AUM Block Geometric Mean Labeling For Star Family with an Application in Coding Theory

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ABSTRACT
This paper introduces a new concept of AUM block geometric mean labeling for star, bistar and U- star graphs. AUM block geometric mean labeling, a recent development in graph theory, involves assigning labels to blocks so that the geometric mean of the labels on adjacent points and the connecting lines satisfy specific criteria. The use of AUM block geometric mean labeling in cryptography is also presented. A method for encoding and decoding using an AUM block geometric mean labeling is given. Suitable illustrations are also presented.

Keywords: AUM Block Geometric Mean Labeling, Star Graph, Bi – Star Graph, U- Star Graph.
AMS classification: 05C78.

1. INTRODUCTION

Graph labeling was introduced by Rosa in 1967 [1], [5], [9] has since been extensively reviewed by Gallian [3]. Different types of labeling have been developed recently including difference cordiality, graceful labeling, skelom graceful labeling, geometric mean labeling, super geometric mean labeling, arithmetic mean labeling, fuzzy bi- magic and anti-magic, etc., [2], [7], [8], [10], [11], [12], [13], [14]. Labeling of blocks of a graph namely AUM block sum labeling, AUM block labeling [4], [14], [15], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]. have been recently introduced with a large scope of application in various fields. AUM block geometric mean labeling was introduced in [25]. In this paper, we have demonstrated that Star, Bi-Star, and U-Star can be labeled using the AUM block geometric mean labeling method. AUM block geometric mean labeling has vast scope application in various fields. One such application in this field of coding theory is presented in this paper.

2. PRELIMINARIES

Definition 2.1 [14]: Star graph
A Star graph $K_{1,n}$ is a tree with one internal node and n leaves.
Definition 2.2 [14]: Bi-star graph
A Bi- star graph $B_{n,n}$ is a graph obtained by joining the center vertices of two copies of $K_{1,n}$ by the edge.

Definition 2.3 [7]: U- star graph

A U-star graph is a graph obtained from the amalgamation of vertex of a graph P_n a shaped U - with 2-star graphs (S_n) where each vertex has a degree of one on graph P_n is the center of the star graph (S_n).

Definition 2.4 [6]: Plain text
The original secret text from the sender to the receiver which requires to be transformed into some version is called Plaintext.

Definition 2.5 [6]: Cipher text
The required version of our plaintext is called Ciphertext.

Definition 2.6 [26]: AUM block geometric mean labeling
Let G be a graph with p vertices, q edges and b blocks, p, q, b ≥ 1.
Let $V(G) = \{v_1, v_2, \dots, v_p\}$, $E(G) = \{e_1, e_2, \dots, e_q\}$, $B(G) = \{B_1, B_2, \dots, B_b\}$ denote the vertex set, edge set and the block set of G respectively.
We say the graph G admits AUM block geometric mean labeling if there exists a bijection $\Psi: V(G) \to Z^+$ and $\Psi^*: E(G) \to Z^+$ by $\Psi^*(uv) = \lfloor\sqrt{f(u)f(v)}\rfloor$ or $\lceil\sqrt{f(u)f(v)}\rceil$ induced from Ψ and $\Psi^{**}: B(G) \to Z^+$ defined as follows,
Let B_k be incident with the vertices $\upsilon_{k_1}, \upsilon_{k_2}, \dots \upsilon_{k_n}$, $1 \leq k_n \leq p$ and the edges $e_{k_1}, e_{k_2}, \dots e_{k_m}$, $1 \leq k_m \leq q$ then $\Psi^{**}(B_k) = \left\lfloor \sqrt[s]{\prod_{i=1}^{n} \Psi(\upsilon_{k_i}) \prod_{i=1}^{n} \Psi^*(e_{k_i})} \right\rfloor$ or $\left\lceil \sqrt[s]{\prod_{i=1}^{n} \Psi(\upsilon_{k_i}) \prod_{i=1}^{n} \Psi^*(e_{k_i})} \right\rceil$ where, s = sum of the incident number of vertices and edges of B_k and $\Psi^{**}(B_k) \neq \Psi^{**}(B_z)$ for $1 \leq k, z \leq b, k \neq z$.

