FUZZY STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH EXTRA TERMS

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Abstract. In this paper, we consider the generalized Hyers-Ulam stability for the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) + G_f(x,y) = 0.$$

with an extra term G_f which is a functional operator of f.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [8] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [18] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians ([3], [4], [5], [7], and [16]).

Katsaras [11] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In this paper, we use the definition of fuzzy normed spaces given in [2],[14], [15].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on X* if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space*.

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Let (X, N) be a fuzzy normed space. (i) A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. (ii) A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$ and any $t > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$ and all positive integer p.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+||x||}, & \text{if } t > 0 \end{cases}$$

is a fuzzy norm on X.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 1.2. [6] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \frac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

In 2001, Rassias [19] introduced the following cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0 \tag{1.1}$$

and the following cubic functional equations were investigated

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.2}$$

in ([10]). Every solution of a cubic functional equation is called *a cubic mapping* and Kim and Han [12] investigated the following cubic functional equation

$$\begin{aligned} &f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) \\ &+ k[f(mx + y) + f(mx - y) - m[f(x + y) + f(x - y)] - 2(m^3 - m)f(x)] = 0 \end{aligned}$$

for some rational number m and some real number k and proved the stability for it in fuzzy normed spaces.

In this paper, we investigate the following functional equation which is added a term by G_f to (1.1)

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y) = 0,$$

where G_f is a functional operator depending on functions f. The definition of G_f is given in section 2 and prove the stability for it in fuzzy normed spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

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2. Cubic functional equations with extra terms

For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Now consider the functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y) = 0 \tag{2.1}$$

with the functional operator G_f.

Theorem 2.1. *Suppose that the mapping $f : X \longrightarrow Y$ is a solution of (2.1) with $f(0) = 0$. Then f is cubic if and only if $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$.*

Proof. Suppose that $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$. Interchanging x and y in (2.1) , we have

$$f(2x + y) - 3f(x + y) + 3f(y) - f(y - x) - 6f(x) + G_f(y, x) = 0. \tag{2.2}$$

for all $x, y \in X$ and letting $x = -x$ in (2.2) , we have

$$f(-2x + y) - 3f(-x + y) + 3f(y) - f(x + y) - 6f(-x) + G_f(y, -x) = 0. \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f(2x + y) - f(-2x + y) - 2f(x + y) + 2f(y - x) - 6f(x) + 6f(-x) = 0. \tag{2.4}$$

for all $x, y \in X$, because $G_f(y, x) = G_f(y, -x)$. Letting $y = x$ in (2.4), we have

$$f(3x) - 22f(x) + 5f(-x) = 0. \tag{2.5}$$

for all $x \in X$ and letting $y = 2x$ in (2.4), by (2.5), we have

$$f(4x) - 2f(3x) - 4f(x) + 6(-x) = 16f(x) + 16f(-x) = 0.$$

for all $x \in X$, because $f(2x) = 8f(x)$. Hence f is odd and by (2.2) and (2.3), f satisfies (1.2). Thus f is a cubic mapping. The converse is trivial. □

3. The Generalized Hyers-Ulam stability for (2.1)

In this section, we prove the generalized Hyers-Ulam stability of (2.1) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x, y) = f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y)$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow Z$ be a function such that there is a real number L satisfying $0 < L < 1$ and*

$$N'(\phi(2x, 2y), t) \geq N'(8L\phi(x, y), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x, y), t) \geq N'(\phi(x, y), t) \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$ and

$$(3.3)\quad N(f(2x) - 8f(x), t) \geq \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$$

for all $x \in X$, all $t > 0$ and some nonnegative real numbers a, b, c. Further, assume that if g satisfies (2.1), then g is a cubic mapping. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$(3.4)\quad \begin{aligned} &N\Big(f(x) - C(x), \frac{1}{8(1-L)}t\Big) \\ &\geq \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\psi(x,t) = \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \psi(x,t),\ \forall x \in X,\ \forall t > 0\}.$$

Then (S, d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 2^{-3}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (3.1), we have

$$N(Jg(x) - Jh(x), cLt) \geq N(2^{-3}(g(2x) - h(2x)), cLt) \geq \psi(x,t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (3.3), $d(f, Jf) \leq \frac{1}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{-3n} f(2^n x)$ for all $x \in X$ and $d(f, C) \leq \frac{1}{8(1-L)}$ and hence we have (3.4).

