Around Poisson New XLindley Process: Comparison and Application in Actuarial Science

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Abstract. This paper deals with a new version of the non-homogeneous Poisson process (called the Poisson New XLindley Process). Some statistical properties are presented. In addition, a comparative study with the counting process version of Poisson, Poisson Lindley, Poisson XLindley, and Poisson's new XLindley process is given using the ruin model.

Key words: Stochastic processes, Poisson process, stochastic properties, Compound Poisson Lindley process.

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1. Introduction

According to experts, the renewal process (RP) and the non-homogeneous Poisson process (NHPP), of which the homogeneous Poisson process (HPP) is a specific case of these two models, are the most commonly used process models to simulate periodic events. We can use the arrival time between two events according to the independent and identical exponential distribution (i.i.d.) to determine the HPP. The fact that the HPP has independent and stationary increasing properties is its main feature. In RP, the arrival time is calculated by a random distribution with non-negative support. The independent increasing property is preserved in the NHPP while the stationary increasing property of the HPP is lost. This allows for clear results in various applications, which is one of the main advantages of NHPP. Several attempts have been made to generalize this simple counting procedure to more regular procedures. For example, NHPP or HPP are generalized using complex, filtered, high-dimensional, and remarkable Poisson procedures. Semi-Markovian procedures are studied in detail and used in many applications as generalized counting techniques. Although process counting trends are widely used, as mentioned previously, the number of technical counting trends that can be used in different contexts is still very limited, and there is still a significant gap between the need for trending in many programs and the availability of actual trends. NHPP is not always suitable due to these limitations, but if a new version of the counting method that overcomes these limitations is properly developed, it could be used in practice. Counting, which tracks the number of events occurring at a given point in time, can be used in a variety of fields such as finance, epidemiology, and reliability analysis to study events such as customer arrivals, machine failures, or disease outbreaks. These models help us understand and predict the timing and frequency of these events, which is essential for decision making and risk assessment in many real-world situations.

A non-uniform Poisson process is a stochastic model used in probability theory and statistics to describe the occurrence of events over time when the magnitude of the event is not constant. In a traditional Poisson process, events occur at a constant rate, but in a non-uniform Poisson process, the rate of events varies over time. This model is particularly useful in situations where events are more or less likely to occur over time. For example, it can be applied to model customer visits to a store, where the arrival rate may be higher during certain hours or days.

By combining the Poisson and Lindley distributions in 1970, Sankaran [8]created the Poisson-Lindley distribution, of which the probability mass function is

$$f(\xi;\omega)=\frac{\varepsilon^{2}(\xi+\omega+2)}{(1+\omega)^{\xi+3}},\xi=0,1,\dots,\omega>0$$

Several works (see Shanker ([12], [13], [14])), Grine and Zeghdoudi [3], Zeghdoudi and Nedjar ([15], [7]), Seghier et al. [9] introduce new discrete distributions by compound Poisson and others introduce new continuous distributions like Poisson-Amarendra[12], Poisson-Sujatha[13], Poisson-Garima[14], Poisson Quasi-Lindley[3], Poisson Pseudo-Lindley[15], Poisson-Gamma Lindley[10], PoissonXLindley [11], and Poisson- new XLIndley.distributions [9] .

This paper uses the following approach. The new process is designed as a Poisson process with an new XLindley mixture distribution. Based on this construction, the basic properties are determined. The new model allows for a wider range of possible process behaviors than the Poisson process and the Lindley Poisson process (Cha [2]), while remaining quite mathematically tractable. The Poisson-new XLindley (PNXL) process is a statistical model that combines elements of the Poisson process and the new XLindley distribution. This process is used to model random events or quantities in various applications, such as reliability analysis, computational science, and queuing theory.

The motivations for undertaking this work are:

- The Poisson new XLindley process is a statistical model that combines elements of Poisson and new XLindley distribution.
- It is simple and easy to use and most properties can be calculated.
- It is a more flexible model than the standard Poisson, Poisson-Lindley and Poisson-new XLindley processes.

