# Differential equations and inclusions involving mixed fractional derivatives with four-point nonlocal fractional boundary conditions

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**Abstract**

We study a new class of boundary value problems of mixed fractional differential equations and inclusions involving both left Caputo and right Riemann-Liouville fractional derivatives, and nonlocal four-point fractional boundary conditions. We apply the standard tools of the fixed-point theory to obtain the sufficient criteria for the existence and uniqueness of solutions for the problems at hand. Illustrative examples for the obtained results are also presented.

**Keywords**: Fractional differential equations; fractional differential inclusions; fractional derivative; boundary value problem; existence; fixed point theorems.
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## 1 Introduction

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [1, 2, 3]. Riemann-Liouville and Caputo are kinds of fractional derivatives. They all generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. As a result, it makes these derivatives very useful at describing the anomalous phenomena, see [4, 5, 6] and references cited therein.

Some solutions of equations containing left and right fractional derivatives were investigated [7, 8, 9]. The left and the right derivatives found interesting applications in fractional variational principles, fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics. In [10], the existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative

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was discussed. In [11, 12], the authors studied the existence of solutions for fractional boundary value problems involving both the left Riemann-Liouville and the right Caputo fractional derivatives.

In this paper, we investigate the existence and uniqueness of solutions for a mixed fractional differential equation involving both left Caputo and right Riemann-Liouville types fractional derivatives associated with nonlocal four-point fractional boundary conditions. Precisely, we study the following problems:

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) = f(t, y(t)), \quad t \in J := [0, 1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) \in F(t, y(t)), \quad t \in J := [0, 1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{1.2}$$

where ${}^cD^{\alpha}_{1-}$ and $D^{\beta}_{0+}$ denote the left Caputo fractional derivative of order $\alpha \in (1, 2]$ and the right Riemann-Liouville fractional derivative of order $\beta \in (0, 1]$ respectively, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\delta \in \mathbb{R}$ is an appropriate constant. Here we remark that the problem (1.1) with $y'(0) = 0$ in palce of $D^{\beta}_{0+}y(\xi) = 0$, was studied recently in [13].

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and prove a basic result that plays a key role in the forthcoming analysis. Section 3 contains the existence and uniqueness results for the problem (1.1), which rely on fixed point theorems due to Banach, Krasnoselskii and Leray-Schauder nonlinear alternative. In Section 4, we discuss existence results for the problem (1.2), which rely on nonlineqar alternative for Kakutani maps and Covitz and Nadler fixed point theorem. Finally in Section 5 we study illustrative examples for the obtained results.

# 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts [14] that we need in the sequel.

**Definition 2.1** *We define the left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ as*

$$I^{\alpha}_{0+}g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.1}$$

$$I^{\alpha}_{1-}g(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.2}$$

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*provided the right-hand sides are point-wise defined on* $(0, \infty)$*, where* $\Gamma$ *is the Gamma function.*

**Definition 2.2** *The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $g : (0, \infty) \to \mathbb{R}$ *such that* $g \in C^n((0, \infty), \mathbb{R})$ *are respectively given by*

$$D_{0+}^{\alpha} g(t) = \frac{d^n}{dt^n}(I_{0+}^{n-\alpha} g)(t),$$

$${}^{c}D_{1-}^{\alpha} g(t) = (-1)^n I_{1-}^{n-\alpha} g^{(n)}(t),$$

*where* $n - 1 < \alpha < n$.

The following lemma, dealing with a linear variant of the problem (1.1), plays an important role in the forthcoming analysis.

**Lemma 2.3** *Let* $h \in C(J, \mathbb{R})$ *and* $P = [(1 - \delta\eta^{\beta+1}) - (\beta + 1)\xi(1 - \delta\eta^{\beta})] \neq 0$*. The function* $y$ *is a solution of the problem*

$$\begin{cases} {}^{c}D_{1-}^{\alpha} D_{0+}^{\beta} y(t) = h(t), \quad t \in J := [0, 1], \\ y(0) = 0, \; D_{0+}^{\beta} y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{2.3}$$

*if and only if*

$$\begin{aligned} y(t) = {} & I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + \frac{[t^{\beta+1}(1 - \delta\eta^{\beta}) - t^{\beta}(1 - \delta\eta^{\beta+1})]}{P\Gamma(\beta + 1)} I_{1-}^{\alpha} h(t)|_{t=\xi} \\ & + \frac{[t^{\beta+1} - \xi(\beta + 1)t^{\beta}]}{P} \left(\delta I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=\eta} - I_{0+}^{\beta} I_{1-}^{\alpha} h(t)|_{t=1}\right), \end{aligned} \tag{2.4}$$

