DRYGAS FUNCTIONAL EQUATIONS WITH EXTRA TERMS AND ITS STABILITY

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Abstract. In this paper, we consider the generalized Hyers-Ulam stability for the following functional equation with an extra term G_f

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y),$$

where G_f is a functional operator of f.

1. Introduction and preliminaries

In 1940, Ulam [12] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [6] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \ge 0$ and p with $p < 1$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [11] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [10] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for the quadratic functional equation and Park [9] proved the generalized Hyers-Ulam stability of the quadratic functional eqution in Banach modules over a C^*-algebra.

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In this paper, we are interested in what kind of terms can be added to the Drygas functional equation [4]

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation. We denote the added term by $G_f(x,y)$ which can be regarded as a functional operator depending on the variables x, y, and functions f. Then the new functional equation can be written as

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y). \tag{1.2}$$

In fact, the functional operator $G_f(x,y)$ was introduced and considered in the cases of additive, quadratic functional equations with somewhat different point of view by the authors([7], [8]).

2. Solutions of 1.2 as additive-quadratic mappings

Let X and Y be normed spacese. For given $l\in\mathbb{N}$ and any $i\in\{1,2,\cdots,l\}$, let $\sigma_i : X\times X\longrightarrow X$ be a binary operation such that

$$\sigma_i(rx,ry)=r\sigma_i(x,y)$$

for all $x,y\in X$ and all $r\in\mathbb{R}$. It is clear that $\sigma_i(0,0)=0$. Also let $F:Y^l\longrightarrow Y$ be a linear, continuous function. For a map $f:X\longrightarrow Y$, define

$$G_f(x,y)=F(f(\sigma_1(x,y)),f(\sigma_2(x,y)),\cdots,f(\sigma_l(x,y))).$$

From now on, for any mapping $f:X\longrightarrow Y$, we deonte

$$f_o(x)=\frac{f(x)-f(-x)}{2},\ \ f_e(x)=\frac{f(x)+f(-x)}{2}$$

First, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)-cf(y-x)\\ &=(a+b)f(x)-cf(-x)+(a-c)f(y)+bf(-y) \end{aligned} \tag{2.1}$$

for fixed real numbers a,b,c with $a=b-c$ and $a\neq 0$. We can easily show the following lemma.

Lemma 2.1. *Let $f:X\longrightarrow Y$ be a mapping. Then f satisfies (2.1) if and only if f is an additive-quadratic mapping.*

Definition 2.2. The functional operator G is called *additive-quadratic* if whenever $G_h(x,y)=0$ for all $x,y\in X$, h is an additive-quadratic mapping.

Lemma 2.3. *Let $f : X \longrightarrow Y$ be a mapping satisfying (1.2) and G additve-quadratic. Then the following are equivalent :*
(1) f is additive-quadratic,
(2) the following equality

$$G_f(x,y)=-G_f(y,x) \tag{2.2}$$

holds for all x, $y\in X$, and
(3) there exist real numbers b,c such that $b\neq c$ and

$$bG_f(x,y)=cG_f(y,x) \tag{2.3}$$

holds for all x, $y\in X$.

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Proof. (1) (⇒) (2) (⇒) (3) are trivial.

(3) (⇒) (1) By (2.2), we have $f(0) = 0$ and by (1.2), we have

$$\begin{aligned} G_f(x,y) &= 2f(x) + f(y) + f(-y) - f(x+y) - f(x-y), \text{ and} \\ G_f(y,x) &= 2f(y) + f(x) + f(-x) - f(x+y) - f(y-x) \end{aligned}$$

for all $x, y \in X$. Hence by (2.3), we have

$$(b+c)f(x+y)+bf(x-y)-cf(y-x) = (2b+c)f(x)+cf(-x)+(b+2c)f(y)+bf(-y)$$

for all $x, y \in X$ and by Lemma 2.1, we have that f is additive-quadratic. □

3. The generalized Hyers-Ulam stability of (1.2)

In this section, we deal with the generalized Hyers-Ulam stability of (1.2). Throughout this paper, assume that G is additive-quadratic and the following inequalities hold