- It is used to model random events and quantities in a variety of applications such as: reliability analysis, residual models, actuarial science and failure probability models.

The paper is structured as follows: In Section 2, a new counting process model is defined and the distribution of the number of events in a certain time is derived. Furthermore, the variance and mean are derived from the number of occurrences in $(0, t)$. The random strength of the modified counting process model is determined in Section 3. Finally, a simulation study is used to demonstrate the usefulness of the proposed processs in comparison with the Poisson, Poisson-Lindley, and Poisson-XLindley processes.

2 . Poisson NXLindley Process and its fundamental characteristics.

Developing a novel counting process model with well-defined mathematical properties is one of the primary objectives of this work. We use the idea that was used to generate the Poisson NXLindley distribution toward this goal. Let X follow the NXLindley distribution with parameter θ with its probability density function (pdf)

$$f_{NXLD}(x;\theta) = \frac{\theta}{2}(1+\theta x)e^{-\theta x}, x, \theta > 0$$

The r^{th} moment about the origin of the NXLindley distribution is given by (see, e.g., Khodja et al. [6]).

$$\mu'_{(r)} = \frac{(r+2)r!}{2\theta^r}, r = 1,2, \dots$$

A comprehensive treatment of the mathematical properties of the NXLindley distribution including estimation and simulation issues is also provided in [6]. The Poisson NXLindley distribution ([9]) is generated by the mixture of the Poisson distribution with mean Φ, resulting in the corresponding probability mass function

$$\begin{aligned} P_{PNXLD} &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \cdot \left(-\frac{\Phi}{2}\right)(1-\Phi\lambda)e^{\lambda\Phi} d\lambda \\ &= \frac{\theta}{2}\left[\frac{\theta x + 2\theta + 1}{(1+\theta)^{x+2}}\right] \end{aligned}$$

The p.m.f. of the Poisson NXLindley distribution (PNXLD) with parameter θ can be obtained as

$$P_{PNXLD}(x;\theta) = \frac{\theta(\theta x + 2\theta + 1)}{2(1+\theta)^{x+2}}, x = 0,1,2, \dots; \theta > 0 \quad (1)$$

The r^{th} factorial moment about origin of the PNXLD (1) can be obtained as

$$\mu'_{(r)} = \frac{r!\,(r+2)}{2\theta^r}, r = 1,2,3, \dots$$

The first four moments about origin and the variance of PNXLD obtained by Seghier et al. [9] are as follows

$$\mu_1' = \frac{3}{2\theta}, \quad \mu_2' = \frac{8+3\theta}{2\theta^2}, \quad \mu_3' = \frac{3\theta^2+24\theta+30}{2\theta^3}, \quad \mu_4' = \frac{3\theta^3+56\theta^2+180\theta+144}{2\theta^4} \text{ and Var} = \frac{6\theta+7}{4\theta^2}$$

To broaden the scope of this Poisson NXLindley distribution into a model for counting processes, wherein the probability of the number of events is explicitly defined, the concept involves introducing an extra time-dependent component into the mean value of the Poisson distribution during the mixing process described in equation (1). Let us consider an orderly counting process $\{N(t), t \geq 0\}$. In this work, as in [2], we will using the notation

$$\{N(t), t \geq 0\} \sim NHPP(\nu(t))$$

to indicate that the counting process $\{N(t), t \geq 0\}$ follows the NHPP with its intensity function ν. In addition, we will indicate that the continuous random variable θ obeys the NXLindley distribution with parameter θ by using the notation $\Phi \sim NXLD(\theta)$. The definition of the Poisson NXLindley process (PNXLP) is given below.

