Solution of Special type of Integro Differential equation by Laplace Decomposition Method

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Abstract :

This article focuses on a specific class of integro-differential equations and their solutions. A. Ansari et al., in J. Appl. Math. & Informatics utilized the series solution method to derive the approximate numerical solution for Volterra integro-differential equations. In this study, we apply the Laplace decomposition method to these equations, exploring both analytic and approximate solutions. Furthermore, we present a comparative analysis of the exact and approximate solutions for some problems.

Keywords: Integro-Differential Equations, Laplace decomposition method, Adomian Polynomials

MSC(2020) Subject Classification: 35A22, 44A10, 45D05.

1. Introduction

Ordinary and partial differential equations, as well as integro-differential equations, are widely used equations in mathematical modelling of real-world issues. Integro-differential equations are commonly utilised in mathematical models used in physical science, including biological models, fluid dynamics, and economics [1,7,13]. Analysing integro-differential equations may be a tedious job unless we use an effective technique; hence, an effective approximation method is needed to determine the solution. There are several methods that dels with the solution of integro differential equations. In [1, 2, 3, 4], the series solution method is applied to determine the solution integro differential equations. In [5], the Variational iteration method determines the solution of differential and integrodifferential equations. S.S. Handibag et al. in [6] applied LDM and SSM for the solution integro differential equations of the form

$$v^n(x) = \psi(x) + \lambda \int_0^x \mathrm{F}(x,t)\,\mathrm{P}(v(t))\,dt \qquad (1)$$

where $v^n(x)$ indicate the nth derivative of v such as $v^n(x)=\frac{d^n v(x)}{dx^n}$. initial conditions $v^m(0)=k_m; 0\le m\le(n-1)$ such as $v(0), v^{'}(0), v^{''}0), \ldots v^{n-1}(0)$, the function $\psi(x)$ are given real valued functions, $\mathsf{F}(x,t)$ is the kernel of integral equation, A is suitable constant and m, are constants that define the initial conditions. The function $\mathsf{P}(v(t))$ is a non-linear function of $v(x)$.

Various strategies, including the series solution method [10, 11, 12, 13], the Variation Iteration Method (VIM) [13, 14] have been used to address these issues. These approaches combine two efficient methods to get accurate solutions to nonlinear equations. The recommended technique is demonstrated by solving instances of the Volterra integro-differential equations [15] using the established method. The acquired findings are compared to precise answers. The current algorithm worked exceptionally well in all scenarios.

2. Methodology

Combining the Adomian Decomposition and Laplace Transform techniques are also referred to as the Laplace Decomposition method (LDM).This methods main benefits is its ability to find a nonlinear equation's precise or approximate solution [3]. Differential equations can be successfully solved using the Laplace Decomposition method (LDM),which was initially presented by Suheil A. Khuri [4, 5]. When equation (1) is run through both sides using the Laplace transform, the result is

$$L\{v(x)\}-s^{n-1}v(0)-s^{n-2}v^{'}(0)-\ldots-v^{j-1}(0) \;=L\{\psi(x)\}+L\{\mathsf{F}(x-t)\}+L\{\mathsf{P}(v(t))\} \tag{2}$$

and

$$L\{v(x)\}=\frac{1}{s}v(0)+v(0)+\frac{1}{s^2}v^{'}(0)+\ldots+\frac{1}{s^j}v^{(n-1)}(0)+\frac{1}{s^n}L\{\psi(x)\}+\frac{1}{s^n}L\{\mathsf{F}(x-t)\}L\{\mathsf{P}(v(t))\} \tag{3}$$

In order to accomplish this, the linear expression v(x) on the left is first expressed using an endless succession of parts provided by,

$$v(x)=\sum_{n=0}^{\infty}v_n(x) \tag{4}$$

recursively find the components $v_n(x), n\ge 0$.