3. MAIN RESULTS

This section covers AUM block geometric mean labeling for star graph, bi- star graph and U-star graph. We apply AUM block geometric mean labeling for the blocks and provide relevant examples as needed.
Theorem 3.1: Star graph $K_{1,n}, n \geq 4$, admits AUM block geometric mean labeling.
Proof: Consider the star graph $K_{1,n}$ as G with $n \geq 4$. Let $V(G) = \{\upsilon_0, \upsilon_1, \upsilon_2 \dots \upsilon_{n-1}\}$, $E(G) = \{e_1, e_2, \dots e_n\}$, $B(G) = \{b_1, b_2, \dots b_n\}$ in G, respectively.
AUM block geometric mean labeling is defined as follows,
Define $\Psi: V(G) \to Z^+$ by $\Psi(\upsilon_0) = 3, \Psi(\upsilon_i) = 5i$, for $1 \leq i \leq n$.
Ψ^* is defined as $E(G) \to Z^+$
$\Psi^*(\upsilon_0\upsilon_i) = i + 3$, for $1 \leq i \leq n$.
Define $\Psi^{**}: B(G) \to Z^+$ by $\Psi^{**}(B_i) = i + 3$, for $1 \leq i \leq n$.
For $i \neq k$, $\Psi^{**}(B_i) \neq \Psi^{**}(B_k)$.
This shows that block labels are distinct.
Hence, $K_{1,n}$ admits AUM block geometric mean labeling.

Example: 3.2

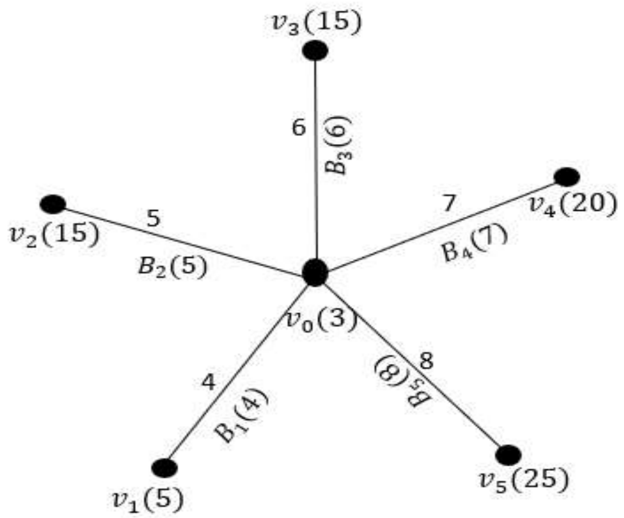

Figure 1: AUM block geometric mean labeling for Star $K_{1,5}$.

Theorem 3.3: Bi- star graph $B_{m,n}$ admits AUM block geometric mean labeling.

Proof: Consider the bi-star graph $B_{m,n}$ as G. Let $V(G) = \{\upsilon_0, \upsilon_1, \upsilon_2 \ldots \upsilon_{n-1}\}$, $E(G) = \{e_1, e_2, \ldots e_n\}$, $B(G) = \{b_1, b_2, \ldots b_n\}$ in G, respectively.

AUM block geometric mean labeling is defined for different cases:

Define Ψ: $V(G) \to Z^+$ by $\Psi(\upsilon_{i-1}) = 5i - 1$, for $1 \leq i \leq m + n + 2$.

Ψ^* is defined as $E(G) \to Z^+$

$\Psi^*(\upsilon_0\upsilon_i) = i + 5$, for $1 \leq i \leq m + 1$.

$\Psi^*(\upsilon_{m+1}\upsilon_{m+1+i}) = \sqrt{\upsilon_{m+1} * \upsilon_{m+1+i}}$, for $1 \leq i \leq n$.

Define Ψ^{**}: $B(G) \to Z^+$ by $\Psi^{**}(B_i) = i + 5$, for $1 \leq i \leq m + 1$.

