MONOPHONIC PEBBLING NUMBER OF DERIVED GRAPHS

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ABSTRACT

Objectives: To find the monophonic pebbling number of some derived graphs. **Methods:** The monophonic pebbling number $\pi_\mu(G)$ of a graph G is the least number of pebbles such that after a sequence of pebbling moves a pebble can be placed on any vertex through a monophonic path. **Findings:** We compute the monophonic pebbling number and monophonic t-pebbling number for generalised petersen graph, Kusudama flower graph and sierpinski triangle.
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1. INTRODUCTION

The game of pebbling was originally presented by Lagarias and Saks to solve a particular number theory problem. Pebbling move involves choosing a vertex with at least two pebbles and removing two pebbles from that vertex and placing one of those pebbles on a neighbouring vertex. Lourdusamy et al. introduced monophonic pebbling in [4]. A chord in a cycle, C, or path, P, refers to an edge uv connecting two vertices, u and v, that are not adjacent in C or P. Any longest chordless path in a graph is called as monophonic path. This paper includes a theorem which determines the monophonic pebbling number of Sierpinski triangle graph S_n which is a fractal. Fractals are geometric figures whose parts have the same statistical nature as a whole.

2.Methodology

Path Identification: Identifying all possible monophonic paths in a graph can be computationally challenging, especially for large or dense graphs.

Move Sequences: Constructing valid sequences of pebbling moves that adhere to the monophonic constraint adds another layer of complexity to the problem.

3.Preliminaries

Definition 3.1. The monophonic pebbling number $\pi_\mu(G, v)$ of a vertex v is the least number such that from every distribution of $\pi_\mu(G, v)$ pebbles on the graph G, a pebble can be moved to any desired vertex v by a sequence of pebbling moves through a monophonic path. The monophonic pebbling number of the graph G, denoted by $\pi_\mu(G)$, is the maximum of $\pi_\mu(G, v)$ over all vertices of G.

Definition 3.2. The monophonic pebbling number $\pi_{\mu t}(G, v)$ of a vertex v is the least number such that from every distribution of $\pi_\mu(G)$ pebbles on the graph G, t pebble(s) can be moved to any desired vertex v by a sequence of pebbling moves through a monophonic path. The monophonic t- pebbling number of the graph G, denoted by $\pi_{\mu t}(G)$ is the maximum is the maximum of $\pi_{\mu t}(G, v)$ over all vertices of G

Definition 3.3. A star polygon $\frac{n}{m}$, with n, m positive integers and $m < \frac{n}{2}$, is a figure formed by connecting every mth point out of n regularly spaced vertices of a cycle C with straight lines.

Definition 3.3. A family of cubic graphs known as the generalized petersen graph GP_n is created by joining the vertices of a regular polygon with the corresponding vertices of a star polygon $\frac{n}{2}$. Let the vertex set of the graph GP_n be $V = v_i, u_i, 0 \leq i \leq n-1$ and the edge set be $E = \{v_i v_{i+1}, v_i u_i, u_i u_{i+2}\}$ where $i \equiv j \ mod \ n, j = \{0,1,2,\ldots,n-1\}$.

Definition 3.4. The Jahangir graph $J_{n,m}, (n \geq 2, m \geq 3)$ is a graph consisting of a cycle C_{nm} such that $V(C_{nm}) = v_0, v_1, \ldots v_{2m-2}, v_{2m-1}$ and an additional vertex x that is adjacent to m vertices of C_{nm} at a distance of n on C_{nm}. Let the center vertex be labeled as v_{nm+1}, and the vertices of cycle C_{nm} be named as $v_1, v_2, \ldots, v_{nm}$.

Definition 3.5. Kusudama flower graph consist of Jahangir graph $J_{2,m}$ with $5m + 1$ vertices. Consider x as the center vertex is connected to $v_1, v_3, \ldots, v_{2m-1}$. The vertex u_i which is connected to $v_i, i = 0,2,\ldots,2m-2$ as well as x. There are two vertices $u_{i,1}, u_{i,2}$ on either sides of the vertex u_i. Both $u_{i,1}, u_{i,2}$ are connected to u_i as well as x.