*where* $I_{1-}^{\alpha} y(s)$ *is defined by (2.2).*

**Proof.** Applying the right fractional integral $I_{1-}^{\alpha}$ to both sides of the equation in the problem (2.3), we get

$$D_{0+}^{\beta} y(t) = I_{1-}^{\alpha} h(t) + c_0 + c_1 t. \tag{2.5}$$

Using the condition $D_{0+}^{\beta} y(\xi) = 0$ in (2.5), we obtain

$$c_0 + c_1 \xi = -I_{1-}^{\alpha} h(t)|_{t=\xi}. \tag{2.6}$$

Next we apply the left fractional integral $I_{0+}^{\beta}$ to the equation (2.5) to get

$$y(t) = I_{0+}^{\beta} I_{1-}^{\alpha} h(t) + c_0 \frac{t^{\beta}}{\Gamma(\beta + 1)} + c_1 \frac{t^{\beta+1}}{\Gamma(\beta + 2)} + c_2 t^{\beta-1}. \tag{2.7}$$

Making use of the conditions $y(0)=0$ and $y(1)=\delta y(\eta)$ in (2.7) yields $c_2=0$ and

$$\frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)}c_0+\frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)}c_1=\delta I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=\eta}-I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=1}. \tag{2.8}$$

Solving (2.7) and (2.8) for $c_0$ and $c_1$, we find that

$$c_0=-\frac{\Gamma(\beta+2)}{P}\Big[\frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)}I_{1-}^{\alpha}h(t)|_{t=\xi}+\xi\left(\delta I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=\eta}-I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=1}\right)\Big],$$

$$c_1=\frac{\Gamma(\beta+2)}{P}\Big[\delta I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=\eta}-I_{0+}^{\beta}I_{1-}^{\alpha}h(t)|_{t=1}+\frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)}I_{1-}^{\alpha}h(t)|_{t=\xi}\Big].$$

Substituting the values of $c_0$ and $c_1$ in (2.6), we get the solution (2.4). By direct computation, we can obtain the converse of this lemma. This completes the proof. □

**Remark 2.4** *Let* $\|h\|=\sup_{t\in[0,1]}|h(t)$. *Then we have the following estimate:*

$$\|y\|\le\|h\|\max_{t\in[0,1]}\left\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}, \tag{2.9}$$

*where*

$$\mu_1(t)=\frac{t^{\beta+1}(1-\delta\eta^{\beta})-t^{\beta}(1-\delta\eta^{\beta+1})}{P\Gamma(\beta+1)},\quad \mu_2(t)=\frac{t^{\beta+1}-\xi(\beta+1)t^{\beta}}{P}. \tag{2.10}$$

*Indeed, we have*

$$\begin{aligned}
|y(t)| &\le \|h\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds+|\mu_1(t)|\|h\|\int_{\xi}^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\quad+|\mu_2(t)|\|h\|\Bigg[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\
&\quad+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\\
&= \|h\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+|\mu_1(t)|\|h\|\int_{\xi}^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\quad+|\mu_2(t)|\|h\|\Bigg[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds\Bigg]\\
&\le \|h\|\max_{t\in[0,1]}\left\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\},
\end{aligned}$$

*where we taken* $(1-s)^{\alpha}\le 1$.

For computation convenience, we introduce the notation:

$$\Lambda=\max_{t\in[0,1]}\Big\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big\}. \tag{2.11}$$

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# 3 Existence and uniqueness results for the problem (1.1)

Let $\mathcal{X} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t \in [0,1]\}$.

In view of Lemma 2.3, we transform the problem (1.1) into a fixed point problem as

$$y = \mathcal{G}y, \tag{3.1}$$

where the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned} \mathcal{G}y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds\Big], \end{aligned} \tag{3.2}$$

where $\mu_1, \mu_2$ are defined by (2.10).

Our first result deals with the existence and uniqueness of solutions for the problem (1.1).

**Theorem 3.1** *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function such that:*

$(H_1)$ $|f(t,y) - f(t,z)| \le L|y-z|$, *for all* $t \in [0,1]$, $y, z \in \mathbb{R}$, $L > 0$.

