On an inequality of Hadamard product and the weighted version of arithmetic and harmonic

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Abstract

In this paper we give inequalities involving the Hadamard product and arithmetic-harmonic means of matrices. Moreover, we prove the trace inequality of the product of the arithmetic and harmonic means.

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1 Introduction

Let M_n denote the set of all $n \times n$ complex matrices. The Hadamard product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is their element-wise product which is given by

$$A \circ B = [a_{ij}b_{ij}].$$

If $A \in M_n$ and all eigenvalues for A are real, $\lambda_i(A)$ denotes the i th largest eigenmvalues of A.

Let A and B be positive definite and $t \in (0, 1)$. Then the weighted version of arithmetic, geometric and harmonic means are defined [8], respectively, by

$$A \nabla_t B = tA + (1 - t)B$$
$$A \sharp_t B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{-1}{2}}$$
$$A !_t B = [tA^{-1} + (1 - t)B^{-1}]^{-1}.$$

For $t = 1/2$, we simply put $A\#B$ for $A\#_{1/2}B$. The usual arithmetic, geometric and harmonic means correspond to $t = 1/2$. The definitions extend to positive semidefinte matrices by continuity. For the rest of this paper, we assume that A and B are positive definite. It is well known that the following inequalities are valid:

$$A!_tB \leq A\#_tB \leq A\nabla_tB,$$

$$(A \circ B) \geq (A\#B) \circ (A\#B)$$

$$(A!_tB)\#(A\nabla_{1-t}B) = A\#B$$

Bapat and Sunder [2] proved that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B), \qquad k = 1, \ldots, n. \tag{1}$$

The Horn theorem [6] says that

$$\prod_{i=k}^{n} \lambda_i(AB) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B), \qquad k = 1, \ldots, n. \tag{2}$$

Bapat and Johnson [5] proved that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \ldots, n. \tag{3}$$

In 2017 Hiai and Lin [4] gave a weighted extension of results of Ando [1]

$$\prod_{i=k} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i \left((A\#_{1-t}B)(A\#_tB)\right) \geq \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \ldots, n.$$

In this paper we have to prove the inequalities

$$\prod_{i=k} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i \left((A\nabla_{1-t}B)(A!_tB)\right) \geq \prod_{i=k}^{n} \lambda_i(AB), \quad k = 1, \ldots, n$$

for $0 \leq t \leq 1$. We also prove some trace inequality of the product of the arithmetic and harmonic means.

2 Main Results

The following lemma are well know, we will apply it to prove the next theorem.

Lemma 1. *[4] Let X be a selfadjoint matrix. For any positive semidefinite A, B, the matrix* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *is positive semidefinite if and only if $XA^{-1}X \leq B$.*

Lemma 2. *[6] Let A, B be positive semidefinite. Then $A\sharp B$ is the maximum of all selfadjoint matrix X for which* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *is positive semidefinite.*

We have to prove some lemmas before proving the main theorem.

Lemma 3. *Let A, B be positive definite matrices. Then for $0 \leq t \leq 1$,* $\begin{bmatrix} A!_tB & A\sharp B \\ A\sharp B & A\nabla_{1-t}B \end{bmatrix}$ *is positive semidefinite.*

Proof. We have to show $A\sharp B\,(A!_tB)^{-1}A\sharp B \leq A\nabla_{1-t}B$. Since we know that

$$
\begin{aligned}
A\sharp B\,(A!_tB)^{-1}A\sharp B &= tA^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}A^{-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \\
&\quad + (1-t)A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-1/2}A^{1/2}B^{-1}A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \\
&= tA^{1/2}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} + (1-t)A \\
&= A\nabla_{1-t}B,
\end{aligned}
$$

by lemma 2 we have $\begin{bmatrix} A!_tB & A\sharp B \\ A\sharp B & A\nabla_{1-t}B \end{bmatrix}$ is positive semidefinite. □

Lemma 4. *Let A, B be positive definite matrices. For all $0 \leq t \leq 1$, $A \circ B \geq (A!_tB) \circ (A\nabla_{1-t}B))$*

Proof. To prove this lemma, we will use the equation obtained from the definition of the geometric mean $(AB^{-1}A)\sharp B = A$ and the inequality $A \circ B \geq$

