Bulletin of Mathematics, vol. 21, no. 3, 1997, pp. 281–287.
[8] Rao, M.K., *Γ-semiring I*, Southeast Asian Bulletin of Mathematics, vol. 19, 1996, pp. 49–54.
[9] Smarandache. L.A., "*Fuzzy sets*", Information and Control 8, 1965, 338-353.che, F., Neutrosophy, Neutrosophic Probability, Set, and Logic, Rehoboth, NM: American Research Press, 1998.14.
[10] Yella Bhargavi, & Akbar Rezaei. (2021). On *Neutrosophic Γ-Semirings*. Neutrosophic Sets and Systems, 47, 338-353. https://fs.unm.edu/nss8/index.php/111/article/view/4156
[11] Zadeh, L.A., *Fuzzy Sets*, Information and Control, vol. 8, 1965, pp. 338–353.