*Then the problem (1.1) has a unique solution on* $[0,1]$ *if*

$$L\Lambda < 1, \tag{3.3}$$

*where* $\Lambda$ *is defined by (2.11).*

**Proof.** Let us define $\sup_{t\in[0,1]} |f(t,0)| = M$ and select $r \ge \dfrac{M\Lambda}{1-L\Lambda}$ to establish that $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \le r\}$ and $\mathcal{G}$ is defined by (3.2). Using the condition $(H_1)$, we have

$$\begin{aligned} |f(t,y)| &= |f(t,y) - f(t,0) + f(t,0)| \le |f(t,y) - f(t,0)| + |f(t,0)| \\ &\le L\|y\| + M \le Lr + M. \end{aligned} \tag{3.4}$$

Then, for $y \in \mathcal{B}_r$, by using Remark 2.4, we obtain

$$\begin{aligned} \|\mathcal{G}y\| &\le (Lr+M)\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds \\ &\quad +|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \end{aligned}$$

$$
\begin{aligned}
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du ds \Big] \Big\} \\
= & \ (Lr+M) \Big\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + |\mu_2(t)| \Big[ \delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} ds \Big] \Big\} \\
\leq & \ (Lr+M)\Lambda < r.
\end{aligned}
$$

This show that $\mathcal{G}y \in \mathcal{B}_r$, $y \in \mathcal{B}_r$. Thus $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$. Next we show that $\mathcal{G}$ is a contraction. For that, let $y, z \in \mathcal{X}$. Then, for each $t \in [0,1]$, we have

$$
\begin{aligned}
& \|(\mathcal{G}y) - (\mathcal{G}z)\| \\
\leq & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,y(u)) - f(u,z(u))| du ds \\
& + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s)) - f(s,z(s))| ds \\
& + |\mu_2(t)| \Big[ \delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,y(u)) - f(u,z(u))| du ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,y(u)) - f(u,z(u))| du ds \Big] \\
\leq & \ L\Lambda \|y - z\|,
\end{aligned}
$$

which, in view of the given condition $L\Lambda < 1$, implies that $\mathcal{G}$ is a contraction. In consequence, it follow by the contraction mapping principle that there exists a unique solution for the problem (1.1) on $[0,1]$. This completes the proof. □

Our next existence result for the problem (1.1) relies on Krasnoselskii's fixed point theorem.

**Lemma 3.2** *(Krasnoselskii's fixed point theorem) [15]. Let $S$ be a closed, bounded, convex and nonempty subset of a Banach space $X$. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be the operators mapping $S$ into $X$ such that (a) $\mathcal{Y}_1 s_1 + \mathcal{Y}_2 s_2 \in S$ whenever $s_1, s_2 \in S$; (b) $\mathcal{Y}_1$ is compact and continuous; (c) $\mathcal{Y}_2$ is a contraction mapping. Then there exists $s_3 \in S$ such that $s_3 = \mathcal{Y}_1 s_3 + \mathcal{Y}_2 s_3$.*

**Theorem 3.3** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition $(H_1)$. In addition we assume that:*

$(H_2)$ *$|f(t,y)| \leq m(t)$, for all $(t,y) \in [0,1] \times \mathbb{R}$ and $m \in C([0,1], \mathbb{R}^+)$.*

*Then there exists at least one solution for the problem (1.1) on* $[0,1]$ *provided that*

$$L\sup_{t\in[0,1]}\left\{\frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+|\mu_1(t)|\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}\right\}<1. \tag{3.5}$$

**Proof.** Setting $\sup_{t\in[0,1]}|m(t)|=\|m\|$, we fix

$$\varrho\geq\|m\|\Lambda, \tag{3.6}$$

where $\Lambda$ is defined by (2.11), and consider $B_\varrho=\{y\in\mathcal{X}:\|y\|\leq\varrho\}$. Introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $B_\varrho$ as follows:

$$\begin{aligned}\mathcal{G}_1y(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}f(u,y(u))duds\\ &\quad+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds,\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_2y(t) &= \mu_2(t)\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}f(u,y(u))duds\\ &\quad+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}f(u,y(u))duds\Bigg].\end{aligned}$$

Observe that $\mathcal{G}=\mathcal{G}_1+\mathcal{G}_2$. Now we verify the hypotheses of Krasnoselskii's fixed point theorem in the following steps.

