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Representation of the Matrix for Conversion between Triangular Bézier Patches and Rectangular Bézier Patches

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Abstract

In this paper we studied Bézier surfaces that are very famous techniques and widely used in Computer Aided Geometric Design. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. In this paper we will give a representation for the conversion matrix which converts one type to another.

1 Introduction

The theory of Bézier curves has an important role and they are numerically the most stable among all polynomial bases currently used in CAD systems. On the other hand in these days Bézier surfaces are very famous techniques and widely used in Computer Aided Geometric Design [1]-[13]. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches and they are defined in terms of the univariate Bernstein polynomials $B_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}$ and the bivariate Bernstein polynomial $B_{i,j,k}^n(u,v,w) = \binom{n}{i,j,k} u^i v^j w^k$ where $u+v+w=1$. A triangular Bézier patch of degree n with control points $T_{i,j,k}$ is defined by

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^n(u,v,w),\ u,v,w \geq 0,\ u+v+w=1.$$

and a rectangular Bézier patch of degree $n \times m$ with control points $P_{i,j}$ is represented by

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t) \quad 0 \leq s,t \leq 1, (\text{see [3]})$$

Since the two patches have different geometric properties it is not easy to use both of them in the same CAD system and conversion of one type to another is needed.

2 Construction of the Conversion Matrices

The following theorem gives the conversion of degree n triangular Bézier patch to degenerate rectangular Bézier patch of degree $n \times n$.

Definition 1 *For all nonnegative integers x the falling factorial is defined by*

$$(x)_n = x(x-1)...(x-n+1) = \prod_{k=1}^{n} (x-(k-1))$$

Theorem 2 *A degree n triangular Bézier patch $T(u,v,w)$ can be represented as a degenerate Bézier patch of degree $n\times n$:*

$$P(s,t)=\sum_{i=0}^{n}\sum_{j=0}^{n}P_{ij}B_i^n(s)B_j^n(t),\qquad 0\le s,t\le 1$$

where the control points P_{ij} are determined by

$$\begin{pmatrix} P_{i0}\\ P_{i1}\\ \vdots \\ P_{in}\end{pmatrix}=A_1A_2...A_i\begin{pmatrix} T_{i0}\\ T_{i1}\\ \vdots \\ T_{i,n-i}\end{pmatrix},\qquad i=0,1,2,...,n.$$

and $A_i (i=0,1,...,n)$ are degree elevation operators in the form

$$A_k=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \dots & 0 & 0\\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k}\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}_{(n-k+2)\times(n-k+1)}$$

Until now no one has studied the generalization of the product $A_1A_2...A_k$ mentioned in the above theorem and indeed the product of these matrices is not easy to calculate for different values of n and k. Here we will give the generalization of this product which will make all the computations easier.

Theorem 3 *The following formula is true*

$$A_1A_2...A_k=\bar{A}_k=\left[\bar{a}_{i,j}^{(k)}\right]_{(n+1)\times(n-k+1)},$$

where

$$\bar{a}_{i,j}^{(k)}=\frac{\binom{i-1}{j-1}(k)_{i-j}(n-k)_{j-1}}{(n)_{i-1}},$$

$$(k)_n=k(k-1)...(k-n+1)=\prod_{j=1}^{n}(k-(j-1))\ \ and$$

$$A_k=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \dots & 0 & 0\\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k}\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}_{(n-k+2)\times(n-k+1)}.$$

Proof. For $k=1$,

$$\bar{A}_1=A_1.$$

Suppose it is true for k, that is

$$A_1A_2...A_k=\bar{A}_k.$$

We will show that it also true for $k+1$, i.e

$$\bar{A}_kA_{k+1}=\bar{A}_{k+1}.$$

Let $c_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $\bar{A}_k A_{k+1}$:

$$\begin{aligned} c_{i,j} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,j}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,j}^{(k+1)}, \end{aligned}$$

where $i = \{1, 2, ..., n+1\}$ and $j = \{1, 2, ..., n-k\}$.

