The pebbling number of thorn graphs

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Abstract

Given a distribution of pebbles on the vertices of a connected graph G, the pebbling number of a graph G, is the least number f(G) such that no matter how these f(G) pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Let $p_1, p_2, \ldots, p_n$ be positive integers and G be a graph such that $|V(G)| = n$. The thorn graph of a graph G with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, $i = 1,2, \ldots, n$. In this paper, the pebbling number of thorn graph of a star , also called as thorn star and the pebbling number of any thorn graph is determined.

Key words: Graphs, Pebbling number, thorn graph, thorn star .

1.Introduction:

Pebbling in graphs was first studied by Chung [1]. A pebbling move consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number f(G,v) such that from every placement of f(G, v) pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the pebbling number of a graph G, denoted by f(G), is the maximum f(G.v) over all the vertices v in G. The t-pebbling number, $f_t(G)$ of a graph G, is the least positive integer such that $f_t(G)$ pebbles are placed on the vertices of G, we can move t pebbles to any vertex by a sequence of pebbling moves, each move removes two pebbles off one vertex and placing one on an adjacent vertex. Given a configuration of pebbles placed on G, let p(G) be the number of pebbles placed on the graph G, q be the number of vertices with atleast one pebble and let r be the number of vertices with an odd number of pebbles.

2.Preliminaries:

Definition 2.1: [2]

Let $p_1, p_2, \dots, p_n$ be positive integers and G be a graph with $|V(G)| = n$. The thorn graph of the graph G, with parameters $p_1, p_2, \dots, p_n$ is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, $i = 1,2, \dots, n$.

The thorn graph of the graph G will be denoted by G^* or by G^* $(p_1, p_2, \dots, p_n)$, if the respective parameters need to be specified. In this paper, we will consider the thorn graph with every $p_i \geq 2$ $(i = 1,2, \dots, n)$.

Definition 2.2:

A bipartite graph is a graph whose vertex set can be partitioned into two subsets $V_1$and V_2 such that each edge has one endpoint in V_1 and the other endpoint in V_2. Such a partition (V_1, V_2) is called a bipartition of the graph.

A complete bipartite graph is a simple bipartite graph with bipartition (V_1, V_2) in which each vertex of V_1 is joined to each vertex of V_2. If $|V_1|$ = m and $|V_2|$ = n, then a complete bipartite graph with bipartition (V_1, V_2) is denoted by $K_{m,n}$. The graph $K_{1,n}$ or $K_{1,n}$ is called a star.

3.The pebbling number of thorn star.

Theorem 3.1:

Let $K_{1,n}$ be a star graph. Then the pebbling number of the thorn star $K^*_{1,n}$ is $f(K^*_{1,n}) = 2n + \sum p_i + 10, n > 1$.

Proof:

Let $V(K_{1,n}) = U \cup W$ where $U = \{v_0\}$ and $W = \{v_1, v_2, \dots, v_n\}$.

Let $X_i = \{x_{ij}: x_{ij}$ is adjacent to $v_i, i = 0,1, \dots, n$ and $j = 1,2, \dots, p_i$ and $\deg(x_{ij}) = 1\}$. Let x_{11} be our target vertex and $p(x_{11}) = 0$.

Consider the following distribution of $2n + \sum p_i + 9$ pebbles on the thorn star.

1. Let $p(K_{1,n}) = 0$.
2. Place p_0 pebbles, one pebble each at the vertices of X_0.
3. Place $p_1 - 1$ pebbles, one pebble each at the vertices of $X_1 - \{x_{11}\}$.
4. Place $p_2 - 1$ pebbles, one pebble each at the vertices of $X_2 - \{x_{21}\}$ such that $p(x_{21}) = 15$.
5. Place $p_k - 1$ pebbles, one pebble each at the vertices of $X_k - \{x_{k1}\}$ such that $p(x_{k1}) = 3$, $k = 3,4, \dots, n$.