Notation 3.7. Throughout this paper, the number of pebbles on the vertex v is denoted as p(v) and the target vertex is denoted as α. Any monophonic path is denoted by μ_i where i is any positive integer. Let $V(\widetilde{\mu_i})$ be the set of vertices not in μ_i and monophonic distance be denoted by D_μ. The symbol $A \xrightarrow{t} B \xrightarrow{s} C$ refers to the transfer of t pebbles from the set of vertices A to the set of vertices B and then s pebbles from the set of vertices of B to the set of vertices of C.

4 Results and Discussion

Theorem 4.1. *For generalised petersen graph is* $\pi_\mu(GP_n) = 2^{n-1} + 2$.

Proof. Let $\mu_1: v_0, v_1, \ldots, v_{n-2}, u_{n-2}$ be one of the monophonic paths of length $n-1$ in graph GP_n.. $V(\tilde{\mu}_1) = u_0, u_1, \ldots, u_{n-3}, u_{n-1}, v_{n-1}$ be the set of vertices not in μ_1. Let $\alpha = u_{n-2}$. Placing $2^{n-1} - 1$ pebbles on v_0, $0 \leq p(u_{n-1}), p(v_{n-1}) \leq 1$ one pebble each on all the other vertices we cannot move a pebble to the α through the monophonic path μ_1.

Therefore, $\pi_{\mu}(G) = 2^{n-1} + 1$. Let D be any configuration of $2^{n-1} + n$ pebbles on the vertices of GP_n.

Case 1: Let $\alpha = v_i$ or u_i , $0 \le i \le n - 1$.

Without loss of generality, let $\alpha = v_i$. Let μ_2: $\{v_{i \bmod n}, v_{(i+1) \bmod n}, \cdots, v_{(i+n-2) \bmod n}$ $u_{(i+n-2) \bmod n}\}$ be the monophonic path of length $n-1$ then $V(\tilde{\mu}_2) = u_{i \bmod n}, u_{(i+1) \bmod n}, \cdots, u_{(i+n-3) \bmod n}, u_{(i+n-1) \bmod n}$. Clearly, D_{μ} from $v_{i \bmod n}$ to $u_{(i+n-2) \bmod n}$ is $n - 1$. If $p(\mu_2) \ge 2^{n-1}$ and $0 \le p(u_{(i+n-1) \bmod n}), p(u_{(i+n-1) \bmod n}) \le 1$ and all the other pebbles have exactly zero pebbles each we are done. If $p(\mu_2) \le 2^{n-1} - 1$ and $p(V(\tilde{\mu}_2)) \ge 3$. If either $p(u_{n-1}) \ge 2$ or $p(v_{n-1}) \ge 2$ this will contribute at least one pebble to the considered monophonic path μ_2. If $p(V(\tilde{\mu}_2) - \{u_{(i+n-1) \bmod n}, v_{(i+n-1) \bmod n}\} \ge 1$ we can find an alternative monophonic path of length n – 1 with the required number of pebbles to reach α. Similarly, we can prove for all u_i.

Theorem 4.2. *For Kusudama flower graph* $\pi_{\mu}(KF_m) = 2^{2m+2} + 3m - 2$.

Proof. Consider the monophonic path μ_1 : $u_{i \bmod 2m,1}, u_{i \bmod 2m}, v_{i \bmod 2m}, v_{(i+1) \bmod 2m}, v_{(i+2) \bmod 2m}, v_{(i+3) \bmod 2m}, \cdots, v_{(i+2m-2) \bmod 2m}, u_{(i+2m-2) \bmod 2m}, u_{(i+2m-2) \bmod 2m,1}$ where $i = 0, 2, \ldots, 2m - 2$. Then $V(\tilde{\mu}_1) = \{u_{(i+2) \bmod 2m}, u_{(i+4) \bmod 2m}, \cdots, u_{(i+2m-4) \bmod 2m}, u_{(i+2m-2) \bmod 2m,2} u_{(i+2) \bmod 2m,1}, u_{(i+2) \bmod 2m}, \cdots, u_{(i+2) \bmod 2m,1}, u_{(i+2) \bmod 2m}, \cdots, u_{(i+2m-4) \bmod 2m,2}\}$. Placing $2^{2m+2} - 1$ pebbles on $u_{i \bmod 2m,1}$ and one pebble each on the vertices in $V(\tilde{\mu}_1)$ we cannot move a pebble to $u_{(i+2m-2) \bmod 2m,1}$. Therefore, $\mu(KF_m) \ge 2^{2m+2} + 3m - 2$. Let us consider a configuration of $2^{2m+2} + 3m - 2$ pebbles on the vertices of KF_m. To prove the sufficient conditions.