For treating the non-linear component $G(v(x))$, the Adomian polynomial shall be embodied by an endless series,

A_n we apply the Adomian polynomial get around its difficulties [1,7,8] in the format,

$$\mathrm{P}\left(v(x)\right)=\sum_{n=0}^{\infty}A_n(x), \quad (5)$$

where,

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{n}\lambda^j v_j\right)_{\lambda=0,n=0,1,2,...} \quad (5)$$

is obtained for all forms of nonlinearity types. (4) into equation (3) result in

$$L\left(\sum_{j=0}^{n}v_n(x)\right)=\frac{1}{s}v(0)+\frac{1}{s^2}v^{'}(0)+...+\frac{1}{s^j}v^{(n-1)}(0)+\frac{1}{s^n}L\{\psi(x)\}+\frac{1}{s^n}\mathbf{L}\{\mathrm{F}(x-t)\}L\left(\sum_{n=0}^{\infty}A_n(x)\right), \quad (6)$$

with the Adomian decomposition approach, the recursive connection listed below can be used

$$L\{v_0(x)\}=\frac{1}{s}v(0)+\frac{1}{s^2}v^{'}(0)+...+\frac{1}{s^n}v^{(n-1)}(0)+\frac{1}{s^n}L\{\psi(x)\}, \quad (7)$$

and

$$L\{v(x)\}=\frac{1}{s^n}L\{\mathrm{F}(x-t)\}L\{A_n(x)\},n..1. \quad (8)$$

When the first portion of [8] is subjected to the inverse Laplace transform $v_0(x)$ is obtained which defined A_0. Consequently, by using second portion of (7) the components of equation (4) will be fully determined.

3. Application of Methodology

Example.1 Consider the integrodifferential equation,

$v^{''}(x)=-8-\frac{1}{3}\left(x^3-x^4\right)+\int_0^x(x-t)v(t)\,dt$ with the initial conditions

$v(0)=0, v^{'}(0)=2$ (9)

To solve the problem we apply Laplace Decomposition Method as

$$L\{v^{''}(x)\}=L\left\{-8-\frac{1}{3}\left(x^3-x^4\right)+\int_0^x(x-t)v(t)\,dt\right\}$$
$$s^2\left(v(s)\right)-sv(0)-v^{'}(0)=-\frac{8}{s}-\frac{1}{3}\left[\frac{6}{s^4}-\frac{24}{s^5}\right]+\frac{1}{s^2}v(s). \quad (10)$$

Further, we obtain,

$$s^2\left(v(s)\right)-2=-\frac{8}{s}-\frac{2}{s^4}+\frac{8}{s^5}+\frac{1}{s^2}v(s)$$
$$v(s)=\frac{2}{s^2}-\frac{8}{s^3}-\frac{2}{s^6}+\frac{8}{s^7}+\frac{1}{s^4}$$
$$v(s)\left(1-\frac{1}{s^4}\right)v(s)=\frac{2}{s^2}-\frac{8}{s^3}-\frac{2}{s^6}+\frac{8}{s^7} \quad (11)$$
$$v(s)=\frac{s^4}{s^4-1}\left[\frac{2}{s^2}-\frac{8}{s^3}-\frac{2}{s^6}+\frac{8}{s^7}\right].$$

Now, applying inverse Laplace transformation to both sides of equation, we get,

$$L^{-1}\{v(s)\}=L^{-1}\left\{\frac{s^4}{s^4-1}\left[\frac{2}{s^2}-\frac{8}{s^3}-\frac{2}{s^6}+\frac{8}{s^7}\right]\right\}.$$

By performing some mathematical steps, we get,

$v(x)=2x-4x^2$. which is an exact solution, for example (Ex.1).

