# Some symmetric identities for twisted $(p, q)$-$L$-function

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

**Abstract :** The main of this paper is to obtain some interesting symmetric identities for twisted $(p, q)$-$L$-function in complex field. We define the twisted $(p, q)$-$L$-function by generalizing the Carlitz's type twisted $(p, q)$-Euler numbers and polynomials. We give some new symmetric identities for twisted $(p, q)$-$L$-function. We also obtain symmetric identities for Carlitz's type twisted $(p, q)$-Euler numbers and polynomials by using symmetric property for twisted $(p, q)$-$L$-function.

**Key words :** Euler numbers and polynomials, $q$-Euler numbers and polynomials, twisted $q$-Euler numbers and polynomials, twisted $(p, q)$-Euler numbers and polynomials, $q$-$L$-function, twisted $(p, q)$-$L$-function, symmetric identities.

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## 1. Introduction

Many $(p, q)$-extensions of some special numbers, polynomials, and functions have been studied(see [1, 2, 3, 4, 7]). Luo and Zhou [5] introduced the $l$-function and $q$-$L$-function. Ryoo [6] investigated some identities on the higher-order twisted $q$-Euler numbers and polynomials. In [8], Ryoo presented the multiple twisted $(h, q)$-$l$-function. In this paper, we construct twisted $(p, q)$-$L$-function in complex field and Carlitz's type twisted $(p, q)$-Euler numbers and polynomials. We obtain some new symmetric identities for twisted $(p, q)$-$L$-function. We also give symmetric identities for Carlitz's type twisted $(p, q)$-Euler numbers and polynomials of by using symmetric property for twisted $(p, q)$-$L$-function.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. The $(p, q)$-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2 q^{n-3} + pq^{n-2} + q^{n-1}.$$

Note that this number is $q$-number when $p = 1$. By substituting $q$ by $\frac{q}{p}$ in the $q$-number, we can not obtain $(p, q)$-number. Therefore, much research has been developed in the area of special numbers and polynomials, and functions by using $(p, q)$-number(see [1, 2, 3, 4, 7]).

By using $q$-number, Luo and Zhou defined the $q$-$L$-function $L_q(s, a)$ and $q$-$l$-function $l_q(s)$ (see [5])

$$L_q(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+a}}{[n + a]_q^s}, \quad (Re(s) > 1; a \notin \mathbb{Z}_0^-), \text{ and } l_q(s) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^s}, \quad (Re(s) > 1).$$

Inspired by their work, the $(p, q)$-extension of the twisted $q$-$L$-function can be defined as follow: Let $\zeta$ be $r$th root of 1 and $\zeta \neq 1$. For $s, x \in \mathbb{C}$ with $Re(x) > 0$, the twisted $(p, q)$-$L$-function $L_{p,q,\zeta}(s, x)$ is define by

$$L_{p,q,\zeta}(s, x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^m}{[m + x]_{p,q}^s}.$$

## 2. Twisted $(p, q)$-Euler numbers and polynomials

In this section, we define twisted $(p, q)$-Euler numbers and polynomials and provide some of their relevant properties. Let $r$ be a positive integer, and let $\zeta$ be $r$th root of 1.

**Definition 1.** For $0 < q < p \leq 1$, the Carlitz's type twisted $(p, q)$-Euler numbers $E_{n,p,q,\zeta}$ and polynomials $E_{n,p,q,\zeta}(x)$ are defined by means of the generating functions

$$G_{p,q,\zeta}(t) = \sum_{n=0}^{\infty} E_{n,p,q,\zeta} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m e^{[m]_{p,q} t}. \tag{2.1}$$

and

$$G_{p,q,\zeta}(t, x) = \sum_{n=0}^{\infty} E_{n,p,q,\zeta}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m e^{[m+x]_{p,q} t}, \tag{2.2}$$

respectively.

Setting $p = 1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted $q$-Euler number $E_{n,q,\zeta}$ and $q$-Euler polynomials $E_{n,q,\zeta}(x)$, respectively.

