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Review of Nonelementary Integrals in Context of Hypergeometric Functions

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Abstract

Nonelementary integrals, which cannot be expressed in the terms of elementary functions, are generally expressed in special functions and hypergeometric function is one such special function denoted by mFn, which plays an important role in integration. In fact the hypergeometric functions are solution of many nonelementary integrals. The introduction of special functions has ended the scope of research of elementary and nonelementary functions in context of antiderivatives, which directly affects the antiderivative of those elementary functions, which are not expressible in terms of elementary functions in closed form. Due to this, a termination is seen in the development of the properties on such new functions. In this paper many classical nonelementary integral functions have been reviewed and expressed in terms of hypergeometric functions. The study has been concluded with the fact that every nonelementary integral is expressible in terms of hypergeometric functions but its converse is not true. The paper ends with some notes on limitations, further scope of research and acknowledgement.

Key-Words: Elementary and Nonelementary Functions, Hypergeometric Functions, Kernel of the Integral, Nonelementary Integrals.

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Introduction

In general we study two basic types of functions in Calculus: elementary and nonelementary. An elementary function is a function of a single variable (may be real or complex) that is defined as an expression in closed forms using sums, differences, products, divisions, roots, compositions of many polynomial, rational, trigonometric, hyperbolic, exponential, logarithmic functions and their inverses (Cherry, 1985, 1986; Closed-form expression - Wikipedia contributors, 2024; Corliss et al. 1989; Gale et al. 1923**;** Risch, 1969, 2022; Ritt, 2022; Rosenlicht, 1972; Singer et al., 1985; Trager, 2022; Yadav, 2012, 2015, 2023; Elementary function - Wikipedia contributors, 2023). Elementary functions were introduced by Joseph Liouville in a series of papers from 1833 to 1841. For example, 5, e, π, x, x^2, lnx, sinx, sinhx, arcsinx, arcsinhx, etc. are elementary functions (Elementary function - Wikipedia contributors, 2023). A function which is not elementary is called a nonelementary function. Many such interesting functions originate due to integration.

A nonelementary antiderivative of an elementary function is an antiderivative (or indefinite integral), that is not an elementary function. In 1835 Liouville gave the first proof that nonelementary antiderivative exist (Marchisotto et al., 1994; Risch, 1969, 2022; Ritt, 2022; Rosenlicht, 1972; Singer et al., 1985; Trager, 2022; Yadav, 2012, 2015, 2023; Nonelementary integral - Wikipedia contributors, 2024). For example, the functions

$$\sqrt{1-x^4}, \frac{1}{lnx}, e^{-x^2}, \sin(x^2), \frac{sinx}{x}, \frac{e^{-x}}{x}$$

etc. are nonelementary antiderivatives i.e. their antiderivatives are not expressible in terms of elementary functions. In fact the set of elementary functions is always closed under arithmetic operations, root extraction and composition but are not always closed under limits and infinite sums (Cherry, 1985, 1986; Closed-form expression - Wikipedia contributors, 2024; Corliss et al. 1989; Gale et al. 1923; Elementary function - Wikipedia contributors, 2023; Risch, 1969, 2022; Ritt, 2022; Rosenlicht, 1972; Singer et al., 1985; Trager, 2022; Yadav, 2012, 2015, 2023). In our discussion the set of elementary functions are not always closed under integration plays an important role. Such types of functions have been taken into consideration in this paper and an attempt has been made to relate nonelementary integrals with the hypergeometric functions. Hypergeometric functions naturally arise when we solve indefinite integrals (i.e. antiderivatives) and when the antiderivatives don't have elementary antiderivatives.

The hypergeometric function is a special function, which is represented by the hypergeometric series. The term hypergeometric series was first used by John Wallis in 1655 but the full systematic treatment was given by Carl Friedrich Gauss in 1813. Many other mathematicians like Leonhard Euler, Ernst Kummer, Bernhard Riemann, Herman Schwarz, Johann Friedrich Pfaff, Gauss, Srinivasa Ramanujan, etc. worked in this area for further study and applications (Bailey, 1964; Chaundy, 1943; Du et al. 2002; Gosper, 1978; Hannah, 2013; Hypergeometric function - Wikipedia contributors, 2024).