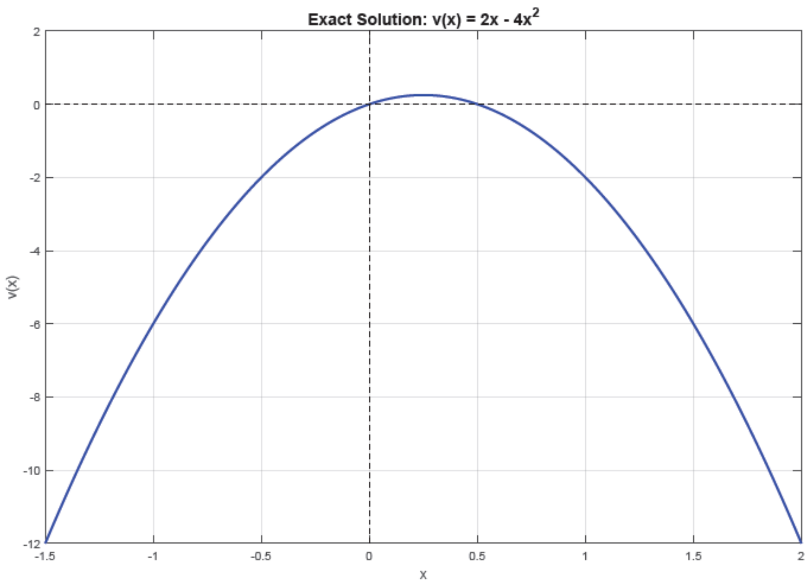

Figure-1: Plot of Exact solution of Example (1).

Example.2 Consider the integrodifferential equation

$$v'(x)=xe^x+\frac{1}{3}x^3e^{2x}-\int_0^x e^{2(x-t)}\, v'(t)^2\,dt, \quad (12)$$

with initial conditions $v(0)=0$.

To solve the problem, we apply Laplace Decomposition Method as

$$L\left\{v'(x)\right\}=L\left\{xe^{x}+\frac{1}{3}x^{3}e^{2x}-\int_{0}^{x}e^{2(x-t)}v'(t)^{2}\,dt\right\}$$
$$s\left(V(s)\right)-v(0)=\frac{1}{(s-1)^{2}}+\frac{2}{(s-2)^{4}}-\frac{1}{(s-2)}L\left\{v'(t)^{2}\right\}. \tag{13}$$

Further, we obtain

$$V(s)=\frac{1}{s(s-1)^{2}}+\frac{2}{s(s-2)^{4}}-\frac{1}{s(s-2)}L\left\{v'(t)^{2}\right\}. \tag{14}$$

Now, applying inverse Laplace transformation to both sides of the equation we get,

$$L^{-1}\left\{V(s)\right\}=L^{-1}\frac{1}{s(s-1)^{2}}+\frac{2}{s(s-2)^{4}}-\frac{1}{s(s-2)}L\left\{v'(t)^{2}\right\}$$
$$v(x)=L^{-1}\left\{\frac{1}{s(s-1)^{2}}\right\}+L^{-1}\left\{\frac{2}{s(s-2)^{4}}\right\}-L^{-1}\left\{\frac{1}{s(s-2)}\right\}L\left\{v'(t)^{2}\right\}$$
$$v(x)=1-e^{x}+xe^{x}+\frac{1}{8}-\frac{1}{8}e^{2x}+\frac{xe^{2x}}{4}-\frac{x^{2}e^{2x}}{4}+\frac{x^{3}e^{2x}}{6}-L^{-1}\left\{\frac{1}{s(s-2)}\right\}L\left\{v'(t)\right\}^{2} \tag{15}$$

Now, consider the first iteration

$$v_{0}(x)=1-e^{x}+xe^{x}+\frac{1}{8}-\frac{1}{8}e^{2x}+\frac{xe^{2x}}{4}-\frac{x^{2}e^{2x}}{4}+\frac{x^{3}e^{2x}}{6}. \tag{16}$$

Now, the first Adomian polynomial is,

$$A_{0}=v'_{0}(x)^{2},$$
$$A_{0}=\left[xe^{x}+\frac{x^{3}e^{2x}}{3}\right]^{2},$$
$$A_{0}=x^{2}e^{2x}+\frac{2}{3}x^{4}e^{3x}+\frac{1}{9}x^{6}e^{4x}.$$