$$\begin{aligned} &\|G_h(x,x)\| \le \|G_h(0,x)\| + \sum_{i=1}^{t} |b_i|\|G_h(\delta_i x,0)\| \ \ if \ \ h : odd, \\ &\|G_h(x,x)\| \le \sum_{i=1}^{r} |p_i|\|G_h(0,\alpha_i x)\| + \sum_{i=1}^{s} |a_i|\|G_h(\lambda_i x,0)\| \ \ if \ \ h : even \end{aligned} \tag{3.1}$$

for some $r, s, t \in \mathbb{N} \cup \{0\}$, some real numbers $p_i, a_i, b_i, \alpha_i, \lambda_i,$ and δ_i and for all $x \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping such that

$$\|f(x+y) + f(x-y) + G_f(x,y) - 2f(x)\| \le \phi(x,y). \tag{3.3}$$

for all $x, y \in X$. Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1}\Big[\phi(2^n x, 2^n x) + \phi(0, 2^n x) + \sum_{i=1}^{t} |b_i|\phi(2^n\delta_i x, 0)\Big]. \tag{3.4}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.4).

Proof. By (3.3), we have

$$\|G_f(x,0)\| \le \phi(x,0), \ \ \|G_f(0,x)\| \le \phi(0,x)$$

for all $x, y \in X$. Setting $y = x$ in (3.3), we have

$$\|f(2x) + G_f(x,x) - 2f(x)\| \le \phi(x,x) \tag{3.5}$$

for all $x \in X$. Hence by (3.1) and (3.5), we have

$$\|f(x) - 2^{-1}f(2x)\| \le 2^{-1}\Big[\phi(x,x) + \phi(0,x) + \sum_{i=1}^{t} |b_i|\phi(\delta_i x, 0)\Big] \tag{3.6}$$

for all $x \in X$. By (3.6), we have

$$\begin{aligned}&\|f(x) - 2^{-n}f(2^n x)\|\\ &\le \sum_{k=0}^{n-1} 2^{-k-1}\Big[\phi(2^k x, 2^k x) + \phi(0, 2^k x) + \sum_{i=1}^{t} |b_i|\phi(2^k\delta_i x, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$,

$$\begin{aligned}&\|2^{-m}f(2^m x) - 2^{-n}f(2^n x)\|\\ &= 2^{-m}\|f(2^m x) - 2^{-(n-m)}f(2^{n-m}(2^m x))\|\\ &\le \sum_{k=m}^{n-1} 2^{-k-1}\Big[\phi(2^k x, 2^k x) + \phi(0, 2^k x) + \sum_{i=1}^{t} |b_i|\phi(2^k\delta_i x, 0)\Big]\end{aligned} \tag{3.7}$$

for all $x \in X$. By (3.2) and (3.7), $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$ for all $x \in X$. By (3.7), we have (3.4).

Replacing x and y by $2^n x$ and $2^n y$ in (3.3), respectively and deviding (3.3) by 2^n, we have

$$\|2^{-n}[f(2^n(x+y)) + f(2^n(x-y)) + G_f(2^n x, 2^n y) - 2f(2^n x)]\| \le 2^{-n}\phi(2^n x, 2^n y)$$

for all $x, y \in X$ and letting $n \to \infty$, we can show that A satisfies (1.2). Since f is odd, A is odd.

Suppose that G_f satisfies (2.2). Then clearly, we can show that G_A satisfies (2.2) and hence by Lemma 2.3, A is an additive-quadratic mapping. Since A is odd, A is an additive mapping.