By (2.1), we get

$$\sum_{n=0}^{\infty} E_{n,p,q,\zeta} \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m e^{[m]_{p,q} t} = \sum_{n=0}^{\infty} \left( [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + \zeta q^l p^{n-l}} \right) \frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

**Theorem 2.** For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\zeta} = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + \zeta p^{n-l} q^l}.$$

By (2.2), we obtain

$$E_{n,p,q,\zeta}(x) = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1 + \zeta p^{n-l} q^l}. \tag{2.3}$$

Next, we introduce Carlitz's type twisted $(h, p, q)$-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$.

**Definition 3.** The Carlitz's type twisted $(h, p, q)$-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$ are defined by

$$E^{(h)}_{n,p,q,\zeta}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m p^{hm} \zeta^m [m + x]^n_{p,q}. \tag{2.4}$$

When $x = 0$, $E^{(h)}_{n,p,q,\zeta} = E^{(h)}_{n,p,q,\zeta}(0)$ are called the twisted $(h, p, q)$-Euler numbers $E^{(h)}_{n,p,q,\zeta}$.

By using (2.4) and $(p, q)$-number, we have the following theorem.

**Theorem 4.** For $n \in \mathbb{Z}_+$, we have

$$E^{(h)}_{n,p,q,\zeta}(x) = [2]_q \left( \frac{1}{p-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{(n-l)x} \frac{1}{1 + \zeta p^{n-l+h} q^l}.$$

By (2.4) and Theorem 2, we have

$$\begin{aligned} E_{n,p,q,\zeta}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{(n-l)x} E^{(l)}_{n-l,p,q,\zeta} [x]^l_{p,q} \\ E_{n,p,q,\zeta}(x + y) &= \sum_{l=0}^{n} \binom{n}{l} p^{xl} q^{(n-l)y} E^{(l)}_{n-l,p,q,\zeta}(x) [y]^l_{p,q}. \end{aligned} \tag{2.5}$$

By (2.1) and (2.2), we get

$$-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}\zeta^{l+n}e^{[l+n]_{p,q}t}+[2]_q\sum_{l=0}^{\infty}(-1)^l\zeta^le^{[l]_{p,q}t}=[2]_q\sum_{l=0}^{n-1}(-1)^l\zeta^le^{[l]_{p,q}t}.$$

Hence we have

$$(-1)^{n+1}\zeta^n\sum_{m=0}^{\infty}E_{m,p,q,\zeta}(n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}E_{m,p,q,\zeta}\frac{t^m}{m!}=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.6}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.6), we have the following theorem.

**Theorem 5.** For $m\in\mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m=\frac{(-1)^{n+1}\zeta^nE_{m,p,q,\zeta}(n)+E_{m,p,q,\zeta}}{[2]_q}.$$

## 3. Twisted $(p,q)$-$l$-function and twisted $(p,q)$-$L$-function

By using twisted $(p,q)$-Euler numbers and polynomials, twisted $(p,q)$-$L$-function is defined. These functions interpolate the twisted $(p,q)$-Euler numbers $E_{n,p,q,\zeta}$, and polynomials $E_{n,p,q,\zeta}(x)$, respectively. From (2.1), we note that

$$\left.\frac{d^k}{dt^k}G_{p,q,\zeta}(t)\right|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^n\zeta^m[m]_{p,q}^k=E_{k,p,q,\zeta},\,(k\in\mathbb{N}).$$

By using the above equation, we are now ready to define twisted $(p,q)$-$l$-function.

**Definition 6.** Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$.

$$l_{p,q,\zeta}(s)=[2]_q\sum_{n=1}^{\infty}\frac{(-1)^n\zeta^n}{[n]_{p,q}^s}. \tag{3.1}$$

Relation between $l_{p,q,\zeta}(s)$ and $E_{k,p,q,\zeta}$ is given by the following theorem.