$\Psi^{**}(B_{m+1+i}) = \sqrt[3]{\upsilon_{m+1} * \upsilon_{m+1+i}(\upsilon_{m+1} * \upsilon_{m+1+i})^{1/2}}$, for $1 \leq i \leq n$.

For $i \neq k$, $\Psi^{**}(B_i) \neq \Psi^{**}(B_k)$.

This shows that block labels are distinct.

Hence, $B_{m,n}$ admits AUM block geometric mean labeling.

Example: 3.4

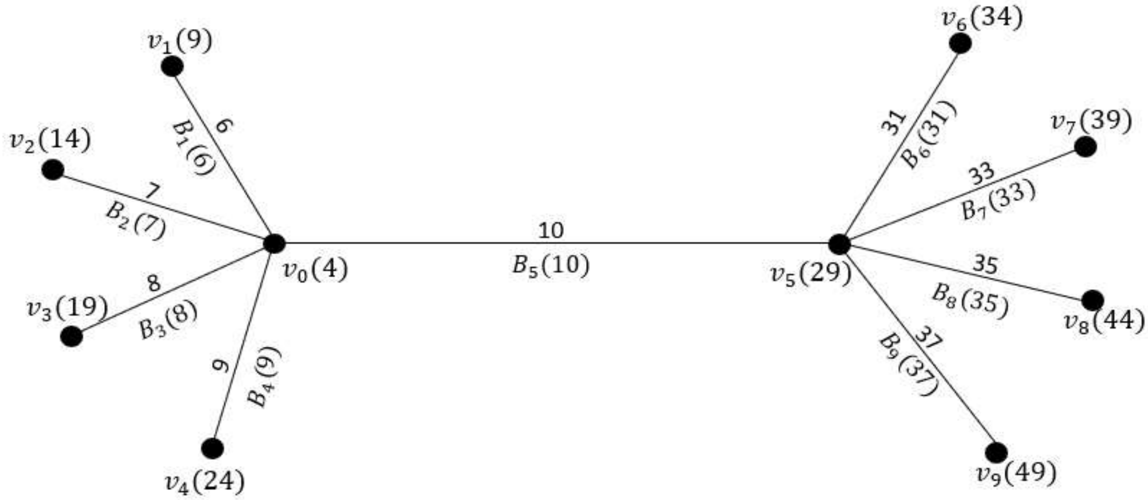

Figure 2: AUM block geometric mean labeling for Bi- Star $B_{4,4}$

Example: 3.5

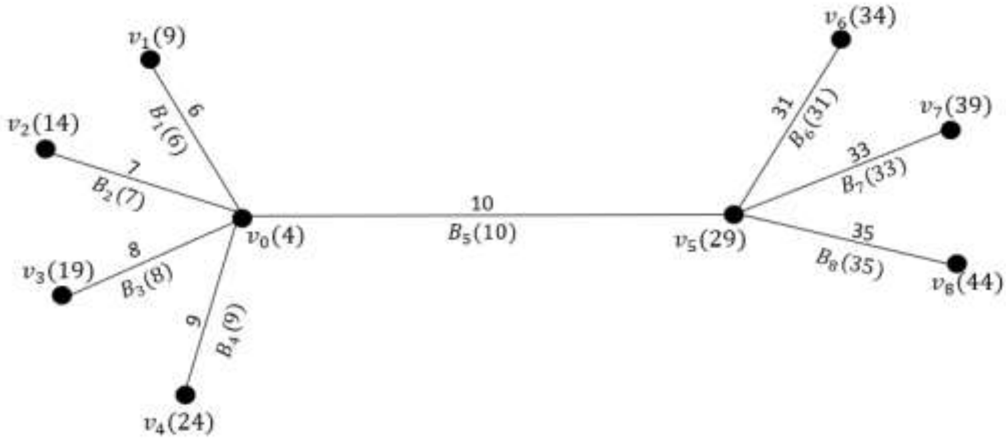

Figure 3: AUM block geometric mean labeling for Bi- Star $B_{4,3}$

Example: 3.6

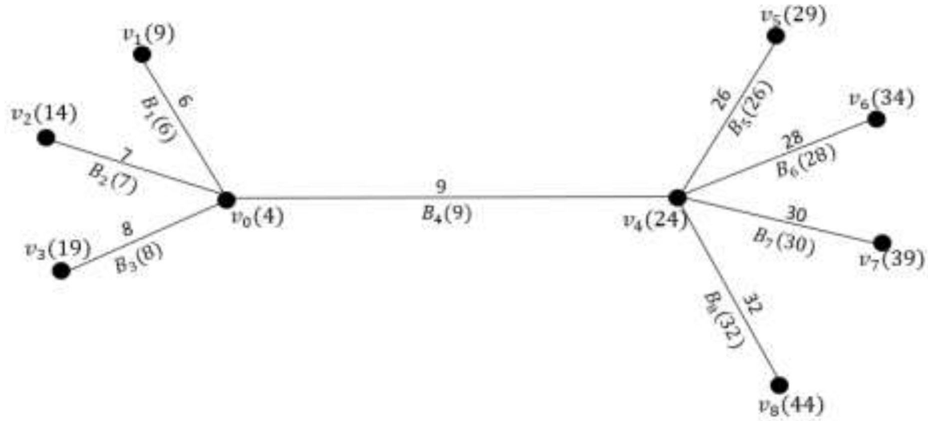

Figure 4: AUM block geometric mean labeling for Bi- Star $B_{3,4}$

Theorem 3.7: U- star $U(S_n)$ graph with $n \geq 2$ admits AUM block geometric mean labeling.
Proof: Consider the U-star graph $U(S_n)$ as G. Let $V(G) = \{ \upsilon_1, \upsilon_2, \dots \upsilon_n\}$, $E(G) = \{e_1, e_2, \dots e_{n-1}\}$, B(G)= $\{ b_1, b_2, \dots b_{n-1}\}$ in G, respectively.