Now, we show the uniqueness of A. Let $E : X \longrightarrow Y$ be an additive mapping with (3.4). Since A and E are additive,

$$\begin{aligned}\|A(x) - E(x)\| &= \|A(2^n x) - E(2^n x)\|\\ &\le 2^{-k}\sum_{n=0}^{\infty} 2^{-n}\Big[\phi(2^n x, 2^n x) + \phi(0, 2^n x) + \sum_{i=1}^{t} |b_i|\phi(2^n\delta_i x, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Hence, letting $k \to \infty$, by (3.2), we have $A = E$. □

Similar to Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n\phi(2^{-n}x, 2^{-n}y) < \infty \tag{3.8}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping satisfying (3.3). Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi(2^{-n}x, 2^{-n}x) + \phi(0, 2^{-n}x) + \sum_{i=1}^{t} |b_i|\phi(2^{-n}\delta_i x, 0)\Big] \tag{3.9}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.9)

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Proof. By (3.3), we have

$$\|G_f(x,0)\| \leq \phi(x,0), \ \ \|G_f(0,x)\| \leq \phi(0,x)$$

for all $x, y \in X$. Setting $y = x = \frac{x}{2}$ in (3.5), we have

$$\Big\| f(x) + G_f\Big(\frac{x}{2}, \frac{x}{2}\Big) - 2f\Big(\frac{x}{2}\Big)\Big\| \leq \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) \tag{3.10}$$

for all $x \in X$. Hence by (3.1), (3.3), and (3.10), we have

$$\Big\| f(x) - 2f\Big(\frac{x}{2}\Big)\Big\| \leq \phi(x,x) + \phi(0,x) + \sum_{i=1}^{t} |b_i|\phi(\delta_i x, 0) \tag{3.11}$$

for all $x \in X$. By (3.11), we have

$$\|f(x) - 2^n f(2^{-n}x)\| \leq \sum_{k=0}^{n-1} 2^k \Big[\phi(2^{-k}x, 2^{-k}x) + \phi(0, 2^{-k}x) + \sum_{i=1}^{t} |b_i|\phi(2^{-k}\delta_i x, 0)\Big]$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$,

$$\begin{aligned} &\|2^m f(2^{-m}x) - 2^n f(2^{-n}x)\| \\ &= 2^m \|f(2^{-m}x) - 2^{(n-m)} f(2^{-(n-m)}(2^{-m}x))\| \\ &\leq \sum_{k=m}^{n-1} 2^k \Big[\phi(2^{-k}x, 2^{-k}x) + \phi(0, 2^{-k}x) + \sum_{i=1}^{t} |b_i|\phi(2^{-k}\delta_i x, 0)\Big] \end{aligned} \tag{3.12}$$

for all $x \in X$. By (3.12), $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. □

Theorem 3.3. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-2n}\phi(2^n x, 2^n y) < \infty \tag{3.13}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping such that

$$\|f(x+y) + f(x-y) + G_f(x,y) - 2f(x) - 2f(y)\| \leq \phi(x,y). \tag{3.14}$$

for all $x, y \in X$. Then there exists an even mapping $Q : X \longrightarrow X$ such that

$$\|Q(x) - f(x)\| \leq \sum_{n=0}^{\infty} 2^{-2n-2}\Big[\phi(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i|\phi(0, 2^n a_i x) + \sum_{i=1}^{s} |a_i|\phi(2^n\lambda_i x, 0)\Big] \tag{3.15}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.15)

Proof. Setting $y = x$ in (3.14), we have

$$\|2^2 f(x) - f(2x) + G_f(x,x)\| \leq \phi(x,x)$$

for all $x \in X$ and by (3.14), we have

$$\|G_f(x,0)\| \leq \phi(x,0), \ \ \|G_f(0,x)\| \leq \phi(0,x)$$

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for all $x \in X$. Since f is even, letting $y = x$ in (3.14), by (3.1), we have