Clearly two pebbles cannot be moved to v_1 and hence one pebble cannot be moved to x_{11}.
Thus, $f(K^*_{1,n}) \geq 2n + \sum p_i + 10$.
Now, let us show that $f(K^*_{1,n}) \leq 2n + \sum p_i + 10$.
Place $2n + \sum p_i + 10$ pebbles on the thorn star.
Case 1: Let x_{ij} , $i = 0,1,2,\ldots,n$ and $j = 1,2,\ldots,p_i$ be our target vertex.
Without loss of generality, let us assume that x_{11} be our target vertex and $p(x_{11}) = 0$.
If any of the vertices of X_1 has atleast four pebbles with $p(v_1) = 0$ or two pebbles with $p(v_1) = 1$ or $p(v_1) \geq 2$, then one pebble can be moved to x_{11}.
Let us assume that $p(x_{1j}) \leq 1 \ \forall\ j = 2,3,\ldots,p_1$. Hence atleast $2n + \sum p_i - p_1 + 11$ pebbles are placed on the remaining vertices. Hence our target is to place two pebbles on v_1 by which one pebble can be moved to x_{11}.
Subcase 1.1: Suppose the star graph has $n + 6$ pebbles then two pebbles can be moved to v_1 as $f_t(K_{1,n}) = 4t + n - 2, n > 1$. Thus, one pebble can be moved to x_{11}.
Subcase 1.2: Suppose $0 \leq p(K_{1,n}) = s < n + 6$, then the number of pebbles on $X - X_1$ is atleast $2n + \sum p_i - p_1 + 11 - s$.
Let r_1 be the number of vertices with odd pebbles on $X - X_1$. Then the number of pebbles that can be brought to $K_{1,n}$ is atleast $\frac{2n+\sum p_i - p_1 + 11 - s - r_1}{2}$.
Subcase 1.2.1: s is even and $\sum p_i - p_1$ is odd.
Then $2n + \sum p_i - p_1 + 11 - s$ is even. Hence placing even number of pebbles in $\sum p_i - p_1$ odd number of vertices gives even number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1 - 1$.

$$\frac{2n + \sum p_i - p_1 + 11 - s - r_1}{2} \geq \frac{2n + \sum p_i - p_1 + 11 - s - \sum p_i + p_1 + 1}{2}$$
$$= \frac{2n - s + 12}{2}.$$

Thus, the total number of pebbles, $K_{1,n}$ can have will be atleast $\frac{2n-s+12}{2} + s \geq n + 6 + \frac{s}{2} \geq n + 6$.
Subcase 1.2.2: s is odd and $\sum p_i - p_1$ is even.
Then $2n + \sum p_i - p_1 + 11 - s$ is even. Hence placing even number of pebbles in $\sum p_i - p_1$ even number of vertices given even number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1$.

$$\frac{2n + \sum p_i - p_1 + 11 - s - r_1}{2} \geq \frac{2n + \sum p_i - p_1 + 11 - s - \sum p_i + p_1}{2}$$
$$= \frac{2n - s + 11}{2}.$$

Thus, the total number of pebbles, $K_{1,n}$ can have will be atleast $\frac{2n-s+11}{2} + s = \frac{2n+s+11}{2}$.
Since s is odd, $s \geq 1$.
Thus, $\frac{2n+s+11}{2} \geq \frac{2n+12}{2} = n + 6$.
Subcase 1.2.3: s is even and $\sum p_i - p_1$ is even.
Then $2n + \sum p_i - p_1 + 11 - s$ is odd. Hence placing odd number of pebbles in even vertices gives odd number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1 - 1$.

$$\frac{2n+\sum p_i - p_1 + 11 - s - r_1}{2} \geq \frac{2n+\sum p_i - p_1 + 11 - s - \sum p_i + p_1 + 1}{2}$$
$$= \frac{2n - s + 12}{2}.$$

Thus, the total number of pebbles, $K_{1,n}$ can have will be atleast $\frac{2n-s+12}{2} + s = \frac{2n+s+12}{2} = n + 6 + \frac{s}{2} \geq n + 6$.