The question of relationship between nonelementary integrals and hypergeometric functions arises because hypergeometric functions generalize many special functions, including those that appear in nonelementary integrals due to the lack of mathematical symbols for many power series solutions. In many integrals, the power series solution cannot be expressed in terms of a finite combination of elementary functions like numbers, algebraic, logarithmic, exponential, trigonometric, inverse trigonometric, hyperbolic, inverse hyperbolic functions. In this paper we have reviewed the relations between these two important concepts of mathematics.

Preliminary Ideas

As we have mentioned earlier that our aim is to relate popular nonelementary integrals in terms of hypergeometric functions, before going to study the relations let us have some brief ideas about antiderivative i.e. indefinite integration and hypergeometric functions:

Indefinite Integration: In 1684 G. W. Leibniz introduced that

$$\int_a^x f(t)dt = \int f(x)dx + K$$

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where K the constant of integration corresponds to the value of the integral for the lower limit a (Integral - Wikipedia contributors, 2024).

Kernel of an Integral: The kernel of an integral refers to the function inside the integral that is being integrated. In the integral $\int e^{x^2} dx$, the kernel is e^{x^2}(Integral - Wikipedia contributors, 2024).

Whereas the basic concepts of hypergeometric functions are as follows:

General Hypergeometric Function: It is denoted by $\mathrm{mFn}(\propto_1, \propto_2, \ldots, \propto_m; \beta_1, \beta_2, \ldots, \beta_n; x)$ and is defined by

$$\mathrm{mFn}(\propto_1, \propto_2, \ldots, \propto_m; \beta_1, \beta_2, \ldots, \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\propto_1)_r . (\propto_2)_r, \ldots, (\propto_m)_r}{(\beta_1)_r, (\beta_2)_r, \ldots, (\beta_n)_r} \frac{x^r}{r!}$$

$$= 1 + \frac{\propto_1 \propto_2 \ldots \propto_m}{\beta_1 \beta_2 \ldots \beta_n} \frac{x^1}{1!} + \frac{\propto_1 (\propto_1 + 1) \propto_2 (\propto_2 + 1) \ldots \propto_m (\propto_m + 1)}{\beta_1(\beta_1 + 1)\beta_2(\beta_2 + 1) \ldots \beta_n(\beta_n + 1)} \frac{x^2}{2!} + \cdots$$

where $(\alpha)_r = \alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + r - 1)$ and $(\alpha)_0 = 1$ popularly known as shifted factorial or rising factorial. The notation $\mathrm{mFn}(\propto_1, \propto_2, \ldots, \propto_m; \beta_1, \beta_2, \ldots, \beta_n; x)$ is called the hypergeometric function and the series on the right hand side is called the hypergeometric series. When one of the parameters $\propto_i$ is equal to –N, where N is a nonnegative integer, the hypergeometric function becomes a finite series and thus a polynomial in x. (Bailey, 1964; Du et al., 2002; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

Confluent Hypergeometric Function: Putting m = n = 1, we get the confluent hypergeometric function, which is denoted by $1F1(\propto, \beta, x)$ and is defined by

$$1F1(\propto, \beta, x) = 1F1(\propto; \beta; x) = \sum_{r=0}^{\infty} \frac{(\propto)_r}{(\beta)_r} \frac{x^r}{r!} = 1 + \frac{\propto}{\beta} \frac{x}{1!} + \frac{\propto (\propto + 1)}{\beta(\beta + 1)} \frac{x^2}{2!} + \cdots$$

It is also known as Kummer Function (Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

Gauss Hypergeometric Function: Putting m = 2, n = 1 in general hypergeometric function, we get it denoted by $F(\propto, \beta; \gamma; x)$ and is defined by

$$F(\propto, \beta; \gamma; x) = 2F1(\propto, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\propto)_r . (\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(\propto)_r . (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} = 1 + \frac{\propto \beta}{\gamma} \frac{x}{1!} + \frac{\propto (\propto +1)\beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{x^2}{2!} + \cdots$$

where x, $\propto$, β, γ may be real or complex, $|x| < 1$ and c is not non-negative integers (Hannah, 2013; Krattenthaler et al., 2003; Maier, 2006; Pearson, 2009; Sao, 2021; Sharma, et al., 2020; Hypergeometric function - Wikipedia contributors, 2024).