*(i)* For $y,z\in B_\varrho$, we have

$$\begin{aligned}&\|\mathcal{G}_1y+\mathcal{G}_2z\|=\sup_{t\in[0,1]}|(\mathcal{G}_1y)(t)+(\mathcal{G}_2z)(t)|\\ \leq\ &\|m\|\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds+|\mu_1(t)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &+|\mu_2(t)|\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\ &+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\}\\ &\leq\|m\|\Lambda\leq\varrho,\end{aligned}$$

where we used (3.6). Thus $\mathcal{G}_1y+\mathcal{G}_2z\in B_\varrho$.

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*(ii)* We show that $\mathcal{G}_1$ is a contraction. Indeed, by using the assumption $(H_1)$ together with (3.5) and the fact that $(1-s)^\alpha < 1, (1 < \alpha \le 2)$ we have

$$\begin{aligned}
|\mathcal{G}_1 y(t) - \mathcal{G}_1 z(t)| &\le L\|y-z\|\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \right.\\
&\quad \left. +|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}\\
&\le L\|y-z\|\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}\\
&\le L\sup_{t\in[0,1]}\left\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\right\}\|y-z\|,
\end{aligned}$$

which implies that

$$\|\mathcal{G}_1 y - \mathcal{G}_1 z\| \le L\sup_{t\in[0,1]}\left\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\right\}\|y-z\|.$$

Hence $\mathcal{G}_1$ is a contraction by (3.5).

*(iii)* Using the continuity of $f$, it is easy to show that the operator $\mathcal{G}_2$ is continuous. Further, $\mathcal{G}_2$ is uniformly bounded on $B_\varrho$ as

$$\|\mathcal{G}_2 x\| = \sup_{t\in[0,1]}|(\mathcal{G}_2 y)(t)| \le \frac{\|m\|M_2(\delta\eta^\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \quad M_2 = \sup_{t\in[0,1]}|\mu_2(t)|.$$

In order to establish that $\mathcal{G}_2$ is compact, we define $\sup_{(t,y)\in[0,1]\times B_\varrho}|f(t,y)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
|(\mathcal{G}_2 y)(t_2) - (\mathcal{G}_2 y)(t_1)| &\le |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\left[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \right.\\
&\quad \left. +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\right]\\
&\le |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \to 0 \text{ as } t_1 \to t_2,
\end{aligned}$$

independent of $y$. This shows that $\mathcal{G}_2$ is relatively compact on $B_\varrho$. As all the conditions of the Arzelá-Ascoli theorem are satisfied, so $\mathcal{G}_2$ is compact on $B_\varrho$. In view of steps *(i)-(iii)*, the conclusion of Krasnoselskii's fixed point theorem applies and hence there exists at least one solution for the problem (1.1) on $[0,1]$. The proof is completed. □

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**Remark 3.4** *Interchanging the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the foregoing result, we can obtain a second result by requiring the condition:*

$$LM_1 \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1, \quad M_1 = \sup_{t\in[0,1]} |\mu_1(t)|,$$

*instead of (3.5).*

The following existence result is based on Leray-Schauder nonlinear alternative.

**Lemma 3.5** *(Nonlinear alternative for single valued maps)[16]. Let $E$ be a Banach space, $C$ a closed, convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of $C$) map. Then either*

*(i) $F$ has a fixed point in $\overline{U}$, or*

*(ii) there is a $u \in \partial U$ (the boundary of $U$ in $C$) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

**Theorem 3.6** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

$(H_3)$ *There exist a function $g \in C([0,1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,y)| \leq g(t)\psi(\|y\|)$, $\forall (t,y) \in [0,1] \times \mathbb{R}$.*

$(H_4)$ *There exists a constant $K > 0$ such that*

$$\frac{K}{\|g\|\psi(K)\Lambda} > 1.$$

*Then the problem (1.1) has at least one solution on $[0,1]$.*

**Proof.** Consider the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ defined by (3.2). We show that $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X} = C([0,1], \mathbb{R})$. For a positive number $r$, let $\mathcal{B}_r = \{y \in C([0,1], \mathbb{R}) : \|y\| \leq r\}$ be a bounded set in $C([0,1], \mathbb{R})$. Then, by using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1 < \alpha \leq 2)$ we have

$$\begin{aligned} |\mathcal{G}y(t)| &\leq \|g\|\psi(r) \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad + |\mu_2(t)| \Bigg[ \delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \Bigg] \Bigg\} \\ &\leq \|g\|\psi(r)\Lambda, \end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|\mathcal{G}y\| \leq \|g\|\psi(r)\Lambda.$$