Replacing x, y, and t by $2^n x$, $2^n y$, and $2^{3n} t$ in (3.2), respectively, we have

$$N(D_f(2^n x, 2^n y), 2^{3n}t) \geq N'(\phi(2^n x, 2^n y), 2^{3n}t) \geq N'(L^n\phi(x,y), t)$$

for all $x, y \in X$ and all $t > 0$. Letting $n \longrightarrow \infty$ in the last inequality, we have

$$C(x + 2y) - 3C(x + y) + 3C(x) - C(x - y) - 6C(y) + G_C(x,y) = 0$$

for all $x, y \in X$ and thus C is a cubic mapping.

Now, we show the uniqueness of C. Let $C_0 : X \longrightarrow Y$ be another cubic mapping with (3.4). Then C_0 is a fixed ponit of J in S and by (3.4), we get

$$d(Jf, C_0) \leq d(Jf, JC) \leq Ld(f, C_0) \leq \frac{L}{8(1-L)} < \infty$$

and by (3) of Theorem 1.2, we have $C = C_0$. □

Similar to Theorem 3.1, we can also have the following theorem.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow Z$ be a function such that there is a real number L satisfying $0 < L < 1$ and*

$$(3.5)\quad N'(\phi(x,y),t) \geq N'\Big(\frac{L}{8}\phi(2x,2y),t\Big)$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$, (3.2), and (3.3). Further, assume that if g satisfies (2.1), then g is a cubic mapping.

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Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality

$$\begin{aligned} & N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big) \\ & \geq \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \end{aligned} \tag{3.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\psi(x,t) = \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \ \geq \ \psi(x,t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S,d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 8g(2^{-1}x)$ for all $x \in X$ and all $g \in S$. Let $g,h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (3.2) and (3.5), we have

$$N(Jg(x) - Jh(x), cLt) \geq N(8(g(2^{-1}x) - h(2^{-1}x)), cLt) \geq \psi(x,t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g,h \in S$ and so J is a strictly contractive mapping. By (3.3), we get

$$N\Big(f(x) - 8f(2^{-1}x), \frac{L}{8}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big) \geq \psi(x,t) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Hence $d(f, Jf) \leq \frac{L}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{3n} f(2^{-n}x)$ for all $x \in X$ and $d(f,C) \leq \frac{L}{8(1-L)}$ and hence we have (3.6). The rest of the proof is similar to that of Theorem 3.1. □

Using Theorem 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.1). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that*

$$\begin{aligned} & N(G_f(0,x),t) \geq \min\{N'(a_1\phi(x,0),t), N'(a_2\phi(0,x),t), N'(a_3\phi(x,-x),t)\}, \\ & N(G_f(x,-x),t) \geq \min\{N'(b_1\phi(x,0),t), N'(b_2\phi(0,x),t), N'(b_3\phi(x,-x),t)\} \end{aligned} \tag{3.8}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1,2,3)$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\begin{aligned} & N\Big(f(x) - C(x), \frac{7}{24(1-L)}t\Big) \\ & \geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned} \tag{3.9}$$

for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. Setting $x = 0$ and $y = x$ in (3.2), we have

$$N(f(2x) - 9f(x) - f(-x) + G_f(0,x), t) \geq N'(\phi(0,x),t) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. Setting $y = -x$ in (3.2), we have

$$N(3f(x) - 5f(-x) - f(2x) + G_f(x,-x), t) \geq N'(\phi(x,-x),t) \tag{3.11}$$

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for all $x \in X$ and all $t > 0$. Hence by (3.10) and (3.11), we get

$$\begin{aligned}(3.12)\qquad &N(6f(x) + 6f(-x) - G_f(0, x) - G_f(x, -x), 2t)\\ &\geq \min\{N'(\phi(0, x), t), N'(\phi(x, -x), t)\}\end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus by (3.8), (3.10), and (3.12), we get

$$\begin{aligned}&N\Big(f(2x) - 8f(x), \frac{7}{3}t\Big)\\ &= \min\Big\{N(f(2x) - 9f(x) - f(-x) + G_f(0, x), t), N\Big(\frac{5}{6}G_f(0, x), \frac{5}{6}t\Big),\\ &\quad N\Big(f(x) + f(-x) - \frac{1}{6}G_f(0, x) - \frac{1}{6}G_f(x, -x), \frac{1}{3}t\Big), N\Big(\frac{1}{6}G_f(x, -x), \frac{1}{6}t\Big)\Big\}\\ &\geq \min\{N'(c_1\phi(x, 0), t), N'(c_2\phi(0, x), t), N'(c_3\phi(x, -x), t)\}\end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.1, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.9). □