Subcase 1.2.4: s is odd and $\sum p_i - p_1$ is odd.

Then $2n + \sum p_i - p_1 + 11 - s$ is odd. Hence placing odd number of pebbles in odd number of vertices gives odd number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1$.

$$\frac{2n+\sum p_i - p_1 + 11 - s - r_1}{2} \geq \frac{2n+\sum p_i - p_1 + 11 - s - \sum p_i + p_1}{2}$$
$$= \frac{2n - s + 11}{2}.$$

Thus, the total number of pebbles, $K_{1,n}$ can have will be atleast $\frac{2n-s+11}{2} + s = \frac{2n+s+11}{2}$.

Since s is odd, $s \geq 1$.

Thus, $\frac{2n+s+11}{2} \geq \frac{2n+12}{2} = n + 6$.

Thus, in all the subcases, $K_{1,n}$ can have atleast $n + 6$ pebbles. Hence two pebbles can be moved to v_1 as $f_2(K_{1,n}) = n + 6$ and by which one pebble can be moved to x_{11}.

Case 2: Let $v_j, j = 0,1, \dots, n$ be any vertex on the star, such that v_j be our target vertex. Without loss of generality, let us assume that v_1 be our target vertex and $p(v_1) = 0$.

Suppose that $p(x_{1j}) \geq 2$ for some $j = 1,2, \dots, p_1$ then one pebble can be moved to v_1.

Assume that $p(x_{1j}) \leq 1 \forall j = 1,2, \dots, p_1$.

If $p(K_{1,n}) \geq n + 2$, then one pebble can be moved to v_1 as $f(K_{1,n}) = n + 2$.

If $0 \leq p(K_{1,n}) = s < n + 2$, then the remaining pebbles are placed on the thorns of the star. Then the pebbles placed on the thorns of the star expect X_1 is atleast $2n + \sum p_i - p_1 + 10 - s$.

Let r_1 be the number of vertices with odd number of pebbles on $X - X_1$. Then $r_1 \leq \sum p_i - p_1$. Then the pebbles that can be brought to $K_{1,n}$ will be atleast

$$\frac{2n+\sum p_i - p_1 + 10 - s - r_1}{2} \geq \frac{2n+\sum p_i - p_1 + 10 - s - \sum p_i + p_1}{2}$$
$$= \frac{2n - s + 10}{2} = n + 5 - \frac{s}{2}.$$

Now, $K_{1,n}$ can have atleast $n + 5 - \frac{s}{2} + s = n + 5 + \frac{s}{2} \geq n + 2$ pebbles. Thus, one pebble can be moved to v_1 as $f(K_{1,n}) = n + 2$.

Therefore, $f(K^*_{1,n}) = 2n + \sum p_i + 10$.

4. The pebbling number of a thorn graph:

In this section, we find the pebbling number of a thorn graph of any graph G with the t-pebbling number $f_t(G)$.

Theorem 4.1:

Let G be a simple connected graph with n vertices. Let G^*be the thorn graph of G with parameter $p_i, i = 1,2, \dots, n$ and each $p_i \geq 2$. Then the pebbling number of the thorn graph G^* is $f(G^*) = 2(f_2(G)) + \sum p_i - 2$.

Proof:

Let G be a simple connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $X_i = \{x_{ij}: x_{ij}$ is adjacent to $v_i, i = 0,1,..,n$ and $j = 1,2, \dots, p_i$ and $\deg(x_{ij}) = 1\}$. Let $X = \cup_{i=1}^{n} X_i$ be the thorn set of G^* .

Let the t-pebbling number of the graph G is $f_t(G)$.