Next we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C([0,1], \mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $C([0,1],\mathbb{R})$. Then, using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1 < \alpha \leq 2)$ and the computations for $\mathcal{G}_2$ in previous theorem, we obtain

$$\begin{aligned}
&|\mathcal{G}y(t_2) - \mathcal{G}y(t_1)| \\
\leq\ & \|g\|\psi(r)\Bigg\{\Bigg|\int_0^{t_1} \frac{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]}{\Gamma(\beta)}ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg| \\
& + |\mu_1(t_2) - \mu_1(t_1)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds + |\mu_2(t_2) - \mu_2(t_1)| \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\} \\
\leq\ & \|g\|\psi(r)\Bigg\{\frac{2(t_2-t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} + |\mu_1(t_2) - \mu_1(t_1)| \frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)} \\
& + |\mu_2(t_2) - \mu_2(t_1)| \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\},
\end{aligned}$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As $\mathcal{G}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $y = \lambda\mathcal{G}y$ is bounded for $\lambda \in [0,1]$. For that, let $y$ be a solution of $y = \lambda\mathcal{G}y$ for $\lambda \in [0,1]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
|y(t)| = |\lambda\mathcal{G}y(t)| \ \leq\ & \Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& + |\mu_2(t)| \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds\Bigg]\Bigg\} |g(t)|\psi(\|y\|) \\
\leq\ & \|g\|\psi(\|y\|)\Lambda,
\end{aligned}$$

which implies that

$$\frac{\|y\|}{\|g\|\psi(\|y\|)\Lambda} \leq 1.$$

In view of $(H_4)$, there is no solution $y$ such that $\|y\| \neq K$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < K\}.$$

The operator $\mathcal{G} : \overline{U} \to \mathcal{X}$ is continuous and completely continuous. From the choice of $U$, there is no $u \in \partial U$ such that $u = \lambda \mathcal{G}(u)$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [16, Theorem 5.2], we deduce that $\mathcal{G}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

# 4 Existence results for the problem (1.2)

Before presenting the existence results for the problem (1.2), we outline the necessary concepts on multi-valued maps [17], [18].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex }\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map $G$ is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in $X$ for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). $G$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of $x_0$ such that $G(\mathcal{N}_0) \subseteq N$. $G$ is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. $G$ has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$. A multivalued map $G : [0, 1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

For each $y \in \mathcal{X}$, define the set of selections of $F$ by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

**Definition 4.1** *A function $y \in C([0, 1], \mathbb{R})$ is said to be a solution of the boundary value problem (1.2) if $y(0) = 0$, $D^{\beta}_{0+}y(\xi) = 0$, $y(1) = \delta y(\eta)$, $0 < \xi, \eta < 1$, and there exists a function $v \in S_{F,y}$ such that $v(t) \in F(t, y(t))$ and*

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-}v(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-}v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-}v(s)ds\Big], \quad t \in [0, 1]. \end{aligned}$$

## 4.1 The upper semicontinuous case

In the case when $F$ has convex values we prove an existence result based on nonlinear alternative of Leray-Schauder type.

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**Definition 4.2** *A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

*(i) $t\longmapsto F(t,y)$ is measurable for each $y\in\mathbb{R}$;*

*(ii) $y\longmapsto F(t,y)$ is upper semicontinuous for almost all $t\in[0,1]$.*

*Further a Carathéodory function $F$ is called $L^1-$Carathéodory if*

*(iii) for each $\rho>0$, there exists $\varphi_\rho\in L^1([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,y)\|=\sup\{|v|:v\in F(t,y)\}\le\varphi_\rho(t)$$

*for all $y\in\mathbb{R}$ with $\|y\|\le\rho$ and for a.e. $t\in[0,1]$.*

We define the graph of $G$ to be the set $Gr(G)=\{(x,y)\in X\times Y: y\in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

**Lemma 4.3** *([17, Proposition 1.2]) If $G:X\to\mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X\times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ and $\{y_n\}_{n\in\mathbb{N}}\subset Y$, if when $n\to\infty$, $x_n\to x_*$, $y_n\to y_*$ and $y_n\in G(x_n)$, then $y_*\in G(x_*)$. Conversely, if $G$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

**Lemma 4.4** *([19]) Let $X$ be a separable Banach space. Let $F:[0,1]\times X\to\mathcal{P}_{cp,c}(X)$ be an $L^1-$ Carathéodory multivalued map and let $\Theta$ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta\circ S_{F,x}:C([0,1],X)\to\mathcal{P}_{cp,c}(C([0,1],X)),\quad y\mapsto(\Theta\circ S_{F,y})(y)=\Theta(S_{F,y})$$

*is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.*

For the forthcoming analysis, we need the following lemma.