For $j = 1$ (first column)

$$\begin{aligned} c_{i,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,1}^{(k+1)}. \end{aligned}$$

For $i = 1$ and $j = 1$

$$\begin{aligned} c_{1,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_{m-1}}{(n)_0} a_{m,1}^{(k+1)} \\ &= a_{1,1}^{(k+1)} = 1 = \bar{a}_{1,1}^{(k+1)}. \end{aligned}$$

For $i = 2$ and $j = 1$,

$$\begin{aligned} c_{2,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{2,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a_{m,1}^{(k+1)} \\ &= \frac{k+1}{n} = \frac{(k+1)_1}{(n)_1}. \end{aligned}$$

For $i = n+1$ and $j = 1$,

$$\begin{aligned} c_{n+1,1} &= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a_{m,1}^{(k+1)} \\ &= \frac{(k+1)k(k-1)...(k-n+2)}{n(n-1)(n-2)...1} \\ &= \frac{(k+1)_n}{(n)_n}. \end{aligned}$$

For $j = 2$ (second solumn), for $i = 1$ and $j = 2$

$$\begin{aligned} c_{1,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,2}^{(k+1)} \\ c_{1,2} &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_0}{(n)_0} a_{m,2}^{(k+1)} \\ c_{1,2} &= a_{1,2}^{(k+1)} = 0 = \bar{a}_{1,2}^{(k+1)}. \end{aligned}$$

For $i = 2$ and $j = 2$

$$\begin{aligned} c_{2,2} &= \sum_{m=1}^{n-k+1} \bar{a}^{(k)}_{i,m} a^{(k+1)}_{m,2} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a^{(k+1)}_{m,2} \\ &= \frac{n-k-1}{n} = \bar{a}^{(k+1)}_{2,2}. \end{aligned}$$

For $i = n+1$ and $j = 2$

$$\begin{aligned} c_{n+1,2} &= \sum_{m=1}^{n-k+1} \bar{a}^{(k)}_{n+1,m} a^{(k+1)}_{m,n+1} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a^{(k+1)}_{m,2} \\ &= \frac{n(k+1)k(k-1)...(k-n+3)(n-k-1)}{n(n-1)(n-2)...1} \\ &= \frac{n(k+1)_{n-1}(n-k-1)_1}{(n)_n} = \bar{a}^{(k+1)}_{n+1,2}. \end{aligned}$$

For $j = n-k$ (last column), for $i = 1$ and $j = n-k$

$$\begin{aligned} c_{1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}^{(k)}_{1,m} a^{(k+1)}_{m,n-k} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_{m-1}}{(n)_0} a^{(k+1)}_{m,n-k} \\ &= a^{(k+1)}_{1,n-k} = 0 = \bar{a}^{(k+1)}_{1,n-k}. \end{aligned}$$

For $i = 2$ and $j = n-k$

$$\begin{aligned} c_{2,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}^{(k)}_{2,m} a^{(k+1)}_{m,j} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a^{(k+1)}_{m,n-k} \\ &= 0 = \bar{a}^{(k+1)}_{2,n-k}. \end{aligned}$$

For $i = n+1$ and $j = n-k$

$$\begin{aligned} c_{n+1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}^{(k)}_{n+1,m} a^{(k+1)}_{m,n-k} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a^{(k+1)}_{m,n-k} \\ &= 1 = \bar{a}^{(k+1)}_{n+1,n-k}. \end{aligned}$$

Hence, $\bar{A}_k A_{k+1} = [c_{i,j}]_{(n+1)\times(n-k)} = \left[\bar{a}^{(k+1)}_{i,j}\right]_{(n+1)\times(n-k)}$, where $\bar{a}^{(k+1)}_{i,j} = \frac{\binom{i-1}{j-1}(k+1)_{i-j}(n-k-1)_{j-1}}{(n)_{i-1}}$. ■

Remark 4 *Sum of the elements in each row of the matrix* $\bar{A}_k$ *is equal to* 1.