AUM block geometric mean labeling is defined as follows,
Define Ψ: $V(G) \to Z^+$ by $\Psi(\upsilon_i) = 4i - 3$, for $1 \leq i \leq 2n + 4$.
Ψ^* is defined as $E(G) \to Z^+$
$\Psi^*(\upsilon_{n+1}\upsilon_{i+1}) = 2(i + 2)$, for $1 \leq i \leq n$.
$\Psi^*(\upsilon_i\upsilon_n) = i + 2$, for $i = 1$.
$\Psi^*(\upsilon_{n+4}\upsilon_{n+4+i}) = \sqrt{(\upsilon_{n+4}\upsilon_{n+4+i})}$, for $1 \leq i \leq n$,
$\Psi^*(\upsilon_{n+i}\upsilon_{n+i+1}) = 2(2i + 5)$, for $1 \leq i \leq 3$
Define Ψ^{**}: $B(G) \to Z^+$ by $\Psi^{**}(B_i) = i + 2$, for $i = 1$.
$\Psi^{**}(B_{i+1}) = 2(i + 2)$, for $1 \leq i \leq n - 1$.
For $i \neq k$, $\Psi^{**}(B_i) \neq \Psi^{**}(B_k)$.

This shows that block labels are distinct.
Hence, $U(S_n)$ admits AUM block geometric mean labeling.

