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Theoretical and Numerical Discussion for the Mixed Integro–Differential Equations

M. E. Nasr [1] and M. A. Abdel-Aty [2]

[1,2] Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt

[1] Department of Mathematics, Collage of Science and Arts -Al Qurayyat, Jouf University, Kingdom of Saudi Arabia

Abstract

In this paper, we tend to apply the proposed modified Laplace Adomian decomposition method that is the coupling of Laplace transform and Adomian decomposition method. The modified Laplace Adomian decomposition method is applied to solve the Fredholm–Volterra integro–differential equations of the second kind in the space $L_2[a,b]$. The nonlinear term will simply be handled with the help of Adomian polynomials. The Laplace decomposition technique is found to be fast and correct. Several examples are tested and also the results of the study are discussed. The obtained results expressly reveal the complete reliability, efficiency, and accuracy of the proposed algorithmic rule for solving the Fredholm–Volterra integro–differential equations and therefore will be extended to other problems of numerous nature.

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1. Introduction

Mathematical modeling of real-life problems usually results in functional equations, such as differential, integral, and integro-differential equations. Many mathematical formulations of physical phenomena reduced to integro-differential equations, like fluid dynamics, biological models, chemical mechanics and contact problems, see [6, 14, 19].

Many problems from physics and engineering and alternative disciplines cause linear and nonlinear integral equations. Now, for the solution of those equations several analytical and numerical methods are introduced, however numerical methods are easier than analytical methods and most of the time numerical methods are used to solve these equations we refer to [1, 2, 18].

Laplace Adomians decomposition method was first introduced by Suheil A. Khuri [16,17] and has been with successfully used to find the solution of differential equations [20]. This method generates a solution in the form of a series whose terms are determined by a recursive relation using the Adomian polynomials.

Most of the nonlinear integro-differential equations don't have an exact analytic solution, therefore approximation and numerical technique should be used, there are only a number of techniques for the solution of integro-differential equations, since it's relatively a new subject in arithmetic.

The modified laplace decomposition technique has applied for solving some nonlinear ordinary, partial differential equations. Recently, the authors have used many methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integrodifferential equations of the second kind [8, 9, 11, 12, 21].

In this paper, we consider the Fredholm–Volterra integro–differential equations of the second kind with continuous kernels with respect to position. We applied Laplace transform and Adomian polynomials to solve nonlinear Fredholm–Volterra integro–differential equations. Al–Towaiq and Kasasbeh [7] have applied the modification of Laplace decomposition method to solve linear interval Fredholm integro–differential equations of the form :

$$u'(x) = f(x) + \int_a^b k(x,t)u(t)dt; \qquad u(a) = \alpha.$$

But in this paper, we will study the modification of Laplace Adomian decomposition method to solve the nonlinear interval Fredholm–Volterra integro–differential equation of the form:

$$\phi(u+q) = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{1}$$

where q is the Phase-Lag is positive, very small and assumed to be intrinsic properties of the medium. The constant parameter λ may be complex and has many physical meanings, the function $\phi(u)$ is unknown in the Banach space and continuous with their derivative with respect to time in the space $L_2[a,b]$, where $[a,b]$ is the domain of integration with respect to the position and it's called the potential function of the mixed integral equation. The kernels $k(u,v), \psi(u,v)$ are positive and continuous in $L_2[a,b]$ and the known function $p(u)$ is continuous and its derivatives with respect to position.

Using Taylor Expansion after neglecting the second derivative in the equation (1) we get,

$$\phi(u) + q\frac{d\phi(u)}{du} = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{2}$$

with initial condition,

$$\phi(a) = \alpha. \tag{3}$$

The equation (2) with initial condition (3) is called Integro-Differential Equation for the Phase-Lag. The Integro-Differential Equation is a kind of functional equation that has associate integral and derivatives of unknown function. These equations were named after the leading mathematicians who have first studied them such as Fredholm, Volterra. Fredholm and Volterra equations are the most encountered types, see [10]. There is, formally only one difference between them, in the Fredholm equation the region of integration is fixed where in the Volterra equation the region is variable. Integro-Differential Equations (IDEs) are given as a combination of differential and integral equations.

2. Preliminaries

In this section, we give some definitions and properties of the Adomian polynomials and Laplace transform.

2.1 Laplace transform

Definition 1. The Laplace transform of a function $\phi(u); u > 0$ is defined as

$$L[\phi(u)] = \Phi(s) = \int_0^{+\infty} e^{-su}\phi(u)du, \tag{4}$$

where s can be either real or complex.

Definition 2. Given two functions ϕ and ψ, we define, for any $u > 0$,

$$(\phi * \psi)(u) = \int_0^u \phi(v)\psi(u-v)dv, \tag{5}$$

the function $\phi * \psi$ is called the convolution of ϕ and ψ.