Case 1: Let $\alpha = v_i$, $i = 0, 2, \ldots, 2m - 2$.

Consider the monophonic path μ_2: $\{v_{i \bmod 2m}, v_{(i+1) \bmod 2m}, v_{(i+2) \bmod 2m}, \cdots, v_{(i+2m-2) \bmod 2m}, u_{(i+2m-2) \bmod 2m,1}\}$. D_μ from $v_{i \bmod 2m}$ to any other vertex of KF_m is at most $2m$. If $p(V(\mu_2)) \ge 2^{2m}$ pebbles we can transfer a pebble to α. Let $p(V(\mu_2)) \le 2^{2m}$. Suppose $p(V(\mu_2)) \le 2^{2m} - 1$. By pigeonhole principle, if $2^{2m} + 3m - 1$ pebbles are distributed on the remaining vertices which are not on μ_2, at least one vertex will receive two pebbles with which we could move at least one pebble to μ_2. Thus, we can reach α. If $u_{i,1}$ or $u_{i,2}$, $i = 0, 2, \ldots, 2m - 4, 2m - 2$ has 8 pebbles to v_i using the path μ_3: $\{u_{i,1}$ or $u_{i,2}, v_x, v_{i-1}$ or $v_{i+1}, v_i\}$. If the path μ_3 has eight pebbles we are done. Suppose $u_i, i = 0, 2, \ldots, 2m - 2$ has at least 2 pebbles we can move a pebble to α. In the monophonic path μ_3, either $u_{i,1}$ or $u_{i,2}$ and v_x has two pebbles and either v_{i-1} or v_{i+1} has one pebble we could move a pebble to the target using the transmitting subgraph. Also either $u_{i,1}$ or $u_{i,2}$ has 4 pebbles and v_x has two pebbles the target could be pebbled.

Case 2: Let $\alpha = v_j$, $j = 1, 3, \ldots, 2m - 1$.

Consider the monophonic path μ_4: $\{v_{j \bmod 2m}, v_{(j+1) \bmod 2m}, v_{(j+2) \bmod 2m}, \cdots, v_{(j+2m-3) \bmod 2m}, u_{(j+2m-3) \bmod 2m}, u_{(j+2m-3) \bmod 2m,1}\}$. If μ_4 has 2^{2m-1} pebbles we are done. Let $p(V(\mu_4)) \le 2^{2m-1} - 1$ and $p(V(\tilde{\mu}_4)) > 7.2^{2m-1} + 3m - 1$. From $V(\tilde{\mu}_4)$ we can move at least one pebble to μ_4 and at most we can move $2^{2m+1} + \frac{3}{2}m - 1$ pebbles. If $u_{(j+1) \bmod 2m}$ or $u_{(j-1) \bmod 2m}$ has 4 pebbles we could move a pebble to the α using path μ_5: $\{u_{(j+1)\ mod\ 2m}$ or $u_{(j-1)\ mod\ 2m}, x, v_{i\ mod\ 2m}\}$. Similarly, if $u_{(j+1)\ mod\ 2m,k,} k = 1,2$ has 4 pebbles we can pebble α.

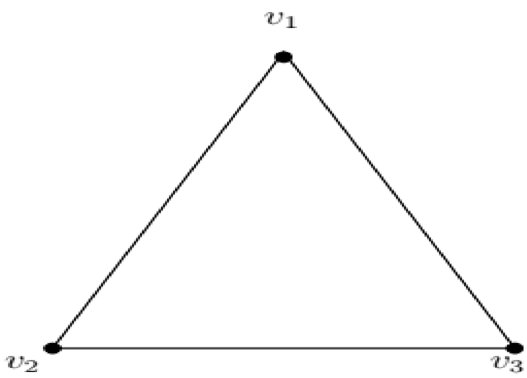
Case 3: Let $\alpha = u_{i\ mod\ 2m},\ i = 0, 2, \dots, 2m$.

