ON COMMUTATIVITY OF GENERALIZED PERIODIC RINGS

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Abstract

Let R be a ring N the set of nilpotent elements of R and C the center of R. A ring R is called periodic if for each $x \in R$, there exist distinct positive integers m and n such that $x^n = x^m$. R is called a generalized periodic ring if for every $x \in R\backslash(N \cup C)$, there exist distinct positive integers m, n of opposite parity such that $x^n = x^m \in N \cap C$. In this paper we prove that every nonzero idempotent e in a 2-torsion free generalized periodic ring R is central and if $a \in N$ then $ea \in C$. Using this result, we show that R is commutative.

Key words: Commutativity, Periodic ring, generalized periodic ring, Centre of a ring.

1 Introduction

Throughout this paper R will denote a ring, N the set of nilpotent, C the centre, J the Jacobson radical and $C(R)$ the commutator ideal of R. The ring R is called generalized periodic if for every $x \in R\backslash(N \cup C)$, there exist distinct positive integers m, n of opposite parity such that $x^n = x^m \in N \cap C$. Bell [2] proved that a generalized periodic ring always has the set N of nilpotent forming an ideal in R. In [1] Bell and yaqub proved that a generalized Boolean ring with central idempotents must be either nil or commutative. In this direction, we prove that all the idempotent elements in a 2-torsion free generalized periodic ring is central.

We need the following lemmas of [2] to prove the main results.

Lemma 1: In a generalized periodic ring R, we have

(i) $C(R) \subseteq J$
(ii) $J \subseteq N \cup C$
(iii) $N \subseteq J$

Lemma 2: Let R be a generalized periodic ring. Then R/N is commutative and hence $C(R) \subseteq N$.

Lemma 3: Let R be a generalized periodic ring and suppose $N \subseteq C$. Then R is commutative.

Lemma 4: If R is a 2-torsion free generalized periodic ring then every nonzero idempotent is central.

Proof: Suppose the idempotent e of R is not central.

Since R is a generalized periodic ring, $e \notin (N \cup C)$ and $-e \notin (N \cup C)$.

Hence $(-e)^n - (-e)^m \in (N \cup C)$, where m, n are opposite parity.

So $(-e)^n - (-e)^m \in C$.

If n, m are either even or odd positive integers, then $0 \in C$.

Otherwise $-2e$ *or* $2e \in C$.

That is, $[2e, x] = 0$ *or* $2[e, x] = 0$, for every $x \in R$.

Since R is a 2-torsion free generalized periodic ring, $[e, x] = 0$.

So $e \in C$, a contradiction.

This contradiction proves that nonzero idempotents are central. ◻

Lemma 5: Let R be a generalized periodic ring. If e is any nonzero idempotent in R and $a \in N$, then $ea \in C$.

Proof: The proof is by contradiction.
Suppose the lemma is false.
Let $a_0 \in N$ and $ea_0 \notin C$. (1)
Since $e \in C$ (by lemma 4.3.9) and $a_0 \in N$, we have ea_0 is nilpotent.
Let $(ea_0)^\alpha \in C, \forall \alpha \geq \alpha_0$, α_0 minimal. (2)
Since $ea_0 \notin C$ we have $\alpha_0 > 1$. Let $a = (ea_0)^{a_0 - 1}$.
Then $a = (ea_0)^{a_0 - 1} \in N$, $a \notin C$ (by the minimality of α_0),
$a^K \in C, \forall k \geq 2$. Since
$e \in C$, and $e^2 = e \neq 0$, $e \notin N$. (3)
Equation 4.3.15 implies that $e + a \notin C$ and $e + a \notin N$ and hence
$(e+a)^{m'} - (e+a)^{n'} \in C$, where m', n' are of opposite parity. (4)
Combining (3) and (4) we see that

$$(m' - n')ea \in C \text{ where } m' - n' \text{ is an odd integer.} \qquad (5)$$

Equation (3) also implies that $(-e + a)$ is not in $N \cup C$.
So $(e + a)^{m''} - (e + a)^{n''} \in N$, where n'', m'' are of opposite parity. (6)
Combining (3) and (6), we see that $(-e)^{m''} - (-e)^{n''} \in N$, (7)
and hence $2e \in N$, since n'', m'' are of opposite parity.
Therefore $(2e)^r = 0$, $r \in Z^+$ and thus $2^r e = 0$, which implies that
$2^r ea \in C$, $r \in Z^+$. (8)
Now combining (5) and (8) and since $(2^r, m' - n') = 1$,
we see that $ea \in C$. Hence by (3), $a = ea \in C$, which contradicts (3).
This contradiction proves the lemma. □

Theorem 6: Suppose R is a generalized periodic ring containing an idempotent element which is not a zero divisor. Then R is commutative.