Now, the second approximation is,

$$v_{1}(x)=-L^{-1}\left\{\frac{1}{s(s-2)}L[A_{0}]\right\}$$
$$v_{1}(x)=-L^{-1}\left\{\frac{1}{s(s-2)}L\left\{v'(t)^{2}\right\}\right\}$$
$$v_{1}(x)=-L^{-1}\left\{\frac{2}{s(s-2)^{4}}+\frac{16}{s(s-2)(s-3)^{5}}+\frac{80}{s(s-2)(s-4)^{7}}\right\}.$$

By performing some mathematical steps, we get,

$$v_{1}(x)=-\frac{1}{8}+\frac{1}{8}e^{2x}-\frac{xe^{2x}}{4}+\frac{x^{2}e^{2x}}{4}-\frac{x^{3}e^{2x}}{6}.$$

$$v(x)=v_0(x)+v_1(x)$$

$$v(x)=\begin{bmatrix}1-e^x+xe^x+\frac{1}{8}-\frac{1}{8}e^{2x}+\frac{xe^{2x}}{4}-\frac{x^2e^{2x}}{4}+\frac{x^3e^{2x}}{6}\\ -\frac{1}{8}+\frac{1}{8}e^{2x}-\frac{xe^{2x}}{4}+\frac{x^2e^{2x}}{4}-\frac{x^3e^{2x}}{6}\end{bmatrix} \quad (17)$$

$$v(x)=e^x(x-1)+1.$$

Now, Hence, $v(x)=e^x(x-1)+1,$ is an exact solution to example (2).

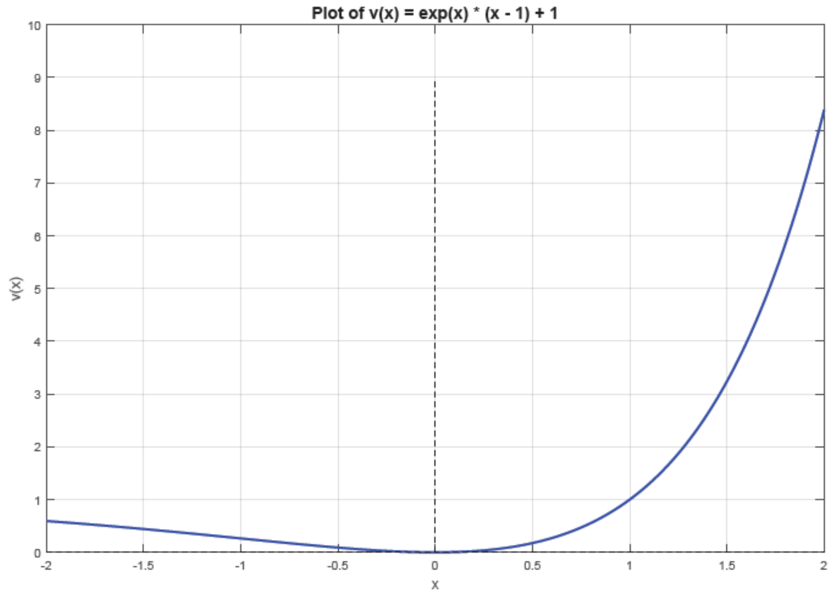

Figure-2: Plot of Exact solution of Example (2)

Example. 3 Consider the integrodifferential equation:

$$v'(x)=-\frac{1}{2}+\int_0^x[v'(t)]^2dt, \qquad \text{with the initial condition } v(0)=0. \quad (18)$$

We apply the Laplace transform of both sides of the equation (18). Since we have

$L\{v(x)\}=V(s)$ and $L\{v'(x)\}=sV(s)-v(0)$. Given that $v(0)=0$, we have:

$$sV(s)=-\frac{1}{2s}+L\left\{\int_0^x[v'(t)]^2dt\right\}.$$

The Laplace transform of the integral term is:

$$L\left\{\int_0^x[v'(t)]^2dt\right\}=\frac{1}{s}L\left\{[v'(x)]^2\right\}.$$

Thus

$$sV(s)=-\frac{1}{2s}+\frac{1}{s}L\left\{[v'(x)]^2\right\}.$$

Rearranging the above equation, we get $V(s)=-\frac{1}{2s^2}+\frac{1}{s^2}L\left\{[v'(x)]^2\right\}$.