Definition 1 (Poisson NXLindley Process).
A counting process $\{N(t), t \geq 0\}$ is called the Poisson NXLindley process (PNXLP) with the set of parameters $(\lambda(t), \theta), \theta > 0, \lambda(t) \geq 0, \forall t \geq 0$, if

$$I\{N(t), t \geq 0\} \mid (\Phi = \phi) \sim NHPP(\phi\lambda(t))$$
$$II\Phi \sim NXLD(\theta)$$

During the entire document, the Poisson NXLindley process with the set of parameters $(\lambda(t), \theta)$ will be denoted by $PNXLP(\lambda(t), \theta)$ and we define$\Lambda(t) = \int_0^t \lambda(x)dx$.

Now we will derive some basic properties ofPNXLP$(\lambda(t), \theta)$. First, the number(s) of events in a given time interval(s)may be of interest when working with a counting process model.

Proposition 2 Let $\{N(t), t \geq 0\}$ be the PNXLP$(\lambda(t), \theta)$. Then, for $t > 0$and $0 \equiv t_0 < t_1 < t_2 < \cdots < t_m$, the following properties hold:

I.

$$P(N(t) = n) = \frac{\theta}{2}\Lambda(t)^n \left[\frac{2\theta + \Lambda(t) + \theta n}{(\theta + \Lambda(t))^{n+2}}\right]$$

II.

$$P(N(t_2) - N(t_1) = n) = \frac{\theta}{2}\left(\Lambda(t_2) - \Lambda(t_1)\right)^n \left[\frac{2\theta + \left(\Lambda(t_2) - \Lambda(t_1)\right) + \theta n}{\left(\theta + \left(\Lambda(t_2) - \Lambda(t_1)\right)\right)^{n+2}}\right]$$

III.

$$P(N(t_i)-N(t_{i-1})=n, i=1,2,\dots,m)=\frac{\theta}{2}\left[\prod_{i=1}^{m}\frac{\left(\Lambda(t_i)-\Lambda(t_{i-1})\right)^{n_i}}{n_i!}\right]\left(\sum_{i=1}^{m}n_i\right)! * \left[\frac{2\theta+\sum_{i=1}^{m}\left(\Lambda(t_i)-\Lambda(t_{i-1})\right)+\theta\sum_{i=1}^{m}n_i}{\left(\theta+\sum_{i=1}^{m}\left(\Lambda(t_i)-\Lambda(t_{i-1})\right)\right)^{\sum_{i=1}^{m}n_i+2}}\right]$$

Proof. see [1].

Proposition 3 Let $\{N(t), t \geq 0\}$ be the PNXLP$(\lambda(t), \theta)$. Thus, the ensuing characteristics are true:

i. $N(t)$'s moment generating function is provided by

$$\psi(s)=\frac{\theta}{2}\left[\frac{1}{(\theta+\Lambda(t)-e^{s}\Lambda(t))}+\frac{\theta}{(\theta+\Lambda(t)-e^{s}\Lambda(t))^{2}}\right], s<ln\left(\frac{\theta+\Lambda(t)}{\Lambda(t)}\right)$$

ii. $N(t)$'s mean and variance are provided by

$$E[N(t)]=\frac{3}{2\theta}\Lambda(t)$$

and

$$\text{Var}[N(t)]=\frac{\Lambda(t)(6\theta+7\Lambda(t))}{4\theta^{2}}$$

Proof. see [1].

Proposition 4 Let $\{N(t), t \geq 0\}$ be the Poisson GD process. Then

$$\text{Var}(N(t))>E[N(t)]$$

Proof. Since the proof is similar to that given in Cha and Mercier [4], it is omitted.

3. Compound PNXLP

The compound Poisson NXLindley process of the stochastic process $\{W(t), t \geq 0\}$ is defined as follows:

$$\left\{W(t)=\sum_{i=1}^{N(t)}X_i, t\geq 0\right\}$$

where$\{N(t), t \geq 0\}$ is the PNXLP, and $\{X_i, i \geq 1\}$ is a family of identically distributed random variables that are independent of $\{N(t), t \geq 0\}$. Let $M_x(s) \equiv E[e^{sX_i}]$, the MGF of X_i. The mean, variance, and moment generating function of $W(t)$ are shown in the following result.