In the present study, we will use the notations $F(\propto, \beta; \gamma; x)$, $2F1(\propto, \beta; \gamma; x)$ and ${}_2F_1(\propto, \beta; \gamma; x)$ for it. For $\propto = \gamma,\ \beta = 1$, the above series becomes the elementary geometric series. For either $\propto =$ 0 or $\beta = 0$, the above series reduces to unity. If any one or both of $\propto$ or β is (are) negative integer, it reduces to a hypergeometric polynomial of degree n in x, which terminates at the $(n+1)^{th}$ term, if either or both is (are) equal to –n, for natural number n. Thus the Binomial sums are a form of terminating hypergeometric series (Bailey, 1964; Chaundy, 1943; Du et al., 2002; Gosper, 1978; Hannah, 2013; Hypergeometric function - Wikipedia contributors, 2024).

In the above definition of general hypergeometric function, we have used m numerator and n denominator parameters. No denominator parameter is taken as zero or a negative integer. If any numerator parameter is zero or negative integer, then the function will be a terminating hypergeometric polynomial. Generally a dash is used to indicate when there is no parameter in either the numerator or the denominator (Hypergeometric function - Wikipedia contributors, 2024).

Integral Representation of Hypergeometric Function: The hypergeometric function is written in terms of integral as

$$2F1(a, b; c; z) = \frac{\lceil c}{\lceil b \lceil (c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} dt$$

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(Hypergeometric function - Wikipedia contributors, 2024).

Previously some study has been done on the relations between elementary, nonelementary and hypergeometric functions but it hasn't been studied the relation between nonelementary integrals and hypergeometric functions. In this paper we studied that relation between some popular nonelementary integral functions and their representation in terms of hypergeometric functions.

Methodology

To express an integral in terms of hypergeometric functions, we can use the following methods:

Whenever an indefinite integral do not have an elementary antiderivative or elementary solution, it is often useful to expand the kernel (integrand) of the integral in a power series and integrate it term by term and thereafter we express the result in terms of hypergeometric functions. This procedure has been used in the study without proper expansion and explanation in the paper (Power series - Wikipedia contributors, 2024).

Sometimes expressing an integral in to a hypergeometric function involves recognizing standard forms or using transformation techniques. Generally we do this by comparing with standard integrals, for example the Beta function is related with hypergeometric function with the following two relations as

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt = \frac{\lceil m \lceil n}{\lceil(m+n)}$$

$$\int_0^1 t^{p-1}(1-t)^{q}dt = \frac{\lceil p \lceil(q+1)}{\lceil(p+q+1)} = 2F1(p, q+1; p+q+1; 1).$$

We can also use Mathematica to express an integral in terms of hypergeometric functions using built in symbolic integration and transformation functions codes. In fact Mathematica automatically expresses an integral in terms of hypergeometric functions, in case the integral is not elementary. In case if the integral doesn't come out in terms of elementary functions, we use the Mathematica command "FunctionExpand[integral]" to express it in terms of hyperbolic

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functions. Sometimes we use the command "Hypergeometric2F1[…]" to write the final integral in terms of hypergeometric functions. In our study we have tried to find the relations between nonelementary integrals and hypergeometric functions available online on public platform.

Discussion

The present work is the study of different nonelementary integrals in context of hypergeometric functions. Since there is no single form of nonelementary integrals, so let us discuss popular nonelementary integrals one by one as following:

Rational Integral: We know that the rational integral given by

$$\int \frac{dt}{(1-t^3)^{2/3}}$$

does not have an elementary antiderivative (Marchisotto et al., 1994; Risch, 1969, 2022; Ritt, 2022; Rosenlicht, 1972; Singer et al., 1985; Trager, 2022), but it can be expressed in terms of hypergeometric functions by expanding the kernel of the integral in series and then integrating it term by term as

$$\int \frac{dt}{(1-t^3)^{2/3}} = t.2F1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};t^3\right)$$