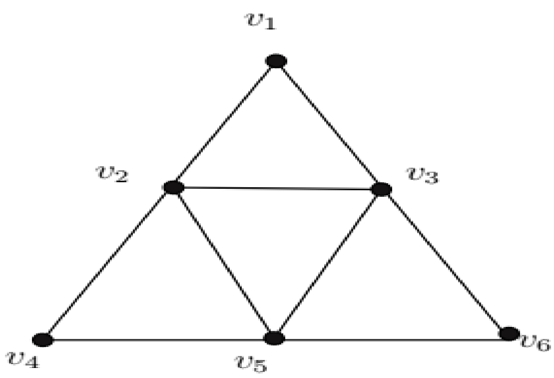
Consider the path μ_6: $\{u_{i\ mod\ 2m}, v_{(i+2m-2)\ mod\ 2m}, \dots, u_{(i+2m-2)\ mod\ 2m}, u_{(i+2m-2)\ mod\ 2m}\}$. The monophonic distance D_μ from $u_{i\ mod\ 2m}$ to any other vertex of KF_m is at most $2m + 1$. So, if the path μ_6 has 2^{2m+1} pebbles we are done. If there is a vertex $u_{j\ mod\ 2m,k}, j \neq i, k = 1,2$ not on μ_6 at least 4 pebbles, then we can move a pebble to α. If the center vertex x has two pebbles or $u_{j\ mod\ 2m}, j \neq i,$ has 2 pebbles we can move a pebble to α.

Case 4: Let $\alpha = u_{i\ mod\ 2m,k}, i = 0,2, \dots, 2m, k = 1,2$

Consider the monophonic path μ_1. If $p(V(\mu_1)) \geq 2^{2m+2}$ we are done. Suppose $p\big(V(\mu_1)\big) \leq 2^{2m+2} - 1$ then, $p(V(\tilde{\mu}_1))$ will be at lease $3m + 3$ pebbles with which we can move at least one pebble to μ_1. If $u_{i\ mod\ 2m,p}, k \neq p, k, p = 1,2$ has four pebbles we can put a pebble on α. If the center vertex x has two pebbles we are done.

Note 4.3. In Sierpinski graph an equilateral triangle is divided in to four smaller equilateral triangles and the center part is removed. This is denoted as S_1, S_2 is derived from S_1 by dividing each triangle in S_1 into four parts and the center part derived in each triangle is removed. By induction we derive S_n from S_{n-1} by using the above process.





Theorem 4.4. *For Sierpinski triangle graph* $\pi_\mu(S_n) = 2^{3^{n-1}} + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$

Proof. Step: 1

Start with an equilateral triangle $S_1 = K_3$. Therefore, $\pi_\mu(S_1) = 3$.

Step: 2

To subdivide the equilateral triangle S_1 we include three more additional vertices $v_4,\ v_5,\ v_6$. Let $V(S_2) = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\}$. The three copies of S_1 in S_2 are named as $S_{2,A}, S_{2,B}$ and $S_{2,C}$. Consider the path μ_1: $\{v_4,\ v_5,\ v_3,\ v_1\}$. Let $V(\tilde{\mu}_1) = \{v_2,\ v_6\}$ be the vertex set complement of μ_1. If 7 pebbles are places on the vertex set v_4 and one pebble each on all the vertices of $V(\tilde{\mu}_1)$ we cannot pebble v_1. Therefore, $\pi_\mu(S_1) \geq 10$. Consider the distribution of 10 pebbles on the vertices of S_2.

Subcase 2.1

Let $\alpha = v_1$. If $p(v_2 \geq 2)$ or $p(v_3 \geq 2)$ we are done since $v_2,\ v_3 \in N(v_1)$. From v_4 we could reach α using μ_1: $\{v_4,\ v_5,\ v_3,\ v_1\}$. Suppose if $p(v_4) \leq 2^3 - 1$ then there must be vertex $v_i \in V(\mu_1)$ such that $p(v_i \geq 2)$. Similarly, from v_6 we can reach α using μ_1: $\{v_6,\ v_5,\ v_2,\ v_1\}$. By symmetry, the proof is similar for the vertices v_4 and v_6.

Subcase 2.2

Let $\alpha = v_2$. $v_1,\ v_3,\ v_4,\ v_5 \in N(v_2)$. If $v_i \in N(v_2)$ has two or more pebbles we are done. If v_6 has four pebbles we can move a pebble to v_2 using μ_3: $\{v_6,\ v_5,\ v_2\}$. By symmetry the proof is similar for the vertices v_3 and v_5. Therefore, $\pi_\mu(S_2) \leq 10$.