Now in the following theorem we consider the inverse process

Theorem 5 *A rectangular Bézier patch* $P(s,t)$ *of degree* $n \times n$ *can be represented as a Triangular Bézier patch* $T(u,v,w)$ *of degree* n :

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k}\ B^n_{i,j,k}(u,v,w), \qquad u,v,w \geq 0, u+v+w=1.$$

where the control points $T_{i,j,k}$ *are determined by*

$$\begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix} = B_i B_{i-1} ... B_1 \begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} \qquad i = 0,1,2,...,n.$$

and $B_i (i = 0,1,...,n)$ *are degree elevation operators in the form*

$$B_k = \begin{bmatrix} 1-t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-t & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-t & t \end{bmatrix}_{(n-k+1)\times(n-k+2)}$$

Proof. Indeed

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B^n_i(s) B^n_j(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B^n_i(s) \left\{ t B^{n-1}_{j-1}(t) + (1-t) B^{n-1}_j(t) \right\} \\ &= \sum_{i=0}^{n} B^n_i(s) \left\{ t \sum_{j=0}^{n} P_{i,j} B^{n-1}_{j-1}(t) + (1-t) \sum_{j=0}^{n} P_{i,j} B^{n-1}_j(t) \right\} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-r} P^r_{i,j} B^n_i(s) B^{n-r}_j(t), \end{aligned}$$

where

$$\begin{aligned} &P^0_{i,j}(t) \equiv P^0_{i,j} = P_{i,j} \\ &P^r_{i,j}(t) = t P^{r-1}_{i,j+1} + (1-t) P^{r-1}_{i,j}. \end{aligned}$$

Let $r = i$,

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P^i_{i,j} B^n_i(s) B^{n-i}_j(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P^i_{i,j} \binom{n}{i} \binom{n-i}{j} s^i (1-s)^{n-i} t^j (1-t)^{n-i-j} \end{aligned}$$

if we use the following reparametrization

$$\begin{cases} s = u \\ t = \frac{v}{1-u} = \frac{v}{v+w} \end{cases}$$

we get

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i} \binom{n}{i}\binom{n-i}{j} u^{i}(1-u)^{n-i}\left(\frac{v}{v+w}\right)^{j}(1-\frac{v}{v+w})^{n-i-j}.$$

Now if $i+j+k=n$

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i} \binom{n}{i}\binom{n-i}{j} u^{i}(1-u)^{n-i}\left(\frac{v}{v+w}\right)^{j}(1-\frac{v}{v+w})^{k} \\ &= \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^{n}(u,v,w) \end{aligned}$$

$$T_{i,j,k} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i}(t).$$

For each value of i, we obtain $(n-i+1)\times(n-i+2)$ matrix B_i. ■

Theorem 6 *The product of the matrices in the above theorem $B_k B_{k-1}...B_1$ can be generalized as follows*

$$Z^{k} = B_k B_{k-1}...B_1 = \begin{bmatrix} b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} \end{bmatrix}$$

where $b_{k,j} = \binom{k}{j} t^{j}(1-t)^{k-j}$ and Z^k is $(n-k+1)\times(n+1)$ matrix.

Proof. For $k=1$,

$$Z^{1} = B_1,$$

suppose it is true for k, that is

$$B_k B_{k-1}...B_1 = Z^{k},$$

we will show that it also true for $k+1$, i.e

$$B_{k+1}B_k B_{k-1}...B_1 = Z^{k+1}.$$

Let $z_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $B_{k+1}B_k B_{k-1}...B_1$,

$$z_{i,j} = \sum_{m=1}^{n-k+1} b_{i,m}^{(k+1)} b_{m,j}^{(k)}$$

where $b_{i,m}^{(k+1)}$ is the element at the i^{th} row, m^{th} column of the matrix B_{k+1}, $b_{m,j}^{(k)}$ is the element at the m^{th} row, j^{th} column of the matrix $B_k B_{k-1}...B_1$, $i=\{1,2,...,n-k\}$ and $j=\{1,2,...,n+1\}$.

For $i=1$(first row)

$$z_{1,j} = \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = 1$ and $j = 1$

$$\begin{aligned} z_{1,1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,1}^{(k)} + b_{1,2}^{(k+1)} b_{2,1}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= (1-t)\, b_{k,0} = (1-t)^{k+1} = Z_{(1,1)}^{k+1}. \end{aligned}$$

For $i = 1$ and $j = 2$,

$$\begin{aligned} z_{1,2} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,2}^{(k)} + b_{1,2}^{(k+1)} b_{2,2}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t)\, b_{k,1} + t b_{k,0} \\ &= b_{k+1,1} = Z_{(1,2)}^{k+1}. \end{aligned}$$