**Theorem 7.** For $k\in\mathbb{N}$, we have

$$l_{p,q,\zeta}(-k)=E_{k,p,q,\zeta}.$$

By using (2.2), we note that

$$\left.\frac{d^k}{dt^k}G_{p,q,\zeta}(t,x)\right|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m[m+x]_{p,q}^k \tag{3.2}$$

and

$$\left.\left(\frac{d}{dt}\right)^k\left(\sum_{n=0}^{\infty}E_{n,p,q,\zeta}(x)\frac{t^n}{n!}\right)\right|_{t=0}=E_{k,p,q,\zeta}(x),\ \text{for } k\in\mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the twisted $(p,q)$-$L$-function.

**Definition 8.** Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$ and $x\notin\mathbb{Z}_0^-$.

$$L_{p,q,\zeta}(s,x)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\zeta^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $L_{p,q,\zeta}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $L_{p,q,\zeta}(s,x)$ and $E_{k,p,q,\zeta}(x)$ is given by the following theorem.

**Theorem 9.** For $k\in\mathbb{N}$, we have $L_{p,q,\zeta}(-k,x)=E_{k,p,q,\zeta}(x)$.

Observe that $L_{p,q,\zeta}(-k,x)$ function interpolates $E_{k,p,q,\zeta}(x)$ numbers at non-negative integers.

## 3. Some symmetric identities for twisted $(p,q)$-$L$-function

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain certain symmetric identities for twisted $(p,q)$-$L$-function.

**Theorem 10.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. Then we obtain

$$\begin{aligned}&[w_2]_{p,q}^s[2]_{q^{w_2}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right)\\&=[w_1]_{p,q}^s[2]_{q^{w_1}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}L_{p^{w_2},q^{w_2},\zeta^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}j\right).\end{aligned}\tag{4.1}$$

**Proof.** Note that $[xy]_q=[x]_{q^y}[y]_q$ for any $x,y\in\mathbb{C}$. In (3.4), by substitute $w_2x+\frac{w_2}{w_1}j$ for $x$ in and replace $q$, $p$, and $\zeta$ by $q^{w_1}$, $p^{w_1}$ and $\zeta^{w_1}$, respectively, we derive next result

$$\begin{aligned}\frac{1}{[2]_{q^{w_1}}}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right)&=\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{w_1 m}}{\left[m+w_2x+\frac{w_2}{w_1}j\right]_{p^{w_1},q^{w_1}}^s}\\&=\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{w_1 m}}{\left[\frac{w_1m+w_1w_2x+w_2j}{w_1}\right]_{p^{w_1},q^{w_1}}^s}\\&=[w_1]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\frac{(-1)^{w_2m+i}\zeta^{w_1(w_2m+i)}}{[w_1(w_2m+i)+w_1w_2x+w_2j]_{p,q}^s}\\&=[w_1]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\frac{(-1)^m(-1)^i\zeta^{w_1w_2m}\zeta^{w_1 i}}{[w_1w_2(x+m)+w_1i+w_2j]_{p,q}^s}.\end{aligned}\tag{4.2}$$

Thus, from (4.2), we can derive the following equation.

$$\begin{aligned}&\frac{[w_2]_{p,q}^s}{[2]_{q^{w_1}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2 j}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right)\\&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\frac{(-1)^{j+i+m}\zeta^{w_1w_2m}\zeta^{w_1i}\zeta^{w_2j}}{[w_1w_2(x+m)+w_1i+w_2j]_{p,q}^s}\end{aligned}\tag{4.3}$$

By using the same method as (4.3), we have

$$\begin{aligned}&\frac{[w_1]_{p,q}^s}{[2]_{q^{w_2}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1 j}L_{p^{w_2},q^{w_2},\zeta^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}j\right)\\&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m=0}^{\infty}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\frac{(-1)^{j+i+m}\zeta^{w_1w_2m}\zeta^{w_2i}\zeta^{w_1j}}{[w_1w_2(x+m)+w_1j+w_2i]_{p,q}^s}\end{aligned}\tag{4.4}$$

Therefore, by (4.3) and (4.4), we have the following theorem. □

Taking $w_2=1$ in Theorem 10, we obtain the following corollary.