Now assume that $x_{11} \in V(G^*)$ be our target vertex and $p(x_{11}) = 0$.

To reach one pebble to the target vertex x_{11} it is enough to reach two pebbles to the vertex v_1.

Now, consider the distribution of $2(f_2(G)) + \sum p_i - 3$ pebbles on the vertices of G^* as follows.

1. $p(G) = 0$.
2. $p(x_{k1}) = 2(f_2(G)) - 1$, where $d(v_k - v_1)$ is the diameter of G.
3. $p(x_{ij}) = 1$ for all $x_{ij} \neq x_{11}, x_{k1}$.

Now, the number of pebbles that can reach G is $f_2(G) - 1$. Thus, two pebbles cannot be moved to the vertex v_1 and hence our target vertex cannot be reached.

Thus, $f(G^*) \geq 2(f_2(G)) + \sum p_i - 2$.

Now let us claim that $f(G^*) \leq 2(f_2(G)) + \sum p_i - 2$.

Let $2(f_2(G)) + \sum p_i - 2$ pebbles be placed on the vertices of G^*. Let x_{ij} be our target vertex for some i, j. Without loss of generality let us assume that x_{11} be our target vertex and $p(x_{11}) = 0$.

Case 1: $p(G) = 0$ and $p(X) = 2(f_2(G)) + \sum p_i - 2$.

Then atleast $f_2(G) + \frac{\sum p_i}{2} - 1 > f_2(G)$ pebbles will reach the vertices of G as $\sum p_i > 2$.

Thus, two pebbles can reach the vertex v_1 and hence one pebble can be moved to x_{11}.

Case 2: $p(G) \geq f_2(G)$.

Then clearly two pebbles can reach the vertex v_1 and hence one pebble can be moved to x_{11}.

Case 3: $0 < p(G) = s < f_2(G)$

If any of the vertices of X_1 has atleast four pebbles with $p(v_1) = 0$ or two pebbles with $p(v_1) = 1$ or $p(v_1) \geq 2$, then one pebble can be moved to x_{11}.

Let us assume that $p(x_{1j}) \leq 1 \ \forall \ j = 2,3, \dots, p_1$.

then the number of pebbles on $X - X_1$ is atleast $2(f_2(G)) + \sum p_i - p_1 - 2 - s$.

Let r_1 be the number of vertices with odd pebbles on $X - X_1$. Then the number of pebbles that can be brought to $K_{1,n}$ is atleast $\frac{2(f_2(G)) + \sum p_i - p_1 - 2 - s - r_1}{2}$.

Subcase 3.1: s is even and $\sum p_i - p_1$ is odd.

Then $2(f_2(G)) + \sum p_i - p_1 - 2 - s$ is odd. Hence placing odd number of pebbles in odd number of vertices gives odd number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1$.

$$\frac{2(f_2(G)) + \sum p_i - p_1 - 2 - s - r_1}{2} \geq \frac{2(f_2(G)) + \sum p_i - p_1 - 2 - s - \sum p_i + p_1}{2}$$

$$= \frac{2\left(f_2(G)\right) - s - 2}{2}.$$

Thus, the total number of pebbles, G can have will be atleast $\frac{2(f_2(G))-s-2}{2} + s \geq f_2(G) + \frac{s-2}{2} \geq f_2(G)$, since s is even, $s \geq 2$.

Subcase 3.2: s is odd and $\sum p_i - p_1$ is even.

Then $2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s$ is odd. Hence placing odd number of pebbles in even vertices gives odd number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1 - 1$.

$$\frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - r_1}{2} \geq \frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - \sum p_i + p_1 + 1}{2}$$
$$= \frac{2f_2(G) - s - 1}{2}$$

Thus, the total number of pebbles, G can have will be atleast $\frac{2f_2(G)-s-1}{2} + s = \frac{2f_2(G)+s-1}{2}$.

Since s is odd, $s \geq 1$.