Example:3.8

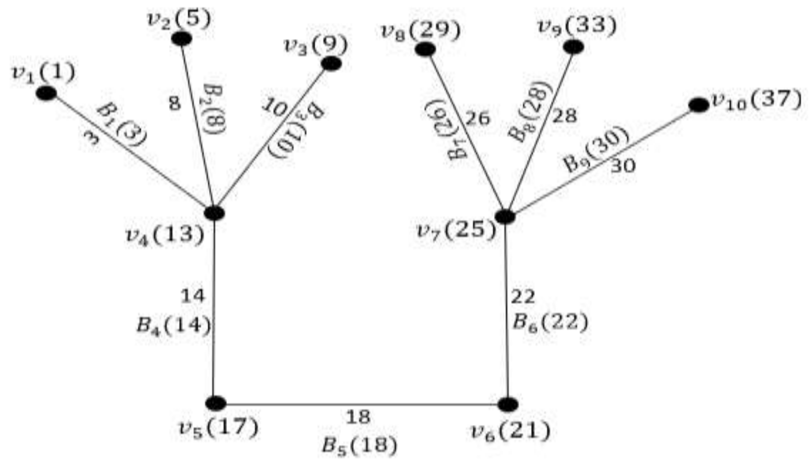

Figure 5: AUM block geometric mean labeling for U- Star $U(S_3)$

4. APPLICATION

In this section, we present new algorithms for encoding and decoding for messages in a more secured transmission channel.

A) New method for Encoding using AUM block geometric mean labeling

- For the first 13 alphabets assign to 1,3,5…25 and the last 13 alphabets assign to 0,2,4, 6, . . . 24.
- To apply shift cipher to each alphabet using $a_i = (a + l)(\text{mod } 26)$, each alphabet is proceeded to l positions. where, l represents the message length and a corresponds to that specific letter.
- Determine the least integer $b_i > 0$ in a way that $GM(a_i, b_i)$ is a natural number and assume c_j for $i = 1$ to n and j= $n + 1, n + 2, \dots 2n$. Such a b_i is assured, if $b_i = a_i$.
- Consider the star graph S_{2n}. First, label the blocks in S_n as b_i. Then, label the rest of blocks in S_{2n} as c_j, arranging them sequentially.
- The Star graph should be labeled using AUM block geometric mean labeling for the 2n blocks.
- The labels of the block are to be reassigned as $BK_i = b_i + BK(i)$ for $i = 1$ to n and $BK_j = c_j + BK(j)$ for j= $n + 1, n + 2, \dots 2n$ where $B(i)$ represents AUM block geometric mean label.
- The encrypted message should be transmitted as a Star graph, using the new label for decryption.

B) New method for Decoding using AUM block geometric mean labeling

- Determine the AUM block geometric mean label for the provided star graph (Key graph).
- Calculate the integers $b_i = BK_i - BK(i)$ for $i = 1$ to n and $c_j = BK_j - BK(j)$ for $j = n + 1$ to $2n$. Here B_i represents the labels in the key graph blocks, and BK(i) denotes the AUM block geometric mean labels for the same graph.
- Determine the integer $a_i > 0$ using $a_i = \frac{c_j^2}{b_i}$ for all $i = 1,2,3, \dots n$ and $j = 1,2,3, \dots 2n$.

- Determine the value of F_i using the formula $F_i = \left[a_i - \left(\frac{\text{no.of blocks}}{2} \right) \right] \pmod{26}$. Refer to the encoding table to map each f_i to its corresponding letter and reveal the plaintext.

C) Decoding key

To decipher the message, the receiver will need both the Star graph and the updated label provided by the sender.

Illustration:

This work includes an example to demonstrate the algorithms discussed.

Consider the message **"BOOK"** that needs to be coded.

Determine the number assigned to each alphabet by referring to the encoding table.

Table 1: Encoding Table

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
1	3	5	7	9	11	13	15	17	19	21	23	25
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	2	4	6	8	10	12	14	16	18	20	22	24

The table below shows the image of each alphabet:

B	O	O	K
3	4	4	21

Length, l = 4.

Using shift cipher $a_i = (a + l) \pmod{26}$, **we get**

B	O	O	K
3	4	4	21
7	8	8	25

$\mathbf{a_1 = 7, a_2 = 8, a_3 = 8, a_4 = 25}$

Determine the least integer $b_i > 0$ in a way that $GM(a_i, b_i)$ is a positive integer and assume c_j for $i = 1$ to n and $j = n + 1$ to $2n$.

$GM(a_1, b_1) = GM(7,7) = 7 = c_5$ $\quad$ $GM(a_2, b_2) = GM(8,2) = 4 = c_6$

$GM(a_3, b_3) = GM(8,16) = 12 = c_7$ $\quad$ $GM(a_4, b_4) = GM(25,1) = 5 = c_8$

Analyse the star graph S_8 and label the first four blocks using b_i values. For the last block of the graph, apply the label c_j as specified.