Step 3

We repeat the same process as step 2 to obtain S_3. We divide the outer 3 triangles $S_{2,A},\ S_{2,B}$ and $S_{2,C}$ in S_2 into 4 congruent equilateral triangles except for the central removed triangle. There are 12 smaller congruent triangles in S_3. In this graph except the removed center part the outer triangles are isomorphic to S_2. Let us name those outer triangles as $S_{(3,A)},\ S_{(3,B)}$ and $S_{(3,C)}$. The cases to move pebbles within $S_{(3,A)}, S_{(3,B)}$ and $S_{(3,C)}$ are similar to what we have discussed in the previous step. Let $\left(S_{(3,A)}\right) = \{v_1,\ v_{A,1},\ v_{A,2},\ v_{A,3},\ v_2,\ v_3\}$, $V\left(S_{(3,B)}\right) = \{v_2,\ v_{B,1},\ v_{B,2},\ v_{B,3},\ v_4,\ v_5\}$ and $V\left(S_{(3,C)}\right) = \{v_3,\ v_{C,1},\ v_{C,2},\ v_{C,3},\ v_5,\ v_6\}$.

Consider μ_4: $\{v_1,\ v_{A,3},\ v_{A,2},\ v_2,\ v_{B,1},\ v_{B,3},\ v_5,\ v_{C,1},\ v_{C,3},\ v_6\}$ then $V(\tilde{\mu}_4) = \{v_{A,1},\ v_3,\ v_{B,2},\ v_4,\ v_{C,2}\}$ be its vertex set complement. Placing $2^9 - 1$ pebbles on v_6 and one pebble each on the vertices of $V(\tilde{\mu}_4)$ we cannot move a pebble to v_1 through μ_4. Therefore, $\pi_\mu(S_3) \geq 2^9 + 5$. Consider the configuration of $2^9 + 5$ pebbles on the vertices of S_3.

Let μ_5: $\{v_1,\ v_{A,1},\ v_{A,2},\ v_3,\ v_{C,3},\ v_{C,2},\ v_5,\ v_{B,3},\ v_{B,1},\ v_4\}$. Let $v_i \in \mu_5$ and $v_j \in S_{(n,B)} \cap \mu_5$ where $j \neq i$. $D_\mu(v_i, v_j) \leq 9$, $V(\tilde{\mu}_5) \in S_n - V(\mu_5)$. If $p(V(\mu_5)) < 2^9$ then there exist pebbles on $p(V(\tilde{\mu}_5))$ from which we can transfer pebbles to the vertices of μ_5. The following sequence of pebbling moves guarantees the reachability of a pebble to α.

then

$$t + p(V(\mu_5)) \geq 2^9$$

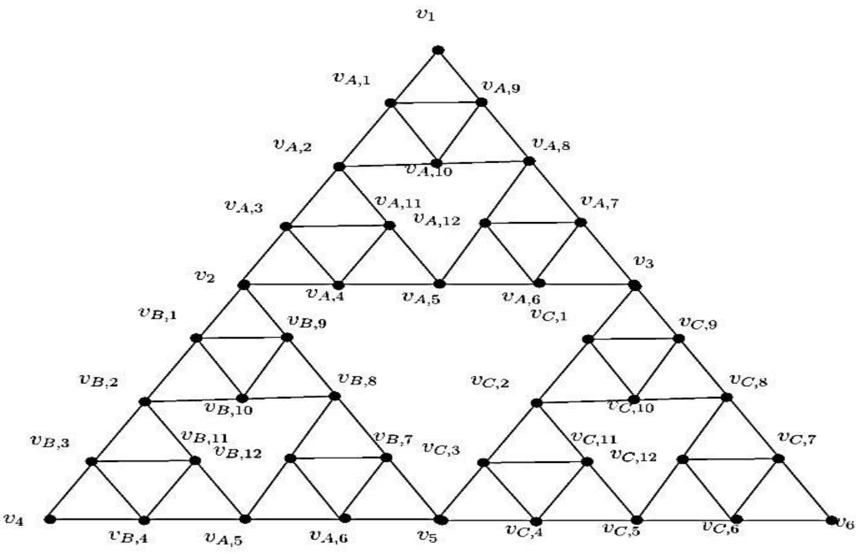
If $p\left(V(\mu_5)\right) = 0$ then $p\left(V(\mu_5)\right) > 2^9$. We can move the required number of pebbles to the vertices of μ_5 in order to reach α by the following sequence of pebbling moves.