Proof: Let e be an idempotent element in R.
Let $a \in N$. By lemma 5, we have $ea \in C$ and hence $[ea, x] = 0$
for all $x \in R$. This implies that $eax - xea = 0$.
Since idempotents are central, so that $eax - exa = 0$.
Therefore $e[a, x] = 0$. (9)
Since e is not a zero divisor, $[a, x] = 0, \forall x \in R, \forall a \in N$.
By lemma 3 and a well-known theorem of Herstein [4], it follows that R is commutative and the theorem is proved. □

Theorem 7: Suppose R is a generalized periodic ring, N the set of nilpotent, and E the set of idempotents of R. Suppose every commutator $[a, b] = ab - ba$ with $a \in N$ and $b \in N$ is potent. Then R is commutative.

Proof: By lemma 2, $C(R) \subseteq N$ and hence $[a, b] \in N$.
By hypothesis $[a, b] = [a, b]^q = [a, b]^{1+\lambda(q-1)}$ for all positive integer λ.
Hence $[a, b] = 0$, Since $[a, b] \in N$.
Thus $[a, b] = 0$ for all $a, b \in N$ that is N is commutative. (10)
We also have $x - x^{n-m+1} \in N$. We proved that for every x in R, we have

$$x - x^k \in N, \text{ for some } k > 1 \text{ or } x \in C \ (x \in R). \qquad (11)$$

Combining equation (10), (11) we see that for all x, y in R,

$$[x - x^k, y - y^r] = 0 \text{ for some } k > 1, r > 1. \qquad (12)$$

As is well known

$$R \cong a \text{ subdirect sum of subdirectly irreducible rings } R_i. \qquad (13)$$

We now see the structure of each of this subdirect summands R_i to prove their commutativity.

Case 1: R_i does not have an identity.

Let $\sigma: R \to R_i$ be the natural homomorphism of R onto R_i and let $\sigma: x \to x_i$.

Let N_i and C_i denote the set of nilpotents and the center of R_i respectively.

We claim that

$$R_i \subseteq N_i \cup C_i. \tag{14}$$

Suppose not, let $x \in R_i$, $x_i \notin N_i$, $x_i \notin C_i$, and let $\sigma: x \to x_i$ ($x \in R$).

Then clearly $x \notin N$ and $x \notin C$ and hence $x^n - x^m \in N$,

for some positive integers n and m, $n \neq m$.

This implies that $x^q = x^q e$ for some positive integer q and some idempotent e in R. In a generalized periodic ring idempotents are central and hence $x^q = x^q e$, $e^2 = e \in C$.

This implies in R_i that

$$x_i^q = x_i^q e_i,\ e_i^2 = e_i \in C_i. \tag{15}$$

Since e_i is a central idempotent in the subdirectly irreducible ring R_i and R_i does not have identity, we have $e_i = 0$. Hence $x_i^q = 0$ a contradiction, since x_i is not nilpotent.

This contradiction proves $R_i \subseteq N_i \cup C_i$.

From (11) we see that $[x_i - x_i^k, y_i - y_i^r] = 0$, $k \geq 1$, $r \geq 1$, $x_i, y_i \in R_i$. (16)

Now by a trivial minimality argument, it is easily verified that 4.3.28 implies

$$[a_i, b_i] = 0, \text{ for all nilpotents } a_i, b_i \text{ in } R_i, \tag{17}$$

i.e., N_i is Commutative.

Combining (14) and (17), we see that R_i is commutative.

Case 2: R_i has an identity.

Since the homomorphic image of a generalized periodic ring is also generalized periodic, it follows that R_i is commutative by corollary 4 of [2].

Since each R_i in the subdirect sum representation 4.3.25 is commutative therefore the ground ring R itself is also commutative and the theorem is proved. □

We consider the following two examples, which show that neither centrality of idempotents nor commutativity of nilpotent elements implies commutativity of a generalized periodic ring. We note that, in each case, central elements are zero divisors.

Example 4.3.13: Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \backslash\ 0,1 \in GF(2) \right\}.$$

It is easily verified that R is a generalized periodic ring with commuting nilpotent but its idempotents are not in the centre.

Example 4.3.14: Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \backslash\ a,b,c \in GF(3) \right\}.$$

It can be seen that R is a generalized periodic ring with central idempotents but its nilpotent do not commute with each other.

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