Theorem 1. *The convolution theorem*

$$L[\phi * \psi](u) = L[\phi(u)] * L[\psi(u)]. \tag{6}$$

Lemma 1. *Laplace Transform of an Integral: If $\Phi(s) = L[\phi(u)]$ then*

$$L\left[\int_0^u \phi(v)dv\right] = \frac{\Phi(s)}{s}. \tag{7}$$

Theorem 2. *The Laplace transform $L[\phi(u)]$ of the derivatives are defined by*

$$L[\phi^{(n)}(u)] = s^n L[\phi(u)] - s^{n-1}\phi(0) - s^{n-2}\phi'(0) - \cdots - \phi^{(n-1)}(0). \tag{8}$$

2.2 Adomians Decomposition method

Consider the general functional equation:

$$\phi = p + N_1\phi + N_2\phi, \tag{9}$$

where N_1, N_2 are a nonlinear operators, p is a known function, and we are seeking the solution ϕ satisfying (9). We assume that for every p , Eq. (9) has one and only one solution.

The Adomians technique consists of approximating the solution of (9) as an infinite series

$$\phi = \sum_{n=0}^{\infty} \phi_n, \tag{10}$$

and decomposing the nonlinear operators N_1, N_2 as respectively

$$N_1\phi = \sum_{n=0}^{\infty} A_n, \qquad N_2\phi = \sum_{n=0}^{\infty} B_n, \tag{11}$$

where A_n, B_n are polynomials (called Adomian polynomials)of $\{\phi_0, \phi_1, \ldots, \phi_n\}$ [4,5] given by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0, 1, 2, \ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_2\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0, 1, 2, \ldots \end{aligned}$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty}\phi_n, \sum_{n=0}^{\infty}A_n$ and $\sum_{n=0}^{\infty}B_n$ are given in [3,13]. Substituting (10) and (11) into (9) yields, we get

$$\sum_{n=0}^{\infty}\phi_n = p + \sum_{n=0}^{\infty}A_n + \sum_{n=0}^{\infty}B_n.$$

Thus, we can identify

$$\begin{aligned} \phi_0 =& p, \\ \phi_{n+1} =& A_n(\phi_0, \phi_1, \ldots, \phi_n) + B_n(\phi_0, \phi_1, \ldots, \phi_n); \quad n = 0, 1, 2, \ldots \end{aligned}$$

Thus all components of ϕ can be calculated once the A_n, B_n are given. We then define the n-terms approximate to the solution ϕ by

$$\Psi_n[\phi] = \sum_{i=0}^{n-1}\phi_i \quad , with \quad \lim_{n\to\infty}\Psi_n[\phi] = \phi.$$

3. Description of the Method

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the Fredholm–Volterra integro–differential equation. Applying the Laplace transform (denoted by L) on the both sides of the equation yield (2), we have

$$\begin{aligned} L[\phi(u)] + qL\left[\frac{d\phi(u)}{du}\right] =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right], \end{aligned} \tag{12}$$

using the differentiation property of Laplace transform (8) we get

$$\begin{aligned} L[\phi(u)] + qsL[\phi(u)] - q\phi(0) =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{13}$$

Thus, the given equation is equivalent to

$$\begin{aligned} L[\phi(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)} L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ + \frac{\lambda}{(1+qs)} L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{14}$$

The Adomian decomposition method and the Adomian polynomials can be used to handle (14) and to address the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$. We first represent the linear term $\phi(u)$ at the left side by an infinite series of components given by

$$\phi(u) = \sum_{n=0}^{\infty} \phi_n(u), \tag{15}$$

where the components $\phi_n; n \geq 0$ will be determined recursively. However, the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$ at the right side of Eq. (14) will be represented by an infinite series of the Adomian polynomials A_n, B_n respectively in the form

$$\mu(v,\phi(v)) = \sum_{n=0}^{\infty} A_n(v), \qquad \nu(v,\phi(v)) = \sum_{n=0}^{\infty} B_n(v), \tag{16}$$

where $A_n, B_n; n \geq 0$ are defined by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\nu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \end{aligned}$$

where the so-called Adomian polynomials A_n, B_n can be evaluated for all forms of nonlinearity [22]. In other words, assuming that the nonlinear function is $\mu(v,\phi(v)), \nu(v,\phi(v))$, therefore the Adomian polynomials are given by

$$\begin{aligned} A_0 &= \mu(\phi_0), & B_0 &= \nu(\phi_0), \\ A_1 &= \phi_1\mu'(\phi_0), & B_1 &= \phi_1\nu'(\phi_0), \\ A_2 &= \phi_2\mu'(\phi_0) + \frac{1}{2}\phi_1^2\mu''(\phi_0), & B_2 &= \phi_2\nu'(\phi_0) + \frac{1}{2}\phi_1^2\nu''(\phi_0). \end{aligned}$$