Next, we decompose $v(x)$ and $v'(x)$ as:

$$v(x)=\sum_{n=0}^{\infty}v_n(x), \quad v'(x)=\sum_{n=0}^{\infty}v_{n'}(x).$$

The Adomian polynomials A_n are given by:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty}\lambda^k v_{k'}(x)\right)^2\right]_{\lambda=0}.$$

From the initial conditions, we have

$$v_0(x)=-\frac{1}{2}x. \tag{19}$$

The first Adomian polynomial is:

$$A_0=\left[v_{0'}(x)\right]^2=\left(-\frac{1}{2}\right)^2=\frac{1}{4}.$$

Now, we compute $v_1(x)$ as

$$v_1(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{A_0\right\}\right\}$$

$$v_1(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{\frac{1}{4}\right\}\right\} \tag{20}$$

$$v_1(x)=\frac{x^2}{8}.$$

The second Adomian polynomial is:

$$A_1=2v_0(x)v_1(x)$$

$$A_1=2\left(-\frac{x}{2}\right)\left(\frac{x^2}{8}\right)$$

$$A_1=-\frac{x^3}{8}.$$

Further, we compute $v_2(x)$ as

$$v_2(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{A_1\right\}\right\}$$

$$v_2(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{-\frac{x^3}{8}\right\}\right\} \tag{21}$$

$$v_2(x)=-\frac{x^5}{160}.$$

The third Adomian polynomial is:

$$A_2=\left[v_1(x)\right]^2+2v_0(x)v_2(x)$$

$$A_2=\left[\frac{x^2}{8}\right]^2+2\left[-\frac{x}{2}\right]\left[-\frac{x^5}{160}\right]$$

$$A_2=\frac{x^6}{160}+\frac{x^4}{64}.$$

The next terms $v_3(x)$ as;

$$v_3(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{A_2\right\}\right\}$$

$$v_3(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{\frac{x^6}{160}+\frac{x^4}{64}\right\}\right\} \tag{22}$$

$$v_3(x)=\frac{x^8}{8960}+\frac{x^6}{1920}.$$

The fourth Adomian polynomial is:

$$A_3=2v_0(x)v_3(x)+2v_1(x)v_2(x)$$

$$A_3=-\frac{x^9}{17920}-\frac{x^7}{480}.$$

The next terms $v_4(x)$ as

$$v_4(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{A_3\right\}\right\}$$

$$v_4(x)=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\mathrm{L}\left\{-\frac{x^9}{17920}-\frac{x^7}{480}\right\}\right\} \tag{23}$$

$$v_4(x)=\frac{x^9}{55296}-\frac{x^{11}}{1917200}.$$

Similarly, the next terms are computed as follows:

$$\begin{aligned}
v_5(x)&=-\frac{x^{11}}{1216512}-\frac{x^{13}}{49847200}+\frac{x^{15}}{2449440000},\\
v_6(x)&=-\frac{x^{13}}{31933440}+\frac{x^{15}}{479001600}-\frac{x^{17}}{22118400000},\\
v_7(x)&=-\frac{x^{15}}{34488115200}+\frac{x^{17}}{1016064000000}-\frac{x^{19}}{85996339200000},\\
v_8(x)&=\frac{x^{17}}{976924160000}-\frac{x^{19}}{71285145600000}+\frac{x^{21}}{14358863902720000}.
\end{aligned} \tag{24}$$