Faddeeva Function: It was introduced by Vera Faddeeva and N. N. Terentyev in 1954. It is denoted and defined by

$$w(x) = e^{-x^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right)$$

It has been also called complex error function or probability integral by Weideman in 1994 and by Baumjohann and Treumann in 1997. It appears as a nameless function in Abramowitz and Stegun's work in 1964. It has been called complex cramp function by Mikhailovskiy in 1975 and by Bogdanov et al. in 1976. The name Faddeeva function was introduced by G. P. M. Poppe and C. M. J. Wijers in 1990 (Faddeeva function – Wikipedia contributors, 2024).

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We know that the integral

$$\mathrm{I} = \int e^{x^2} dx$$

doesn't have an elementary closed form solution i.e. it is a nonelementary integral or function (Yadav, 2012, 2023). But it can be expressed in terms of special function like in terms of hypergeometric function following the same process of expansion of kernel of the integral. Using integral representation of the Kummer's confluent hypergeometric function 1F1(a; b; z), we can write it as

$$\int e^{x^2} dx = x.1F1\left(\frac{1}{2};\frac{3}{2};x^2\right) + K$$

where K is a constant of integration. A standard result exists between such integral and hypergeometric function as

$$\int e^{\lambda x^2} dx = x.1F1\left(\frac{1}{2};\frac{3}{2};\lambda x^2\right)$$

Putting $\lambda = 1$ in it, we can get the above relation.

We also know that $\int_0^x e^{t^2} dt = \int e^{x^2} dx$, ignoring the coefficient of integration i.e. merely some constant, the Faddeeva function can be written in hypergeometric function form as

$$w(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}.x.1F1\left(\frac{1}{2};\frac{3}{2};x^2\right)\right).$$

Sine and Cosine Integrals: These are denoted and defined by

$$\mathrm{Si(x)} = \int \frac{\mathrm{sinx}}{\mathrm{x}} \mathrm{dx}, \text{and } \mathrm{Ci(x)} = \int \frac{\mathrm{cosx}}{\mathrm{x}} \mathrm{dx}$$

In fact sine integral function is defined as

$$\mathrm{Si(x)} = \int_0^{\mathrm{x}} \frac{\mathrm{sint}}{\mathrm{t}} \mathrm{dt}$$

which doesn't have an elementary closed form representation (Nonelementary integral - Wikipedia contributors, 2024) but can be written in terms of generalized hypergeometric function 2F3 and the expansion of Si(x) in terms of hypergeometric series, can be found by expanding $\frac{\mathrm{sint}}{\mathrm{t}}$ using $\frac{\mathrm{sint}}{\mathrm{t}} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$ and integrating term by term, we get a hypergeometric function representation of it, given by

$$\mathrm{Si(x)} = x.2F3\left(\frac{1}{2}, 1; \frac{3}{2}, 2{,}2; -\frac{x^2}{4}\right)$$

Similarly the cosine integral function is defined by

$$\mathrm{Ci(x)} = \int_0^x \frac{\mathrm{cost}}{\mathrm{t}} \mathrm{dt}$$

It is defined in more general form as

$$\mathrm{Ci(x)} = \gamma + \mathrm{lnx} + \int_0^x \frac{\mathrm{cost} - 1}{\mathrm{t}} \mathrm{dt}$$

It also doesn't have an elementary closed form representation (Nonelementary integral - Wikipedia contributors, 2024) but can be written in terms of generalized hypergeometric function 2F3 and the expansion of Ci(x) in terms of hypergeometric series, can be found by expanding $\frac{\mathrm{cost}-1}{\mathrm{t}}$ using $Ci(\mathrm{x}) = \gamma + \mathrm{lnx} - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,(2n)!}$ and integrating term by term, we get a hypergeometric function representation of it, given by

$$\mathrm{Ci(x)} = \gamma + \mathrm{lnx} + x^2.2F3\left(1{,}1; \frac{3}{2}, 2{,}2; -\frac{x^2}{4}\right)$$

where γ is a constant.