$$\begin{aligned} &\|f(x) - 2^{-2}f(2x)\| \\ &\leq 2^{-2}\Big[\phi(x,x) + \|G_f(x,x)\|\Big] \\ &\leq 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r}|p_i|\|G_f(0,\alpha_i x)\| + \sum_{i=1}^{s}|a_i|\|G_f(\lambda_i x,0)\|\Big] \\ &\leq 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r}|p_i|\phi(0,\alpha_i x) + \sum_{i=1}^{s}|a_i|\phi(\lambda_i x,0)\Big] \end{aligned}$$

for all $x \in X$. Hence we have

$$\begin{aligned} &\|f(x) - 2^{-2n}f(2^n x)\| \\ &\leq \sum_{k=0}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r}|p_i|\phi(0, 2^k a_i x) + \sum_{i=1}^{s}|a_i|\phi(2^k\lambda_i x, 0)\Big] \end{aligned} \tag{3.16}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$, by (3.16)

$$\begin{aligned} &\|2^{-2m}f(2^m x) - 2^{-2n}f(2^n x)\| \\ &= 2^{-2m}\|f(2^m x) - 2^{-2(n-m)}f(2^{n-m}(2^m x))\| \\ &\leq \sum_{k=m}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r}|p_i|\phi(0, 2^k a_i x) + \sum_{i=1}^{s}|a_i|\phi(2^k\lambda_i x, 0)\Big] \end{aligned} \tag{3.17}$$

for all $x \in X$. By (3.17), $\{2^{-2n}f(2^n x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. □

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{2n}\phi(2^{-n}x, 2^{-n}y) < \infty \tag{3.18}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping satisfying (3.14). Then there exists an even mapping $Q : X \longrightarrow X$ such that

$$\|Q(x)-f(x)\| \leq \sum_{n=0}^{\infty} 2^{2n}\Big[\phi(2^{-n}x, 2^{-n}x)+\sum_{i=1}^{r}|p_i|\phi(0, 2^{-n}a_i x)+\sum_{i=1}^{s}|a_i|\phi(2^{-n}\lambda_i x, 0)\Big] \tag{3.19}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.19)

Proof. Setting $y = x = \frac{x}{2}$ in (3.14), we have

$$\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x) + G_f\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\| \leq \phi\Big(\frac{x}{2}, \frac{x}{2}\Big)$$

for all $x \in X$. By (3.14), we have

$$\|G_f(x,0)\| \leq \phi(x,0), \;\; \|G_f(0,x)\| \leq \phi(0,x)$$

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for all $x \in X$ and so, we have

$$\begin{aligned}\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x)\Big\| &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \Big\|G_f\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \Big\|G_f\Big(0, \alpha_i \frac{x}{2}\Big)\Big\| + \sum_{i=1}^{s} |a_i| \Big\|G_f\Big(\lambda_i \frac{x}{2}, 0\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \phi\Big(0, \alpha_i \frac{x}{2}\Big) + \sum_{i=1}^{s} |a_i| \phi\Big(\lambda_i \frac{x}{2}, 0\Big)\end{aligned}$$

for all $x \in X$. Similar to Theorem 3.1, we have the result. □

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (3.2). Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exists a mapping $F : X \longrightarrow X$ such that F satisfies (1.2) and*

$$\begin{aligned} &\|F(x) - f(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n x) + \sum_{i=1}^{s} |a_i| \phi_1(\lambda_i 2^n x, 0)\Big] \\ &\quad + \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(\delta_i 2^n x, 0)\Big] \end{aligned} \tag{3.20}$$

for all $x \in X$, where $\phi_1(x, y) = \frac{1}{2}\Big[\phi(x, y) + \phi(-x, -y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.20)