$$p(V(\tilde{\mu}_5)) \xrightarrow{t} V(\mu_5) \xrightarrow{1} \alpha$$

Otherwise, with the pebbles on $V(\tilde{\mu}_5)$ we can have an alternative monophonic path of length 9. Similarly, Consider the monophonic path μ_6: $\{v_1,\ v_{A,3},\ v_{A,2},\ v_2,\ v_{B,3},\ v_{B,2},\ v_5,\ v_{C,1},\ v_{C,3},\ v_6\}$. D_μ from $v_i \in \mu_6$ to any other vertex $v_j \in S_{(n,C)} \cap \mu_6$ where $j \neq i$ and the vertex $v_{B,2}$ can be deduced from the path μ_6.

There are also two longest monophonic paths starting from $S_{(3,B)}$ and two other starting from $S_{(3,C)}$. By symmetry the proof is similar.

Step 4



S_4

In this step we divide the outer 3 triangles $S_{(3,A)}$, $S_{(3,B)}$ and $S_{(3,C)}$ in S_3 into 12 congruent equilateral triangles except the center part. $V(S_{(4,A)}) = \{v_1,\ v_{A,1},\ v_{A,2},\ v_{A,3},\ v_{A,4},\ v_{A,5},\ v_{A,6},\ v_{A,7},\ v_{A,8},\ v_{A,9},\ v_{A,10},\ v_{A,11},\ v_{A,12},\ v_2,\ v_3\}$, $V(S_{(4,B)}) = \{v_2,\ v_{B,1},\ v_{B,2},\ v_{B,3},\ v_{B,4},\ v_{B,5},\ v_{B,6},\ v_{B,7},\ v_{B,8},\ v_{B,9},\ v_{B,10},\ v_{B,11},\ v_{B,12},\ v_4,\ v_5\}$, $V(S_{(4,C)}) = \{v_3,\ v_{C,1},\ v_{C,2},\ v_{C,3},\ v_{C,4},\ v_{C,5},\ v_{C,6},\ v_{C,7},\ v_{C,8},\ v_{C,9},\ v_{C,10},\ v_{C,11},\ v_{C,12},\ v_5,\ v_6\}$. Consider the monophonic path μ_4: $\{v_1,\ v_{A,1},\ v_{A,10},\ v_{A,8},\ v_{A,7},\ v_{A,6},\ v_{A,5},\ v_{A,11},\ v_{A,3},\ v_2,\ v_{B,9},\ v_{B,10},\ v_{B,2},\ v_{B,3},\ v_{B,4},\ v_{B,5},\ v_{B,12},\ v_{B,7},\ v_5,\ v_{C,3},\ v_{C,11},\ v_{C,2},\ v_{C,9},\ v_{C,8},\ v_{C,12},\ v_{C,6},\ v_6\}$ then $V(\tilde{\mu}_4)$: $\{v_{A,9},\ v_{A,2},\ v_{A,12},\ v_{A,4},\ v_3,\ v_{B,1},\ v_{B,8},\ v_{B,11},\ v_4,\ v_{B,6},\ v_{C,4},\ v_{C,5},\ v_{C,7},\ v_{C,10}\}$. Placing $2^{27} + 11$ on $v_{c,1}$ and one pebble each on the vertices of $V(\tilde{\mu}_4)$ we cannot move a pebble to v_1 through μ_4. Therefore, $\pi_\mu(S_3) \geq 2^{27} + 11$. Consider the configuration of $2^{27} + 11$ pebbles on the vertices of S_4. The outer three triangles are isomorphic to S_3. Let μ_5: $\{v_1, v_{A,1},\ v_{A,10},\ v_{A,8}, v_{A,7}, v_{A,6},\ v_{A,5},\ v_{A,11},\ v_{A,3},\ v_2,\ v_{B,9},\ v_{B,2}, v_{B,3}, v_{B,4}, v_{B,5}, v_{B,12}, v_{B,7} v_5, v_{C,3}, v_{C,11}, v_{C,2}, v_{C,1}, v_{C,9}, v_{C,8}, v_{C,12}, v_{C,6}, v_6\}$. μ_4 and μ_5 are the longest monophonic paths starting from $S_{(4,A)}$. We arrive at having the monophonic path of length at most 27 from v_1. The monophonic distance between any two vertices in this graph can be deduced using these two paths. There are also two long monophonic paths starting from $S_{(4,B)}$ and $S_{(4,C)}$.