**Corollary 11.** Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$L_{p,q,\zeta}(s, w_1x)=\frac{[2]_q}{[2]_{q^{w_1}}[w_1]_{p,q}^s}\sum_{j=0}^{w_1-1}(-1)^j\zeta^jL_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, x+\frac{j}{w_1}\right).$$

Let us take $s = -n$ in Theorem 10. For $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for twisted $(p, q)$-Euler polynomials.

**Theorem 12.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$[w_1]_{p,q}^n [2]_{q^{w_2}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} E_{n,p^{w_1},q^{w_1},\zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right)$$
$$= [w_2]_{p,q}^n [2]_{q^{w_1}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} E_{n,p^{w_2},q^{w_2},\zeta^{w_2}} \left( w_1 x + \frac{w_1}{w_2} j \right).$$

Taking $w_2 = 1$ in Theorem 12, we obtain the following distribution relation.

**Corollary 13.** Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$E_{n,p,q,\zeta}(w_1 x) = \frac{[2]_q}{[2]_{q^{w_1}}} [w_1]_{p,q}^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^j E_{n,p^{w_1},q^{w_1},\zeta^{w_1}} \left( s, x + \frac{j}{w_1} \right).$$

By (2.5), we have

$$\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} E_{n,p^{w_1},q^{w_1},\zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right)$$
$$= \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \sum_{i=0}^{n} \binom{n}{i} q^{w_2 j(n-i)} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \left[ \frac{w_2}{w_1} j \right]^i_{p^{w_1},q^{w_1}}$$
$$= \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \sum_{i=0}^{n} \binom{n}{i} q^{w_2 j(n-i)} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \left( \frac{[w_2]_{p,q}}{[w_1]_{p,q}} \right)^i [j]^i_{p^{w_2},q^{w_2}}$$

Hence we have the following theorem.

**Theorem 14.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} E_{n,p^{w_1},q^{w_1},\zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right)$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_2]^i_{p,q} [w_1]^{-i}_{p,q} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2(n-i)j} [j]^i_{p^{w_2},q^{w_2}}.$$

For each integer $n \geq 0$, let $\mathcal{A}_{n,i,p,q,\zeta}(w) = \sum_{j=0}^{w-1} (-1)^j \zeta^j q^{j(n-i)} [j]^i_{p,q}$. The sum $\mathcal{A}_{n,i,p,q,\zeta}(w)$ is called the alternating twisted $(p, q)$-power sums.

By Theorem 14, we have

$$\begin{aligned} &[2]_{q^{w_2}} [w_1]^n_{p,q} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} E_{n,p^{w_1},q^{w_1},\zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right) \\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]^i_{p,q} [w_1]^{n-i}_{p,q} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1) \end{aligned} \tag{4.5}$$

By using the same method as in (4.5), we have

$$\begin{aligned} &[2]_{q^{w_1}} [w_2]^n_{p,q} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} E_{n,p^{w_2},q^{w_2},\zeta^{w_2}} \left( w_1 x + \frac{w_1}{w_2} j \right) \\ &= [2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]^i_{p,q} [w_2]^{n-i}_{p,q} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1 x) \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2) \end{aligned} \tag{4.6}$$

Therefore, by (4.5) and (4.6) and Theorem 12, we have the following theorem.

**Theorem 15.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} E_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}^{(i)}(w_1 x) \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} E_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}^{(i)}(w_2 x) \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1).\end{aligned}$$

By Theorem 15, we obtain the interesting symmetric identity for the twisted $(h, p, q)$-Euler numbers $E_{n,p,q,\zeta}^{(h)}$ in complex field.

**Corollary 16.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2) E_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}^{(i)}\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1) E_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}^{(i)}.\end{aligned}$$

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