Proof. By (3.3), we have

$$\|f_e(x + y) + f_e(x - y) + G_{f_e}(x, y) - 2f_e(x) - 2f_e(y)\| \le \phi_1(x, y) \tag{3.21}$$

for all $x, y \in X$. By Theorem 3.3, there exists an even mapping $Q : X \longrightarrow Y$ such that $Q(x) = \lim_{n \longrightarrow \infty} 2^{-2n} f_e(2^n x)$ for all $x \in X$,

$$Q(x + y) + Q(x - y) + G_Q(x, y) = 2Q(x) + 2Q(y) \tag{3.22}$$

for all $x, y \in X$, and

$$\begin{aligned} &\|Q(x) - f_e(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n a_i x) + \sum_{i=1}^{s} |a_i| \phi_1(2^n \lambda_i x, 0)\Big] \end{aligned} \tag{3.23}$$

for all $x \in X$. Similarly, there exists an odd mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n \longrightarrow \infty} 2^{-n} f_o(2^n x)$ for all $x \in X$,

$$A(x + y) + A(x - y) + G_A(x, y) - 2A(x) = 0 \tag{3.24}$$

for all $x, y \in X$, and

$$\|A(x) - f_o(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(2^n \delta_i x, 0)\Big] \tag{3.25}$$

for all $x \in X$.

Let $F = Q + A$. Since Q is even and A is odd, $2Q(y) = F(y) + F(-y)$ and by (3.22) and (3.24), F satisfies (1.2). Since $\|F(x) - f(x)\| \le \|Q(x) - f_e(x)\| + \|A(x) - f_o(x)\|$, by (3.23) and (3.25), we have (3.20).

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Suppose that G_f satisfies (2.2). Then clearly, we can show that G_F satisfies (2.2) and hence by Lemma 2.3, F is an additive-quadratic mapping. The proof of the uniqueness of F is similar to Theorem 3.1. □

Theorem 3.6. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n \phi(2^{-n}x, 2^{-n}y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exist a mapping $F : X \longrightarrow X$ such that

(3.26)

$$\begin{aligned}&\|F(x) - f(x)\| \\ \leq &\sum_{n=0}^{\infty} 2^{2n-2}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \sum_{i=1}^{r} |p_i|\phi_1(0, 2^{-n}x) + \sum_{i=1}^{s} |a_i|\phi_1(\lambda_i 2^{-n}x, 0)\Big] \\ &+ \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \phi_1(0, 2^{-n}x) + \sum_{i=1}^{t} |b_i|\phi_1(\delta_i 2^{-n}x, 0)\Big]\end{aligned}$$

for all $x \in X$, where $\phi_1(x, y) = \frac{1}{2}\Big[\phi(x, y) + \phi(-x, -y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.26).

4. Applicaions

In this section, we illustrate how the theorems in section 3 work well for the generalized Hyers-Ulam stability of various additive-quadratic functional equations.

As examples of $\phi(x, y)$ in Theorem 3.5 and Theorem 3.6, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following theorem :

Theorem 4.1. *Assume that all of the conditions in Theorem 3.1 hold and G_f satisfies (2.2). Let p be a real number with $0 < p < \frac{1}{2}, 1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that*

(4.1)

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)+G_f(x,y)\| \leq \epsilon(\|x\|^p\|y\|^p+\|x\|^{2p}+\|x\|^{2p})$$

for all $x, y \in X$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \begin{cases} \Psi_1(x), & \text{if } 0 < p < \frac{1}{2} \\ \Psi_2(x), & \text{if } 1 < p \end{cases}$$

for all $x \in X$, where

$$\Psi_1(x) = \Big[3 + \sum_{i=1}^{r} |p_i| + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p}\Big]\frac{\epsilon}{4 - 4^p}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t} |b_i||\delta_i|^{2p}\Big]\frac{\epsilon}{2 - 4^p}\|x\|^{2p}$$

and

$$\Psi_2(x) = \Big[3 + \sum_{i=1}^{r} |p_i| + \sum_{i=1}^{s} |a_i||\lambda_i|^{2p}\Big]\frac{4^{p-1}\epsilon}{4^p - 4}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t} |b_i||\delta_i|^{2p}\Big]\frac{2^{2p-1}\epsilon}{4^p - 2}\|x\|^{2p}$$

Lemma 4.2. *Let G be the operator defined by*

$$G_f(x, y) = f(2x + y) - f(x + 2y) + f(x - y) - f(y - x) - 3f(x) + 3f(y)$$

for all mapping $f : X \longrightarrow Y$. Then G is additive-quadratic.