Corollary 3.4. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.5). Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that (3.8) hold. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality*

$$\begin{aligned}(3.13)\qquad &N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big)\\ &\geq \min\{N'(c_1\phi(x, 0), t), N'(c_2\phi(0, x), t), N'(c_3\phi(x, -x), t)\}\end{aligned}$$

holds for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. By (**??**), we get

$$\begin{aligned}&N\Big(f(x) - 8f(2^{-1}x), \frac{7L}{24}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big)\\ &\geq \min\{N'(c_1\phi(x, 0), t), N'(c_2\phi(0, x), t), N'(c_3\phi(x, -x), t)\}\end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.2, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.13). □

From now on, we consider the following functional equation

$$\begin{aligned}(3.14)\qquad &f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y)\\ &+ k[f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)] = 0\end{aligned}$$

for some positive real number k.

Lemma 3.5. [12] *A mapping $f : X \longrightarrow Y$ satisfies (3.14) if and only if f is a cubic mapping.*

Using Theorem 2.1, Theorem 3.1, and Theorem 3.2, we have the following example.

Example 3.6. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

(3.15)

$$\begin{aligned}&N(f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + k[f(2x + y) + f(2x - y)\\ &- 2f(x + y) - 2f(x - y) - 12f(x)], t) \geq \frac{t}{t + \|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p}\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and some positive real numbers k, p with $p \neq \frac{3}{2}$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2k|8 - 2^{2p}|t}{2k|8 - 2^{2p}|t + \|x\|^{2p}} \tag{3.16}$$

for all $x \in X$.

Proof. Let $G_f(x, y) = k[f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)]$ and $\phi(x, y) = \|x\|^{2p} + \|y\|^{2p} + \|x\|^{2p}\|y\|^p$. Then $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$ and f satisfies (3.2). Letting $y = 0$ in (3.15), we have

$$N(f(2x) - 8f(x), t) \geq N'\Big(\frac{1}{2k}\phi(x, 0), t\Big)$$

for all $x \in X$ and all $t > 0$, where

$$N'(r, t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+|r|}, & \text{if } t > 0 \end{cases}$$

for all $r \in \mathbb{R}$. By Theorem 3.1, and Theorem 3.2, there exists an unique mapping $C : X \longrightarrow Y$ with (2.1) and (3.16). Since $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$, $G_C(y, x) = G_C(y, -x)$ for all $x, y \in X$ and letting $y = 0$ in $D_C(x, y) = 0$, we have $C(2x) = 8C(x)$ for all $x \in X$. By Theorem 2.1, we have the result. □

We can use Corollary 3.3 and Corollary 3.4 to get a classical result in the framework of normed spaces. As an example of $\phi(x, y)$ in Corollary 3.3 and Corollary 3.4, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following example.

Example 3.7. Let X be a normed space and Y a Banach space. Suppose that if g satisfies (2.1), then g is a cubic mapping. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x, y)\| \leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{3.17}$$

for all $x, y \in X$ and a fixed positive real numbers p, ϵ with $p \neq \frac{3}{2}$. Suppose that

$$\|G_f(0, x)\| \leq \epsilon \max\{a_1, a_2, a_3\}\|x\|^{2p}, \;\; \|G_f(x, -x)\| \leq \epsilon \max\{b_1, b_2, b_3\}\|x\|^{2p}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1, 2, 3)$. Then there is an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{7\epsilon}{3|8 - 2^{2p}|} \max\{3, a_1, a_2, 3a_3, b_1, b_2, 3b_3\}\|x\|^{2p}$$

for all $x \in X$.

Proof. Define a fuzzy norm N' on $\mathbb{R}$ by

$$N_{\mathbb{R}}(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Similary we can define a fuzzy norm N_Y on Y. Then (Y, N_Y) is a fuzzy Banach space. Let $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then by definitions N_Y and N', the following inequality holds :

$$N_Y(Df(x, y), t) \geq N_{\mathbb{R}}(\phi(x, y), t)$$

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for all $x, y \in X$ and all $t > 0$. By Corollary 3.3 and Corollary 3.4, we have the result. □

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