$(A\#B) \circ (A\#B)$. Then we have that,

$$
\begin{aligned}
(A!_tB) \circ (A\nabla_{1-t}B) &= ((tA^{-1} + (1-t)B^{-1})^{-1} \circ ((1-t)A + tB) \\
&= t^{-1}[A - (1-t)A\left((1-t)A + tB\right)^{-1}A] \circ \left((1-t)A + tB\right) \\
&= [t^{-1}(1-t)A \circ A + A \circ B] \\
&\quad - t^{-1}(1-t)A\left((1-t)A + tB\right)^{-1}A \circ \left((1-t)A + tB\right) \\
&\leq [t^{-1}(1-t)A \circ A + A \circ B] - t^{-1}(1-t)A \circ A \\
&= A \circ B.
\end{aligned}
$$

□

Theorem 5. *Let A, B be positive definite matrices. Then for* $0 \leq t \leq 1$

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i\left((A\nabla_{1-t}B)(A!_tB)\right) \geq \prod_{i=k}^{n} \lambda_i(AB), \quad k = 1, \ldots, n.$$

Proof. By Lemma 4 and the inequality (2), we get

$$\lambda_i(A \circ B) \geq \lambda_i\left((A\#_{1-t}B) \circ (A!_tB)\right), \quad i = 1, \ldots, n$$

and by the inequality (2) and (3) for any $k = 1, \ldots, n$, we have

$$
\begin{aligned}
\prod_{i=k}^{n} \lambda_i(A \circ B) &\geq \prod_{i=k}^{n} \lambda_i\left((A\nabla_{1-t}B) \circ (A!_tB)\right) \\
&\geq \prod_{i=k}^{n} \lambda_i\left((A\nabla_{1-t}B)(A!_tB)\right) \\
&\geq \prod_{i=k}^{n} \lambda_i(A\nabla_{1-t}B)\lambda_i(A!_tB). \qquad (4)
\end{aligned}
$$

We proceed to prove the second inequality of this theorem by using lemma 3 and theorem 2.5 of [4]. As for any $k = 1, \ldots, n$

$$
\begin{aligned}
\prod_{i=k}^{n} \lambda_i(A\nabla_{1-t}B) &= \prod_{i=k}^{n} \lambda_i\left((A!_tB)^{-1}\right) \lambda_i\left((A\#B)(A\#B)\right) \\
&\geq \prod_{i=k}^{n} \lambda_i(A!_tB)^{-1}\lambda_i(AB). \qquad (5)
\end{aligned}
$$

$i=k$

□

That is, $\prod_{i=k}^{n} \lambda_i(A \nabla_{1-t} B)\lambda_i(A !_t B) \geq \prod_{i=k}^{n} \lambda_i(AB)$.

Next we will prove some trace inequality of the product of the arithmetic and harmonic means.

Lemma 6. *Let A, B be positive definite matrices. Then*

$$\det((A!_t B)(A\nabla_{1-t}B)) = \det(AB).$$

Proof. For any the positive definite matrices A and B, $\det(A\#B)$ can be directly calculated as follows

$$\begin{aligned}\det(A\#B) &= \det\left(A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}\right)\\ &= \det(A^{1/2})\det(B^{1/2}). \qquad (6)\end{aligned}$$

Applying this result, it follows that

$$\det((A!_t B)\#(A\nabla_{1-t}B)) = \det((A!_t B)^{1/2})\det((A\nabla_{1-t}B)^{1/2}).$$

But we know that $(A!_t B)\#(A\nabla_{1-t}B) = A\#B$. Thus

$$\det((A!_t B)(A\nabla_{1-t}B)) = \det(AB).$$

□

Theorem 7. *Let A, B be positive definite matrices. Then*

$$tr((A\nabla_{1-t}B)(A!_t B)) \le tr(AB).$$

Proof. Apply lemma 6 by using the relation of the determinant and eigenvalue, we get

$$\prod_{i=1}^{n}\lambda_i((A!_t B)(A\nabla_{1-t}B)) = \prod_{i=1}^{n}\lambda_i(AB).$$

Thus the second inequality of theorem 5 is equivalent to

$$\prod_{i=1}^{k}\lambda_i((A!_t B)(A\nabla_{1-t}B)) \le \prod_{i=1}^{k}\lambda_i(AB), \qquad k = 1\ldots n.$$

Next, we use the example II.3.5(vii) of [3] implies

$$\mathrm{tr}((A\nabla_{1-t}B)(A!_t B)) \le \mathrm{tr}(AB).$$

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