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Proof. Suppose that $G_f(x, y) = 0$ for all $x, y \in X$. Then we have

$$f(2x + y) - f(x + 2y) + f(x - y) - f(y - x) - 3f(x) + 3f(y) = 0. \tag{4.2}$$

and so we have

$$f_e(2x + y) - f_e(x + 2y) - 3f_e(x) + 3f_e(y) = 0 \tag{4.3}$$

for all $x, y \in X$ and letting $y = y - x$ in (4.3), we have

$$f_e(x + y) - f_e(x - 2y) - 3f_e(x) + 3f_e(x - y) = 0 \tag{4.4}$$

for all $x, y \in X$. Letting $y = -y$ in (4.4), we have

$$f_e(x - y) - f_e(x + 2y) - 3f_e(x) + 3f_e(x + y) = 0 \tag{4.5}$$

for all $x, y \in X$. By (4.4) and (4.5), we have

$$f_e(x+2y)+f_e(x-2y)-2f_e(x)-8f_e(y)-4[f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)] = 0$$

for all $x, y \in X$ and so f_e is quadratic.

Since f_o is an odd mapping, by (4.2), we have

$$f_o(2x + y) - f_o(x + 2y) + 2f_o(x - y) - 3f_o(x) + 3f_o(y) = 0 \tag{4.6}$$

for all $x, y \in X$ and letting $y = -x - y$ in (4.6), we have

$$f_o(x - y) + f_o(x + 2y) + 2f_o(2x + y) - 3f_o(x) - 3f_o(x + y) = 0 \tag{4.7}$$

for all $x, y \in X$. By (4.6) and (4.7), we have

$$f_o(2x + y) + f_o(x - y) - 2f_o(x) + f_o(y) - f_o(x + y) = 0 \tag{4.8}$$

for all $x, y \in X$ and letting $y = -y$ in (4.10), we have

$$f_o(2x - y) + f_o(x + y) - 2f_o(x) - f_o(y) - f_o(x - y) = 0 \tag{4.9}$$

for all $x, y \in X$. By (4.10) and (4.9), we have

$$f_o(2x + y) + f_o(2x - y) - 4f_o(x) = 0 \tag{4.10}$$

for all $x, y \in X$ and hence f_o is additive. Thus f is an additive-quadratic mapping. □

By Lemma 2.3, Theorem 4.1, and Lemma 4.2, we have the following theorem :

Theorem 4.3. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|f(x + 2y) - f(2x + y) + f(x + y) + f(y - x) + f(x) - 4f(y) - f(-y)\|$$
$$\leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|x\|^{2p})$$

for all $x, y \in X$ and some a real number p with $0 < p < \frac{1}{2}, 1 < p$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \begin{cases} \left[\frac{3}{4-4^p} + \frac{4}{2-4^p}\right]\epsilon\|x\|^{2p}, & if\ 0 < p < \frac{1}{2} \\ \left[\frac{3\times4^{p-1}}{4^p-4} + \frac{2\times4^p}{4^p-2}\right]\epsilon\|x\|^{2p}, & if\ 1 < p \end{cases}$$

for all $x \in X$.

Proof. For a mapping $h : X \longrightarrow Y$, let $G_h(x, y) = h(2x + y) - h(x + 2y) + h(x - y) - h(y - x) - 3h(x) + 3h(y)$. By Lemma 4.2, G is additive-quadratic and f, G satisfiy (4.1). Since G_f satisfies (2.2) in Lemma 2.3, by Theorem 4.1, we have the result. □

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