**Lemma 4.5** *(Nonlinear alternative for Kakutani maps)[16]. Let $E$ be a Banach space, $C$ a closed convex subset of $E$, $U$ an open subset of $C$ and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

*(i) $F$ has a fixed point in $\overline{U}$, or*

*(ii) there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.*

**Theorem 4.6** *Assume that:*

*$(B_1)$ $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is $L^1$-Carathéodory and has nonempty compact and convex values;*

$(B_2)$ *there exist a function* $\phi \in C([0,1], \mathbb{R}^+)$, *and a nondecreasing function* $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|w| : w \in F(t,y)\} \leq \phi(t)\Omega(\|y\|)$$

*for each* $(t,y) \in [0,1] \times \mathbb{R}$;

$(B_3)$ *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\|\phi\|\Lambda\Omega(M)} > 1,$$

*where* $\Lambda$ *is defined by (2.11).*

*Then the boundary value problem (1.2) has at least one solution on* $[0,1]$.

**Proof.** Define an operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\Omega_F(y) = \{h \in \mathcal{X} : h(t) = N(y)(t)\}$$

where

$$\begin{aligned} N(y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t) \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

We will show that $\Omega_F$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega_F$ *is convex for each* $y \in \mathcal{X}$. This step is obvious since $S_{F,y}$ is convex ($F$ has convex values), and therefore we omit the proof.

In the second step, we show that $\Omega_F$ *maps bounded sets (balls) into bounded sets in* $\mathcal{X}$. For a positive number $\rho$, let $B_\rho = \{y \in \mathcal{X} : \|y\| \leq \rho\}$ be a bounded ball in $\mathcal{X}$. Then, for each $h \in \Omega_F(y), y \in B_\rho$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds + \mu_1(t) \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Bigg[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds\Bigg]. \end{aligned}$$

Then, by using the fact that $(1-s)^{\alpha-1} \leq 1 \ (1 < \alpha \leq 2)$ we have

$$|h(t)| \leq \|g\|\Omega(r)\Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$\begin{aligned} &+|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\ &+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\} \\ \leq\ & \|\phi\|\Omega(r)\Lambda, \end{aligned}$$

which, on taking the norm for $t \in [0,1]$. yields

$$\|h\| \leq \|\phi\|\Omega(r)\Lambda.$$

Now we show that $\Omega_F$ *maps bounded sets into equicontinuous sets of* $\mathcal{X}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in B_\rho$. For each $h \in \Omega_F(y)$, using the fact that $(1-s)^{\alpha-1} \leq 1$ $(1 < \alpha \leq 2)$, we obtain

$$\begin{aligned} &|h(t_2)-h(t_1)| \\ \leq\ & \|\phi\|\Omega(r)\Bigg\{\Bigg|\int_0^{t_1} \frac{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)}ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds\Bigg| \\ &+|\mu_1(t_2)-\mu_1(t_1)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds + |\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\} \\ \leq\ & \|\phi\|\Omega(r)\Bigg\{\frac{2(t_2-t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)} + |\mu_1(t_2)-\mu_1(t_1)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)} \\ &+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\}, \end{aligned}$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As $\Omega_F$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega_F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

In our next step, we show that $\Omega_F$ *is upper semicontinuous.* To this end it is sufficient to show that $\Omega_F$ *has a closed graph,* by Lemma 4.3. Let $y_n \to y_*$, $h_n \in \Omega_F(y_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(y_*)$. Associated with $h_n \in \Omega_F(y_n)$, there exists $v_n \in S_{F,y_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_n(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_n(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_n(s)ds\Big]. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds$$

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$$+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds\Big].$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to \mathcal{X}$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds\Big]. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t) - h_*(t)\| \\ = &\ \Big\|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n - v_*)(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} (v_n - v_*)(s)ds \\ &+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n - v_*)(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n - v_*)(s)ds\Big]\Big\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4.4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds\Big]. \end{aligned}$$

Finally, we show there exists an open set $U \subseteq \mathcal{X}$ with $y \notin \theta\Omega_F(y)$ for any $\theta \in (0,1)$ and all $y \in \partial U$. Let $\theta \in (0,1)$ and $y \in \theta\Omega_F(y)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0,1]$, we can obtain

$$\begin{aligned} &|y(t)| = |\theta\Omega_F(y)(t)| \\ \leq &\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \\ &+|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds \\ &+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds\Bigg] \\ \leq &\ \|\phi\|\Omega(\|y\|)\Lambda, \end{aligned}$$

which implies that

$$\frac{\|y\|}{\|\phi\|\Omega(\|y\|)\Lambda} \leq 1.$$

In view of $(B_3)$, there exists $M$ such that $\|y\| \neq M$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(\mathcal{X})$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $y \in \partial U$ such that $y \in \theta\Omega_F(y)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.5), we deduce that $\Omega_F$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.2). This completes the proof. □

## 4.2 The Lipschitz case

We prove in this subsection the existence of solutions for the problem (1.2) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [21].