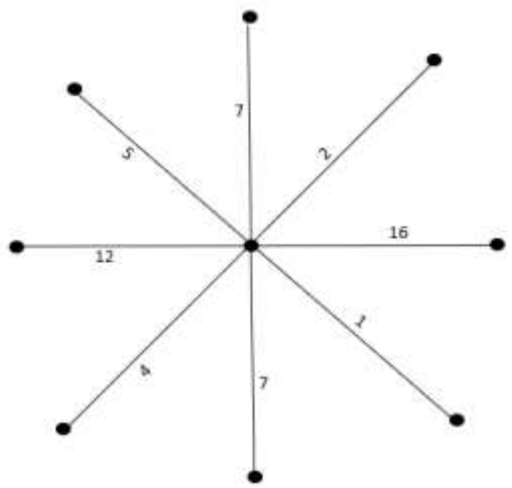

Figure 1 Using the AUM block geometric mean labeling for S_{2n} blocks, as in Figure 2.

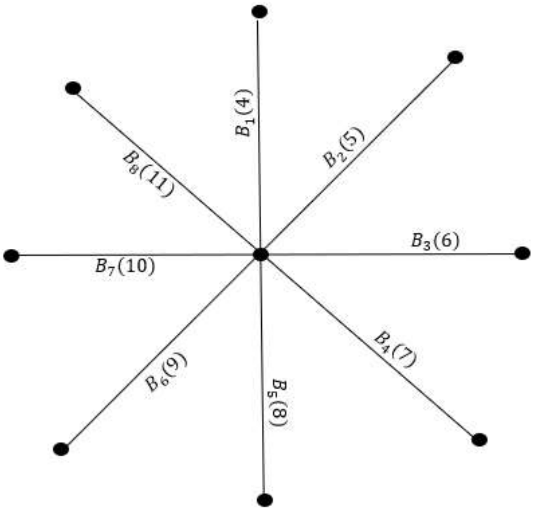

Figure 2 Reallocate the block labeling as $B\,K_i = b_i + B\,K(i)$ for $i = 1,2,3, \ldots n$ and $B\,k_j = c_j + B\,K(j)$ for j= 5,6,7,8 where $B\,(i)$ represents AUM block geometric mean label.

$B\,K_1 = b_1 + B\,K\,(1) = 7 + 4 = 11$

$B\,K_2 = b_2 + B\,K\,(2) = 2 + 5 = 7$

$B\,K_3 = b_3 + B\,K\,(3) = 18 + 6 = 24$

$B\,K_4 = b_4 + B\,K\,(4) = 1 + 7 = 8$

$B\,K_5 = c_5 + B\,K\,(5) = 7 + 8 = 15$

$B\,K_6 = c_6 + B\,K\,(6) = 4 + 9 = 13$

$B\,K_7 = c_7 + B\,K\,(7) = 12 + 10 = 22$

$B_{\,8} = c_8 + B\,K\,(8) = 5 + 11 = 16$

The figure 3 shows relabelled graph.

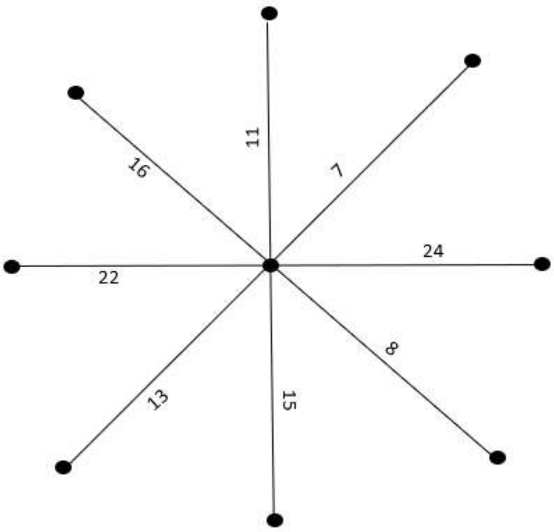

Figure 3 Transmit the assigned graph to the receiver for decryption.