Theorem (t) 's moment generating function, represented by $M_{W(t)}(s)$, is as follows:

$$M_{W(t)}(s) = \frac{\theta}{2}\left[\frac{1}{(\theta + \Lambda(t) - M_X(s)\Lambda(t))} + \frac{\theta}{(\theta + \Lambda(t) - M_X(s)\Lambda(t))^2}\right]$$

where (t) 's mean and variance are

$$E[W(t)] = \frac{3}{2\theta} E[X]\Lambda(t)$$

and

$$\begin{aligned}\mathrm{Var}[W(t)] &= \frac{3}{2\theta} E[X^2]\Lambda(t) + \frac{9}{4\theta^2}(E[X]\Lambda(t))^2 \\ &= \frac{9(E[X]\Lambda(t))^2 + 6\theta E[X^2]\Lambda(t)}{4\theta^2}\end{aligned}$$

Proof. Upon conditioning on $N(t)$ we get,

$$\begin{aligned}M_{W(t)}(s) &= \sum_{n=0}^{\infty} E\left[e^{sW(t)} \mid N(t) = n\right] P(N(t) = n) \\ &= \sum_{n=0}^{\infty} E\left[e^{s(X_1+X_2+\cdots+X_n)} \mid N(t) = n\right] P(N(t) = n) \\ &= \sum_{n=0}^{\infty} E\left[e^{s(X_1+X_2+\cdots+X_n)}\right] P(N(t) = n) \\ &= \sum_{n=0}^{\infty} (M_X(s))^n P(N(t) = n).\end{aligned}$$

Subsequently, by employing analogous reasoning to that elucidated in the proof of Proposition 2, we obtain the anticipated outcomes.

4. Simulation Study: Ruin model

We take the continuous-time model is one where the losses are modeled by a compound Poisson XLindley process: for $> 0 U_t = u + ct - W(t)$, where

$$W(t) = \sum \sum_{i=1}^{N(t)} X_i, t \geq 0$$

Or

- u represents the initial capital,
- c is the premium per unit of time,
- $W(t)$represents the aggregate losses up to time t, with
  $N(t)$ is Poisson XLindley process of intensity λ,
  the X_i represent the individual losses, they are assumed to be i.i.d. of expectation $E[X_i]$ and independent of $N(t)$.
  We have

$$E[U_t] = u + \left(c - (\theta(2+\theta) + 2)\lambda \frac{E[X_i]}{\theta(1+\theta)^2}\right) t, \mathrm{Var}(U_t) = t\mathrm{Var}[W(t)]$$

For compound Poisson new XLindley process:

$$E[U_t] = u + (c - 3\lambda E[X_i]/2\theta)t, \mathrm{Var}(U_t) = t\mathrm{Var}[W(t)]$$

Remark 6 (Compound Poisson XLindley process) We can show that if $c \leq (\theta(2+\theta) + 2)\lambda \frac{E[X_i]}{\theta(1+\theta)^2}$, then the company is sure to be ruined.

We defined $L(u)P$ (it exists $t \geq 0: U_t < 0/U_0 = u) = 1$. In the case where $c > (\theta(2+\theta) + 2)\lambda \frac{E[X_i]}{\theta(1+\theta)^2}$, using the law of large numbers, we have $\frac{W(t)}{t} = \frac{N(t)}{t} \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \to (\theta(2+\theta) + 2)\lambda \frac{E[X_i]}{\theta(1+\theta)^2}$ when $t \to \infty$ and therefore $U_t = u + t\left(c - \frac{W(t)}{t}\right) \to +\infty$, when $t \to +\infty, L(u) < 1$.

(Compound Poisson new XLindley process) We can show that if $\leq 3\lambda E[X_i]/2\theta$, then the company is sure to be ruined.