Exponential Integral: It is denoted and defined by

$$\mathrm{Ei(x)} = \int \frac{e^x}{x} \mathrm{dx}$$

Sometimes it is written as

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^{t}}{t}\,dt, \qquad x > 0$$

Which is nonelementary integral (Nonelementary integral - Wikipedia contributors, 2024). Applying the same procedure of expansion of the kernel of the integral and integrating term by term and then using limits, we get

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^{n}}{n.\,n!} = \gamma + \ln x + x.\,2F2(1,1;2,2;-x)$$

where γ is a constant known as Euler – Mascheroni constant.

Dawson Integral: It was presented by ***Abramowitz*** and ***Stegun*** in **1965**. It is also called Dawson's function and is denoted and defined by

$$F(x) = e^{-x^{2}} \int_{0}^{x} e^{t^{2}}\,dt$$

This is a nonelementary integral (Nonelementary integral - Wikipedia contributors, 2024). Dawson's integral can be expressed in the form of confluent hypergeometric function 1F1 as follows

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n+1}}{(2n+1)!} = x.\,1F1\left(1;\frac{3}{2};-x^{2}\right)$$

Another nonelementary integral function Y(x) given by Fried and Conte in 1961 is defined by

$$Y(x) = \frac{e^{-x^{2}}}{x} \int_{0}^{x} e^{t^{2}}\,dt$$

(Nonelementary integral - Wikipedia contributors, 2024). This can be expressed in hypergeometric function as

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$$\mathrm{Y(x)} = 1F1\left(1; \frac{3}{2}; -x^2\right)$$

Sitenko Function: It was given by ***Sitenko*** in **1982** and is denoted and defined by

$$\varphi(\mathrm{x}) = 2\mathrm{xe}^{-\mathrm{x}^2}\int_0^{\mathrm{x}} \mathrm{e}^{\mathrm{t}^2}\,\mathrm{dt}$$

(Nonelementary integral - Wikipedia contributors, 2024). This can be written in the form of hypergeometric function as

$$\varphi(\mathrm{x}) = 2\mathrm{xe}^{-\mathrm{x}^2}\int_0^{\mathrm{x}} \mathrm{e}^{\mathrm{t}^2}\,\mathrm{dt} = 2x^2.\,1F1\left(1; \frac{3}{2}; -x^2\right)$$

Error Function: It was introduced by *Gauss*. Mikhailovskiy (in 1975) and Bogdanov et al. (in 1976) called it a Cramp function. It is denoted and defined by

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\int_0^{\mathrm{x}} \mathrm{e}^{-\mathrm{t}^2}\,\mathrm{dt}, \text{or } \mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\int \mathrm{e}^{-\mathrm{x}^2}\mathrm{dx}$$

(Nonelementary integral - Wikipedia contributors, 2024). Expanding the kernel of the above integral and using term by term integration, the error function can be written in hypergeometric function form as

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\int_0^{\mathrm{x}} \mathrm{e}^{-\mathrm{t}^2}\,\mathrm{dt} = \frac{2\mathrm{x}}{\sqrt{\pi}}\,1\mathrm{F}1\left(\frac{1}{2}; \frac{3}{2}; -\mathrm{x}^2\right)$$

Fresnel Functions: It was introduced by Abramowitz and Stegun in **1965** and are denoted and defined by

$$\mathrm{C(x)} + \mathrm{iS(x)} = \int_0^{\mathrm{x}} \mathrm{e}^{\frac{\mathrm{i}\pi\mathrm{t}^2}{2}}\,\mathrm{dt}$$

Its alternative forms are called **Fresnel integrals** and are denoted and defined by

$$x(t) = \int_0^t \cos(x^2)\,dx, \text{and } y(t) = \int_0^t \sin(x^2)\,dx$$

These are nonelementary integrals (Nonelementary integral - Wikipedia contributors, 2024). Fresnel's Integrals are written in hypergeometric function form as

$$x(t) = \int_0^t \cos(x^2)\,dx = x.\,1F2\left(\frac{1}{4}; \frac{1}{2}, \frac{5}{4}; -\frac{x^4}{4}\right)$$

$$y(t) = \int_0^t \sin(x^2)\,dx = \frac{x^3}{3}\,1F2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}; -\frac{x^4}{4}\right)$$