Decoding key: The labeled graph above serves as the key for decoding the message.
Decoding: Once we receive the labeled graph, use decoding alogrithm B to the determine the plaintext.
Refer figure 2.

Calculate the integers b_i and c_j using the star graph (figure 3) along with AUM block geometric mean labeling. For $b_i = BK_i - BK(i)\ for\ i = 1\ to\ n\ and\ c_j = BK_j - BK(j)\ for\ j = n + 1\ to\ 2n$.
Here B_i represents the labels in the key graph blocks, and $B(i)$ denotes the AUM block geometric mean labels for the same graph.

$$b_1 = BK_1 - BK(1) = 11 - 4 = 7 \qquad c_5 = BK_5 - BK(5) = 15 - 8 = 7$$
$$b_2 = BK_2 - BK(2) = 7 - 5 = 2 \qquad c_6 = BK_6 - BK(6) = 13 - 9 = 4$$
$$b_3 = BK_3 - BK(3) = 24 - 6 = 18 \qquad c_7 = BK_7 - BK(7) = 22 - 10 = 12$$
$$b_4 = BK_4 - BK(4) = 8 - 7 = 1 \qquad c_8 = BK_8 - BK(8) = 16 - 11 = 5$$

The integer a_i using $\mathrm{a_i} = \frac{\mathrm{c_j^2}}{\mathrm{b_i}}$ for all i = 1,2,3,4 and j = 5,6,7,8.

$$\mathrm{a_1} = \frac{\mathrm{c_5^2}}{\mathrm{b_1}} = \frac{7^2}{7} = 7 \qquad \mathrm{a_3} = \frac{\mathrm{c_7^2}}{\mathrm{b_3}} = \frac{12^2}{18} = 8$$
$$\mathrm{a_2} = \frac{\mathrm{c_6^2}}{\mathrm{b_2}} = \frac{4^2}{2} = 8 \qquad \mathrm{a_4} = \frac{\mathrm{c_8^2}}{\mathrm{b_4}} = \frac{5^2}{1} = 25$$

To determine the plain text, use $\mathrm{F_i} = \left[\mathrm{a_i} - \left(\frac{\text{no.of blocks}}{2}\right)\right] (\mathrm{mod}\ 26)$

$$\mathrm{F_1} = \left[\mathrm{a_1} - \left(\frac{\text{no. of blocks}}{2}\right)\right] (\mathrm{mod}\ 26) = \left[7 - \frac{8}{2}\right] (\mathrm{mod}\ 26) = 3$$
$$\mathrm{F_2} = \left[\mathrm{a_2} - \left(\frac{\text{no. of blocks}}{2}\right)\right] (\mathrm{mod}\ 26) = \left[8 - \frac{8}{2}\right] (\mathrm{mod}\ 26) = 4$$
$$\mathrm{F_3} = \left[\mathrm{a_3} - \left(\frac{\text{no. of blocks}}{2}\right)\right] (\mathrm{mod}\ 26) = \left[8 - \frac{8}{2}\right] (\mathrm{mod}\ 26) = 4$$
$$\mathrm{F_4} = \left[\mathrm{a_4} - \left(\frac{\text{no. of blocks}}{2}\right)\right] (\mathrm{mod}\ 26) = \left[25 - \frac{8}{2}\right] (\mathrm{mod}\ 26) = 21$$

From Encoding table, the plain text is **BOOK**.

5. CONCLUSION

This paper introduces the AUM block geometric mean for star graphs, bi-star graphs, and U-star graphs. It also presents new methods for encryption and decryption, which transform plain text into cipher text using numbered alphabets. The integration of AUM block geometric mean labeling on star graphs enhances these coding techniques, making them applicable to complex scenarios involving the conversion of cipher text back to plain text. An illustrative example is provided. In the future, we aim to create new coding methods using different types of graphs.

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