Thus, $f_2(G) + \frac{s-1}{2} \geq f_2(G)$.

Subcase 3.3: s is even and $\sum p_i - p_1$ is even.

Then $2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s$ is even. Hence placing even number of pebbles in $\sum p_i - p_1$ even number of vertices given even number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1$.

$$\frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - r_1}{2} \geq \frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - \sum p_i + p_1}{2}$$
$$= \frac{2f_2(G) - s - 2}{2}.$$

Thus, the total number of pebbles, G can have will be atleast $\frac{2f_2(G)-s-2}{2} + s = \frac{2f_2(G)+s-2}{2} = f_2(G) + \frac{s-2}{2} \geq f_2(G)$, since s is even, $s \geq 2$.

Subcase 3.4: s is odd and $\sum p_i - p_1$ is odd.

Then $2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s$ is even Hence placing even number of pebbles in $\sum p_i - p_1$ odd number of vertices gives even number of vertices with odd pebbles. Thus, $r_1 \leq \sum p_i - p_1 - 1$.

$$\frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - r_1}{2} \geq \frac{2\left(f_2(G)\right) + \sum p_i - p_1 - 2 - s - \sum p_i + p_1 + 1}{2}$$
$$= \frac{2f_2(G) - s - 1}{2}.$$

Thus, the total number of pebbles, $K_{1,n}$ can have will be atleast $\frac{2f_2(G)-s-1}{2} + s = \frac{2f_2(G)+s-1}{2}$.

Since s is odd, $s \geq 1$.

Thus, $\frac{2f_2(G)+s-1}{2} = f_2(G) + \frac{s-1}{2} \geq f_2(G)$

Thus, in all the subcases, G can have atleast $f_2(G)$ pebbles. Hence two pebbles can be moved to v_1 and by which one pebble can be moved to x_{11}.

Now, let us assume that $v \in V(G)$ be our target vertex.

Without loss of generality, let us assume that v_1 is our target vertex and $p(v_1) = 0$.

Suppose that $p(x_{1j}) \geq 2$ for some $j = 1,2, \dots, p_1$ then one pebble can be moved to v_1.
Assume that $p(x_{1j}) \leq 1 \forall j = 1,2, \dots, p_1$.
If $p(G) \geq f(G)$, then one pebble can be moved to the target vertex v_1.
If $p(G) = 0$, then all pebbles are placed on the thorn set X. Then by case 1, the target vertex can be easily pebbled.
If $0 < p(G) = s < f(G)$, then the remaining pebbles $2(f_2(G)) + \sum p_i - 2 - s$ are placed on the thorns of G. Let r_1 be the number of vertices with odd number of pebbles on $X - X_1$.
Then $r_1 \leq \sum p_i - p_1$. Thus, from the thorn set except X_1, the number of pebbles that can reach the graph G will be atleast

$$\frac{2f_2(G) + \sum p_i - p_1 - 2 - s - r_1}{2} \geq \frac{2f_2(G) + \sum p_i - p_1 - 2 - s - \sum p_i + p_1}{2}$$

$$= \frac{2f_2(G) - 2 - s}{2} = f_2(G) - \frac{s+2}{2}.$$

Thus, the total number of pebbles G can have will be atleast $f_2(G) - \frac{s+2}{2} + s = f_2(G) + \frac{s-2}{2}$.
Also, $f_2(G) + \frac{s-2}{2} \geq f_2(G)$, since s is even, $s \geq 2$.
Thus, one pebble can be moved to the target vertex v_1.
Therefore, $f(G^*) \leq 2(f_2(G)) + \sum p_i - 2$.
Hence $f(G^*) = 2(f_2(G)) + \sum p_i - 2$.

Conclusion:

In this paper, we determine the pebbling number of thorn graph of star and thorn graph of any graph G if the t-pebbling number of the graph G is known. The t-pebbling number of thorn graph of any graph is an open problem.

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