Let $(X, d)$ be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by $H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [20]).

**Definition 4.7** *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called* (**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leq \gamma d(x, y)$ for each $x, y \in X$ and* (**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

**Lemma 4.8** *([21]) Let $(X, d)$ be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

**Theorem 4.9** *Assume that:*

$(A_1)$ *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, y(t)) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$;*

$(A_2)$ *$H_d(F(t, y), F(t, \bar{y}) \leq q(t)|y - \bar{y}|$ for almost all $t \in [0,1]$ and $y, \bar{y} \in \mathbb{R}$ with $q \in C([0,1], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq q(t)$ for almost all $t \in [0,1]$.*

*Then the problem (1.2) has at least one solution on $[0,1]$ if*

$$\|q\|\Lambda < 1, \tag{4.1}$$

*where $\Lambda$ is defined by (2.11).*

**Proof.** Consider the operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined in the beginning of the proof of Theorem 4.6. Observe that the set $S_{F,y}$ is nonempty for each $y \in \mathcal{X}$ by the assumption $(A_1)$, so $F$ has a measurable selection (see Theorem III.6 [22]). Now we show that the operator $\Omega_F$ satisfies the assumptions of Lemma 4.8. To show that $\Omega_F(y) \in \mathcal{P}_{cl}(\mathcal{X})$

for each $y \in \mathcal{X}$, let $\{u_n\}_{n\geq 0} \in \Omega_F(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{X}$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &+\mu_2(t)\Big[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds\Big]. \end{aligned}$$

As $F$ has compact values, we pass onto a subsequence (if necessary) to obtain that $v_n$ converges to $v$ in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &+\mu_2(t)\Big[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

Hence, $u \in \Omega_F(y)$.

Next we show that there exists $\hat{\theta} := \|q\|\Lambda < 1$ such that

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \hat{\theta}\|y-\bar{y}\| \;\text{ for each }\; y, \bar{y} \in \mathcal{X}.$$

Let $y, \bar{y} \in \mathcal{X}$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds \\ &+\mu_2(t)\Big[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds\Big]. \end{aligned}$$

By $(A_2)$, we have

$$H_d(F(t,y), F(t,\bar{y}) \leq q(t)|y-\bar{y}|.$$

So, there exists $w \in F(t,\bar{y})$ such that

$$|v_1(t) - w| \leq q(t)|y(t)-\bar{y}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq q(t)|y(t)-\bar{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t,\bar{y})$ is measurable (Proposition III.4 [22]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t,\bar{y})$. So $v_2(t) \in F(t,\bar{y})$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq q(t)|y(t)-\bar{y}(t)|$. For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_2(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds$$

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$$+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_2(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_2(s)ds\Big].$$

Thus

$$\begin{aligned} & |h_1(t)-h_2(t)| \\ \le\ & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|v_1-v_2|(s)ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|v_1-v_2|(s)ds \\ & +|\mu_2(t)|\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|v_1-v_2|(s)ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|v_1-v_2|(s)ds\Big] \\ \le\ & \|q\|\Lambda\|y-\bar{y}\|, \end{aligned}$$

which yields $\|h_1-h_2\| \le \|q\|\Lambda\|y-\bar{y}\|$.

Analogously, interchanging the roles of $y$ and $\overline{y}$, we can obtain

$$H_d(\Omega_F(y),\Omega_F(\bar{y})) \le \|q\|\Lambda\|y-\bar{y}\|.$$

By the condition (4.1), it follows that $\Omega_F$ is a contraction and hence it has a fixed point $y$ by Lemma 4.8, which is a solution of the problem (1.2). This completes the proof.□

# 5 Examples

**(a)** We construct examples for the illustration of the results obtained in Section 3. For that, we consider the following problem:

$$\begin{cases} D_{1-}^{7/4} D_{0+}^{3/4} y(t) = f(t,y(t)),\ t\in J := [0,1], \\ y(0)=0,\ D_{0+}^{3/4} y(\xi)=0,\quad y(1)=(5/2)y(2/3), \end{cases} \tag{5.1}$$