We defined $L(u)P($ it exists $t \geq 0: U_t < 0/U_0 = u) = 1$. In the case where $> 3\lambda E[X_i]/2\theta$, using the law of large numbers, we have $\frac{W(t)}{t} = \frac{N(t)}{t} \frac{1}{N(t)} \sum_{i=1}^{N(t)} X_i \to 3\lambda E[X_i]/2\theta$ when $t \to \infty$ and therefore $U_t = u + t\left(c - \frac{W(t)}{t}\right) \to +\infty$, when $t \to +\infty, L(u) < 1$.

Now, if $X_i \rightsquigarrow$ exponential distribution where $E[X_i] = V[X_i] = 1$ and $E\left[X_i^2\right] = 2$.

Table 1: Compound Poisson New XLindley Process

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.025	2.0625
50	5	0.5	1	1	54.25	2.0625
75	10	1	2	1	84.25	2.0625
75	30	1	2	1	104.25	2.0625
75	5	10	2	1	72.5	71.25
100	20	2	5	2	138.8	3.12
100	20	0.2	0.5	2	138.8	3.12
150	50	1	3	2	249.0	2.5
150	10	1	3	2	169.0	2.5
150	1	5	1	2	137.0	142.5

Table 2: Compound Poisson XLindley Process

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.153	2.412
50	5	0.5	1	1	54.375	1.64
75	10	1	2	1	84.444	1.419
75	30	1	2	1	104.44	1.419
75	5	10	2	1	74.444	41.975
100	20	2	5	2	139.18	0.991
100	20	0.2	0.5	2	138.84	1.489
150	50	1	3	2	249.29	0.833
150	10	1	3	2	169.29	0.833
150	1	5	1	2	139.5	51.563

Table 3: Compound Poisson Lindley Process

u	c	λ	θ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	0.2	1	10.083	2.326
50	5	0.5	1	1	54.25	1.937
75	10	1	2	1	84.333	1.722
75	30	1	2	1	104.33	1.722
75	5	10	2	1	73.333	52.222
100	20	2	5	2	139.07	1.142
100	20	0.2	0.5	2	138.67	1.635
150	50	1	3	2	249.17	0.993
150	10	1	3	2	169.17	0.993
150	1	5	1	2	137	58.75

Table 4: Compound Poisson Lindley Process

u	c	λ	T	$E[U_t]$	$V[U_t]$
10	1	0.1	1	10.9	0.2
50	5	0.5	1	54.5	1
75	10	1	1	84.0	2
75	30	1	1	104.0	2
75	5	10	1	70	20
100	20	2	2	139.6	8
100	20	0.2	2	139.6	0.8
150	50	1	2	248.0	4
150	10	1	2	168.0	4
150	1	5	2	142	20

Tables 1, 2, 3 and 4 presents the expectation and variance of U_t using Poisson new XLindley, Poisson XLindley, Poisson Lindley process.

According to tables 1-4, the compound Poisson new XLindley, compound Poisson Lindley and compound Poisson XLindley process gives satisfactory results with respect to compound Poisson process because the proposed process has more parameters which can overcome the disadvantage of compound Poisson process. Moreover, remark 6 has been affirmed. Also, the compound Poisson new XLindley process gives satisfactory results with respect to compound Poisson Lindley and compound Poisson XLindley process.

5. Conclusion

We proposed a Poisson NXLindley process in this work. Some properties for this new process are shown. In addition, a suggestion to apply this process using the ruin model and a simulation study to compare the proposal process with Poisson, Poisson Lindley and Poisson XLindley process are given. The proposed process provides efficient results as compared to a Poisson, Poisson Lindley and Poisson XLindley process. In future studies, We use the surplus process model of an insurance company.

We have

$$U_t = u + P_t - W(t)$$

where:

- U_t represents the surplus of the company at time t.;
- u is the initial capital ($U_0 = u$);
- P_t is the gain process (premiums received, interest from investments and all other sources);
- $W(t)$is the loss process (compensation paid, interest from credits, etc.)..

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