For $i = 1$ and $j = n+1$,

$$\begin{aligned} z_{1,n+1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{1,2}^{(k+1)} b_{2,n+1}^{(k)} + ... + b_{1,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= 0 = Z_{(1,n+1)}^{k+1}. \end{aligned}$$

For $i = 2$ (second row)

$$z_{2,j} = \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = 2$ and $j = 1$,

$$\begin{aligned} z_{2,1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,1}^{(k)} + b_{2,2}^{(k+1)} b_{2,1}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= 0 = Z_{(2,1)}^{k+1}. \end{aligned}$$

For $i = 2$ and $j = 2$

$$\begin{aligned} z_{2,2} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,2}^{(k)} + b_{2,2}^{(k+1)} b_{2,2}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t) b_{k,0} \\ &= (1-t)^{k+1} = Z_{2,2}^{k+1}. \end{aligned}$$

For $i = 2$ and $j = n+1$,

$$\begin{aligned} z_{2,n+1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{2,2}^{(k+1)} b_{2,n+1}^{(k)} + ... + b_{2,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= t b_{k,k} = Z_{2,n+1}^{k+1}. \end{aligned}$$

For $i = n-k+1$

$$z_{n-k+1,j} = \sum_{m=1}^{n-k+1} b_{n-k+1,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = n-k$ and $j = 1$

$$\begin{aligned} z_{n-k,1} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,1}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,1}^{(k)} + ... + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= 0 = Z_{n-k,1}^{k+1}. \end{aligned}$$

For $i = n-k$ and $j = 2$

$$\begin{aligned} z_{n-k,2} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,2}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,2}^{(k)} + ... + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= 0 = Z_{n-k,2}^{k+1}. \end{aligned}$$

For $i = n-k$ and $j = n+1$

$$\begin{aligned} z_{n-k,n+1} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,n+1}^{(k)} + ... + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= tb_{k,k} = Z_{n-k,n+1}^{k+1}. \end{aligned}$$

$$\bar{A}_k A_{k+1} = \bar{A}_{k+1} = \left[\begin{array}{cc} 1 & 0 \\ \frac{k+1}{n} & \frac{n-k-1}{n} \\ \frac{(k+1)k}{n(n-1)} & \frac{2(k+1)(n-k-1)}{n(n-1)} \\ \frac{(k+1)(k)(k-1)}{n(n-1)(n-2)} & \frac{3(k+1)k(n-k-1)}{n(n-1)(n-2)} \\ \frac{(k+1)k(k-1)(k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)k(k-1)(n-k-1)}{n(n-1)(n-2)(n-3)} \\ \frac{(k+1)k(k-1)(k-2)(k-3)}{n(n-1)(n-2)(n-3)} & \frac{5(k+1)k(k-1)(k-2)(n-k-1)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \frac{(k+1)k...(k-n+3)}{n(n-1)(n-2)...2} & \frac{(n-1)(k+1)k...(k-n+4)(n-k-1)}{n(n-1)(n-2)...2} \\ \frac{(k+1)k...(k-n+2)}{n(n-1)(n-2)...1} & \frac{n(k+1)k...(k-n+3)(n-k-1)}{n(n-1)(n-2)...1} \end{array} \right.$$

$$\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{(n-k-1)(n-k-2)}{n(n-1)} & 0 \\ \frac{3(k+1)(n-k-1)(n-k-2)}{n(n-1)(n-2)} & \frac{(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)} \\ \frac{6(k+1)k(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)} \\ \frac{10(k+1)k(k-1)(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{10(k+1)k(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$$
\left.\begin{array}{cccccc}
0 & 0 & & & & \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)} & 0 & 0 & \ldots & 0 & 0 \\
\frac{5(k+1)(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)(n-k-5)}{n(n-1)(n-2)(n-3)(n-4)} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
\vdots & \vdots & & & \frac{n-k-1}{n} & \frac{k+1}{n} \\
\vdots & \vdots & & & 0 & 1
\end{array}\right] \blacksquare
$$

Conclusion 7 *As we mentioned before, mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. These two types of patches have different geometric properties so it is difficult to use both of them in the same CAD system. One may need to convert one type to another and here in this paper we studied on the conversion matrix to convert triangular Bézier patch to a rectangular Bézier patch and a rectangular Bézier patch to a triangular Bézier patch. We found simple representations for these two matrices which will allow the conversion in one step.*

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