Here $\alpha = 7/4, \beta = 3/4, \xi = 1/3, \eta = 2/3, \delta = 5/2$, and

$$f(t,y) = \frac{1}{2\sqrt{t^2+81}}\Big(\cos y + \frac{|y|}{1+|y|}\Big) + \frac{e^{-2t}}{t+4}. \tag{5.2}$$

With the given data, it is found that

$$\begin{aligned} & P = [(1-\delta\eta^{\beta+1}) - (\beta+1)\xi(1-\delta\eta^\beta)] \approx 0.262961 \neq 0, \\ & \sup_{t\in[0,1]}\Big\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Big\} \approx 1.454491, \end{aligned}$$

and $\Lambda \approx 4.503584$ ($\Lambda$ is given by (2.11)). Furthermore, $|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$ with $L = 1/9$ so that $L\Lambda \approx 0.0.500398 < 1$. Clearly the hypothesis of Theorem 3.1 is satisfied and hence the problem (5.1) has a unique solution by the conclusion of Theorem 3.1.

In order to illustrate Theorem 3.3, we notice that (3.5) is satisfied as

$$L\Big\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}+|\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Big\}\approx 0.161610<1,$$

and

$$|f(t,y)|\le m(t)=\frac{1}{\sqrt{t^2+81}}+\frac{e^{-2t}}{t+4}.$$

As all the assumptions of Theorem 3.3 hold true, we deduce from the conclusion of Theorem 3.3 that the problem (5.1) has at least one solution on $[0,1]$.

Now we demonstrate the application of Theorem 3.6 by considering the nonlinear function

$$f(t,y)=\frac{e^{-t}}{\sqrt{t+36}}\Big(y+\frac{2}{\pi}\tan^{-1}y+\frac{1}{10}\Big). \tag{5.3}$$

Clearly $|f(t,y)|\le g(t)\psi(\|y\|)$, where $g(t)=\frac{e^{-t}}{\sqrt{t+36}}$, $\psi(\|y\|=(\frac{11}{10}+\|y\|)$. By the condition $(H_4)$, we find that $K>3.310535$. Thus all the conditions of Theorem 3.6 are satisfied and consequently, the problem (5.1) with $f(t,y)$ given by (5.3) has has at least one solution on $[0,1]$.

**(b)** Here we illustrate the results obtained in Section 4. Let us consider the following fractional differential inclusion involving both left Caputo and right Riemann-Liouville types fractional derivatives equipped with fractional boundary conditions:

$$\begin{cases} D_{1-}^{7/4}D_{0+}^{3/4}y(t)\in F(t,y(t)),\ t\in J:=[0,1],\\ y(0)=0,\ D_{0+}^{3/4}y(\xi)=0,\quad y(1)=(5/2)y(2/3), \end{cases} \tag{5.4}$$

In order to illustrate Theorem 4.6, we take

$$F(t,y(t))=\Bigg[\frac{1}{\sqrt{t^2+49}}\Big(\frac{|y(t)|}{2(1+|y(t)|)}+|y(t)|+\frac{1}{2}\Big),\ \frac{e^{-t}}{9+t}\Big(\sin y(t)+\frac{1}{80}\Big)\Bigg]. \tag{5.5}$$

Clearly $|F(t,y(t))|\le\phi(t)\Omega(\|y\|)$, where $\phi(t)=\frac{1}{\sqrt{t^2+49}}$ and $\Omega(\|y\|)=\|y\|+1$. Using the condition $(B_3)$, we find that $M>1.804018$. As the hypothesis of Theorem 4.6 is satisfied, the problem (5.4) with $F(t,y(t))$ given by (5.5) has at least one solution on $[0,1]$.

Now we illustrate Theorem 4.9 by considering

$$F(t,x(t))=\Bigg[\frac{1}{\sqrt{100+t^2}},\ \frac{\sin x(t)}{(6+t)}+\frac{1}{50}\Bigg]. \tag{5.6}$$

Obviously $q(t)=1(6+t)$ with $\|q\|=1/6$ and $d(0,F(t,0))\le q(t)$ for almost all $t\in[0,1]$. Moreover, $\|q\|\Lambda\approx 0.750597$. Thus all the assumptions of Theorem 4.9 hold true and consequently its conclusion applies to the problem (5.4) with $F(t,y(t))$ given by (5.6).

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