The approximate solution up to $v_8(x)$ is:

$$ADM_{v(x)} = \begin{bmatrix} -\frac{x}{2}+\frac{x^2}{8}-\frac{x^5}{160}+\frac{x^6}{1920}+\frac{x^8}{8960}-\frac{x^9}{55296} \\ -\left(\frac{1}{1917200}+\frac{1}{1216512}\right)x^{11}-\left(\frac{1}{49847200}+\frac{1}{31933440}\right)x^{13} \\ +\left(\frac{1}{2449440000}+\frac{1}{479001600}-\frac{1}{34488115200}\right)x^{15} \\ +\left(\frac{1}{1016064000000}-\frac{1}{22118400000}+\frac{1}{976924160000}\right)x^{17} \\ -\left(\frac{1}{85996339200000}+\frac{1}{71285145600000}\right)x^{19}+\frac{x^{21}}{14358863902720000} \end{bmatrix}.$$

which is closer to the exact solution $Exact_{v(x)} = -\ln\left(\frac{1}{2}x+1\right)$, at limiting case.

The approximate solution in [10] for Example (3) by series solution method is $SSM_{v(x)} = -\frac{x}{2}+\frac{x^3}{12}+\frac{x^7}{1008}+\text{L}$

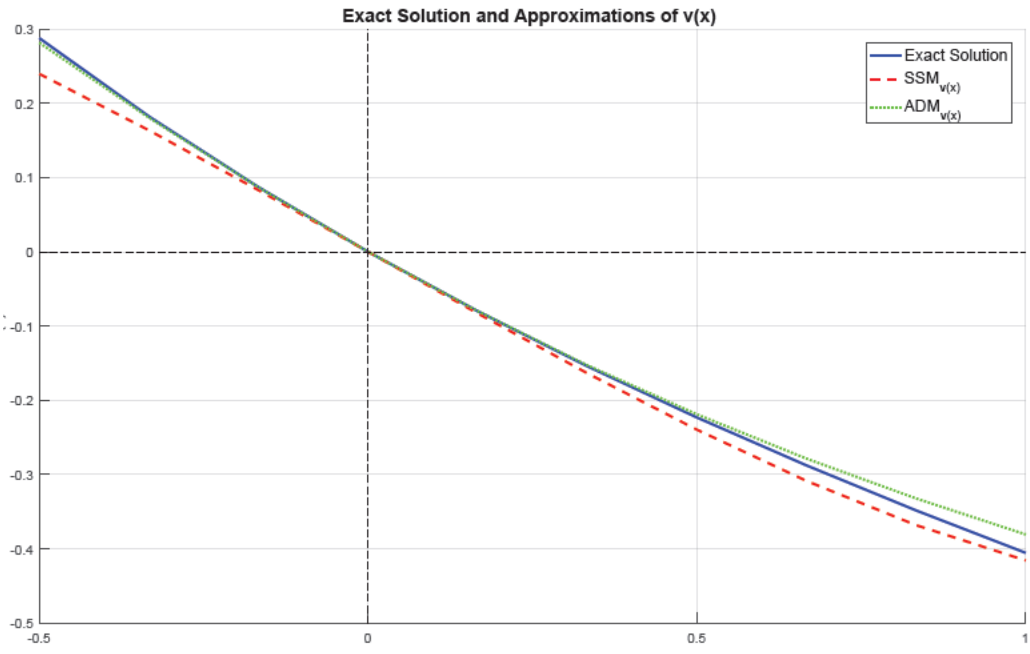

Figure-3:Plot of Exact and approximate solution of Example (3)

Figure 3 represents the exact and approximate solutions of Example 3.3. The graph shows that the Laplace decomposition method is more accurate and suitable for problem 3.3 than the series solution method.

4. Conclusion

This study employed the Laplace Decomposition Method (LDM) to solve linear and nonlinear integro-differential equations, as outlined in [10]. To address the nonlinear cases, we utilised Adomian polynomials. In the referenced article, equations Example 1 to Example 3 were solved using the series solution method, which yielded approximate results. In contrast, our application of the LDM to Example 1 and Example 2 produced exact solutions within one or two iterations, outperforming the series method, which only provided approximations. For Example 3, although we obtained an approximate solution using the LDM, it was closer to the exact solution than the one derived from the series method. A comparison of these solutions is shown in Figure 3. This shows that for Examples (1) to Example (3), the Laplace Decomposition Method is more effective than the series solution method.

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