Plasma Dispersion Function: It was given by ***Fried*** and ***Conte*** in **1961**. It is also known as **Fried-Conte function** and is denoted and defined by

$$Z(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - x}$$

(Nonelementary integral - Wikipedia contributors, 2024). It is related to the Faddeeva function w(x) and this can also be written in the confluent hypergeometric function form as

$$Z(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - x} = \pi e^{-x^2}.\,1F1\left(\frac{1}{2}; \frac{3}{2}; i\,x^2\right)$$

Logarithmic Integral Function: It is denoted and defined by

$$Li(x) = \int_0^x \frac{dt}{\ln t}, \qquad x > 1;\ \text{or}\ Li(x) = \int \frac{dx}{\ln x}$$

(Nonelementary integral - Wikipedia contributors, 2024). This function appears in number theory particularly in the study of prime numbers. For large x, the logarithmic integral function can be expanded in a series as

$$Li(x) = \gamma + \ln\ln x + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n.\,n!}$$

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which is directly related to the generalized hypergeometric function 2F2 as

$$\mathrm{Li}(x) = \gamma + \ln\ln x + (\ln x)\,.\,2F2(1,1;2,2;\ln x)$$

where γ is a constant. There are many more nonelementary integral functions which can be expressed in form of hypergeometric functions as:

Complete Elliptic Integrals can be written as

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$$

(Nonelementary integral - Wikipedia contributors, 2024).

Bessel Function of First Kind can be written as

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma(n+1)}\,0F1\left(-;n+1;-\frac{x^2}{4}\right)$$

(Nonelementary integral - Wikipedia contributors, 2024).

Hyperbolic Sine Integral is given by

$$\mathrm{Shi}(x) = \int \frac{\sinh x}{x}dx = x.\,1F2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{x^2}{4}\right)$$

(Nonelementary integral - Wikipedia contributors, 2024).

We see that the above nonelementary integrals contain rational functions, exponential functions, trigonometric etc. and they lead to the representations involving different hypergeometric functions. The nonelementary integrals involving the exponential functions lead to the confluent hypergeometric functions. The classical nonelementary integral, the error function have been written in terms of hypergeometric functions. The exponential and logarithmic nonelementary integrals are expressed in terms of hypergeometric functions and so on. Thus we see that the hypergeometric functions naturally generalize the power series solutions of nonelementary integrals i.e. they cover a broad class of indefinite integral solutions that are nonelementary. So

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we see that the hypergeometric functions and their generalizations provide us powerful mathematical tools for representing and analyzing the nonelementary integrals. Thus all the nonelementary integrals are expressible in terms of hypergeometric functions i.e. they are indirectly hypergeometric functions. Obviously every hypergeometric function is not a nonelementary integral function.

Justification: As we know that there were two different approaches to integration, one by Newton and another by Leibnitz. Newton allowed series solution of integration whereas Leibnitz followed integration in finite terms. Newton accepted term by term integration using power series solution (Yadav & Sen, 2013).

In our discussion, we can find the power series of the integrand (kernel of the integral) and then doing term by term integration, we can find another series of integrals of all terms. Then we can find its equivalent form of hypergeometric function or any special function. But since all most all special functions are expressible in terms of hypergeometric function, which follows that our conclusion that all nonelementary integrals are expressible in terms of hypergeometric function is justified.

Conclusion

We can conclude that all nonelementary integral are expressible in terms of different hypergeometric functions i.e. every nonelementary integral function is indirectly a hypergeometric function but its converse is not true.

Limitations

Although many popular nonelementary integral functions have been written in terms of hypergeometric functions but we have not mentioned all such nonelementary integrals. This is the big limitation of the study that many more such integrals have been left for the study.

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Scope of Research

As stated above that all nonelementary integral functions have not been covered in the study, so there is a lots of scope of further research in this area.

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It is acknowledge that we have used the public online platforms to collect data and relations between different functions, whenever it was needed. We have also used Mathematica to justify the results found manually.

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