Substituting (15) and (16) into (14), we will get

$$\begin{aligned} L\left[\sum_{0}^{\infty}\phi_n(u)\right] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)\sum_{0}^{\infty}A_n(v)dv\right] \\ + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)\sum_{0}^{\infty}B_n(v)dv\right]. \end{aligned} \tag{17}$$

The Adomian decomposition method presents the recursive relation

$$L[\phi_0(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}, \tag{18}$$

$$L[\phi_1(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_0(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_0(v)dv\right], \tag{19}$$

$$L[\phi_2(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_1(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_1(v)dv\right]. \tag{20}$$

In general, the recursive relation is given by

$$\begin{aligned} L[\phi_{n+1}(u)] =& \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right] \\ &+ \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right], \quad n = 0,1,2,\ldots \end{aligned} \tag{21}$$

A necessary condition for Eq. (21) to work is that

$$\lim_{s\to\infty}\frac{\lambda}{(1+qs)} = 0.$$

Applying inverse Laplace transform to Eqs. (18)–(21), so our required recursive relation

$$\phi_0(u) = G(u), \tag{22}$$

and

$$\begin{aligned} \phi_{n+1}(u) =& L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right]\right] \\ &+ L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right], \end{aligned} \tag{23}$$

where $G(u)$ may be a function that arises from the source term and also the prescribed initial conditions, the initial solution is very important, the choice of (22) as the initial solution always leads to noise oscillation during the iteration procedure, the modified laplace decomposition method [15] suggests that the operate $G(u)$ defined above in (18) be rotten into two parts:

$$G(u) = G_1(u) + G_2(u).$$

Instead of iteration procedure (22) and (23), we suggest the following modification

$$\begin{aligned}
\phi_0(u) &= G_1(u),\\
\phi_1(u) &= G_2(u) + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_0(v)dv\right]\right]\\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_0(v)dv\right]\right],\\
\phi_{n+1}(u) &= L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right]\right]\\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right], \quad n=0,1,2,\ldots
\end{aligned}$$

We then define the n-terms approximate to the solution $\phi(u)$ by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1}\phi_i(u), \quad with \quad \lim_{n\to\infty}\Psi_n[\phi(u)] = \phi(u).$$

In this paper, the obtained series solution converges to the exact solution.

3.1 A Test of Convergence

In fact, on every interval the inequality $\|\phi_{i+1}\|_2 < \beta\|\phi_i\|_2$ is required to hold for $i = 0, 1, \ldots, n$, wherever $0 < \beta < 1$ may be a constant and n is that the maximum order of the approximate used in the computation. Of course, this is often only a necessary condition for convergence, as a result of it might be necessary to compute $\|\phi_i\|_2$ for each $i = 0, 1, \ldots, n$ so as to conclude that the series is convergent.

4. Application of the Laplace transform–Adomian decomposition method

In this section, the Laplace transform–Adomian decomposition method for solving Fredholm–Volterra integro–differential equation is illustrated in the two examples given below. To show the high accuracy of the solution results from applying the present method to our problem (2) compared with the exact solution, the maximum error is defined as:

$$R_n = \|\phi_{Exact}(u) - \Psi_n[\phi(u)]\|_\infty,$$

where $n = 1, 2, \ldots$ represents the number of iterations.

Example 1

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u+0.2) = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv, \tag{24}$$

where

$$p(u) = \frac{1}{12}(-3 - u^3 cos(u)).$$

Using Taylor Expansion after neglecting the second derivative in the equation (24) we get,

$$\phi(u) + 0.2\frac{d\phi(u)}{du} = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv; \quad \phi(0) = 0. \tag{25}$$

The exact solution for this problem is

$$\phi(u) = cos(u) - sin(u).$$

First, we apply the Laplace transform to both sides of (25)

$$L[\phi(u)] + 0.2L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{26}$$

Using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.2sL[\phi(u)] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{27}$$

or equivalently

$$L[\phi(u)] = \frac{L[p(u)]}{1+0.2s} + \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u \phi^3(v)dv\right]. \tag{28}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u), \phi^3(u)$ as given above in (15) and (16) respectively into Eq. (28) we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] &= \frac{L[p(u)]}{1+0.2s} + \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\sum_{n=0}^{\infty}A_n(v)dv\right] \\ &\quad + \frac{1}{1+0.2s}L\left[\int_0^u \sum_{n=0}^{\infty}B_n(v)dv\right]. \end{aligned} \tag{29}$$

The recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{L[p(u)]}{1+0.2s}, \\ L[\phi_1(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_0(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_n(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_n(v)dv\right], \end{aligned} \tag{30}$$

where A_n, B_n are the Adomian polynomials for the nonlinear terms $\phi^2(u), \phi^3(u)$ respectively. The Adomian polynomials for $\mu(v,\phi(v)) = \phi^2(u), \nu(v,\phi(v)) = \phi^3(u)$ are given by

$$\begin{aligned} A_0 &= \phi_0^2, & B_0 &= \phi_0^3, \\ A_1 &= 2\phi_0\phi_1, & B_1 &= 3\phi_0^2\phi_1, \\ A_2 &= 2\phi_0\phi_2 + \phi_1^2, & B_2 &= 3\phi_0^2\phi_2 + 3\phi_0\phi_1^2, \\ A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, & B_3 &= 3\phi_0^2\phi_3 + 6\phi_0\phi_1\phi_2 + \phi_1^3. \end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (30), and using the recursive relation (30) gives

$$\begin{aligned} \phi_0(u) &= 1 - u - u^2 + \frac{1}{2}u^3 + \frac{1}{12}u^4 - \dots \\ \phi_1(u) &= \frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{1}{8}u^4 + \frac{1}{6}u^5 + \dots \\ \phi_2(u) &= \frac{1}{12}u^4 - \frac{1}{12}u^5 + \dots \end{aligned} \tag{31}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1} \phi_i(u) = \left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right) \quad n = 1, 2, \dots$$

$$\phi(u) = \lim_{n\to\infty} \Psi_n[\phi(u)] = \lim_{n\to\infty} \left[\left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = cos(u) - sin(u).$$

Example 2

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u + 0.01) = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv, \tag{32}$$

where

$$p(u) = 1 - \frac{1}{4}e^{-u} + 0.0100502e^u.$$

Using Taylor Expansion after neglecting the second derivative in the equation (32) we get,

$$\phi(u) + 0.01\frac{d\phi(u)}{du} = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv; \quad \phi(0) = 1. \tag{33}$$

The exact solution for this problem is

$$\phi(u) = e^u.$$

First, we apply the Laplace transform to both sides of (33)

$$L[\phi(u)] + 0.01L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{34}$$

using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.01sL[\phi(u)] - 0.01 = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{35}$$

or equivalently

$$\begin{aligned} L[\phi(u)] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \phi(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\phi^2(v)dv\right]. \end{aligned} \tag{36}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u)$ as given above in (15) and (16) respectively into above equation, we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \sum_{n=0}^{\infty}\phi_n(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\sum_{n=0}^{\infty}A_n(v)dv\right], \end{aligned} \tag{37}$$

the recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s}, \\ L[\phi_1(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_0(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_n(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_n(v)dv\right], \end{aligned} \tag{38}$$

where A_n are the Adomian polynomials for the nonlinear terms $\phi^2(u)$. The Adomian polynomials for $\mu(v,\phi(v)) = \phi^2(u)$ is given by

$$\begin{aligned} A_0 &= \phi_0^2, \\ A_1 &= 2\phi_0\phi_1, \\ A_2 &= 2\phi_0\phi_2 + \phi_1^2, \\ A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, \end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (38), and using the recursive relation (38) gives

$$\begin{aligned}\phi_0(u) &= 1 + u - \frac{1}{2}u^3 - \frac{1}{2}u^4 - \frac{13}{40}u^5 + \dots \\ \phi_1(u) &= \frac{1}{2}u^2 + \frac{2}{3}u^3 + \frac{5}{12}u^4 + \frac{7}{120}u^5 + \dots \\ \phi_2(u) &= \frac{1}{8}u^4 + \frac{11}{40}u^5 + \dots\end{aligned} \tag{39}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1} \phi_i(u) = \left(1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5 + \dots\right) \quad n = 1, 2, \dots$$

$$\phi(u) = \lim_{n\to\infty} \Psi_n[\phi(u)] = \lim_{n\to\infty} \left[\left(1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5 + \dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = e^u.$$

5. Conclusions

In this work, the Laplace decomposition technique has been successfully applied to finding the approximate solution of the nonlinear Fredholm–Volterra integro–differential equation. The method is extremely powerful and efficient find analytical moreover as numerical solutions for wide classes of nonlinear Fredholm–Volterra integro–differential equations. It provides a lot of realistic series solutions that converge very rapidly in real physical issues.

The main advantage of this technique is that the fact that it provides the analytical solution. Some examples are given and therefore the results reveal that the method is extremely effective. some of the nonlinear equations are examined by the modified technique to Illustrate the effectiveness and convenience of this technique, and in all cases, the modified technique performed excellently. The results reveal that the proposed technique is extremely effective and easy.

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