If we go on iterating like this we will get a maximal monophonic path μ_n of length 3^{n-1} and there will be $3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$ vertices which does not pass through the monophonic path. We arrive at the result $2^{3^{n-1}} + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$ by induction. Hence the proof.

Theorem 4.5. *For generalized Petersen graph* $\pi_{\mu t}(GP_n) = t2^{n-1} + 2$.

Proof. Without loss of generality, consider the monophonic path $\mu_1 = \{v_0, v_1, \ldots, v_{n-2}, u_{n-2}\}$ then $V(\tilde{\mu}_1) = \{u_0, u_1, \ldots, u_{n-3}, u_{n-1}, v_{n-1}\}$. Suppose if $p(u_{n-2}) = t2^{n-1} + 1$, $p(u_{n-1}) = p(v_{n-1}) = 1$ and all the other vertices have exactly zero pebbles, we cannot transfer t pebbles to u_0. Thus $\pi_{\mu t}(GP_n) \geq 2^{n-1}t + 2$.

When $t = 1$ Theorem 3.1 holds. Assume that the finding is true for $2 \leq t' \leq t - 1$. Evidently, $V(GP_n)$ has at least $2^n + 2$ pebbles. Hence, we can move a pebble to the target with a maximum expense of 2^{n-1} pebbles. Thus the remaining pebbles on $V(GP_n)$ is $\pi_{(t-1)\mu}(GP_n) = t2^{n-1} + 2 - 2^n = (t-1)2^{n-1} + n$ and hence $t - 1$ extra pebbles can be shifted to α by the method of induction. Thus, $\pi_{\mu t}(GP_n) \leq 2^{n-1}t + 2$.

Theorem 4.6. *For Kusudama flower graph* $\pi_{\mu t}(KF_m) = t2^{2m+2} + 3m + 2$.

Proof: Without loss of generality, consider the monophonic path μ_1: $\{u_{0,1}, u_0, v_0, v_1, v_2, \ldots, v_{2m-2}, u_{2m-2}, u_{2m-2,1}\}$ then $V(\tilde{\mu}_1) = \{u_2, u_4, \ldots, u_{2m-4}, u_{0,2}, u_{2m-2,1}, u_{2,1}, u_{2,2} \ldots, u_{2m-4,1}, u_{2m-4 \, mod \, 2m,2}\}$ be the vertex set complement of μ_1. Placing $t2^{2m+2} - 1$ pebbles on $u_{0,2}$ and one pebble each on $V(\tilde{\mu}_1)$ we cannot move t pebble(s) to $u_{2m-2,1}$. Therefore, $\pi_\mu(KF_m) \geq t2^{2m+2} + 3m + 2$.

When $t = 1$ the result is true by Theorem 3.2. Assume that the finding is true for $2 \leq t' \leq t - 1$. Evidently, the graph KF_m has at least $2^{2m+3} + 3m + 2$ pebbles. Hence we can move a pebble to the target with a maximum expense of 2^{2m+3} pebbles. Then the remaining number of pebbles on $V(KF_m)$ is $t2^{2m+3} + 3m + 2 - 2^{2m-3} = (t-1)2^{2m+2} + 3m + 2 = \pi_{(t-1)\mu}(KF_m)$ and hence we can transfer $t - 1$ extra pebbles to the target vertex by the method of induction. Thus, $\pi_\mu(KF_m) \leq t2^{2m+2} + 3m + 2$.

Theorem 4.7. *For seirpinski triangle graph* $\pi_{\mu t}(S_n) = 2^{3^{n-1}}t + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$

***Proof*:** Consider any one of the maximal monophonic path μ_i starting from v_1 and ending with v_6 in S_n. Let $\alpha = v_1$ place $2^{3^{n-1}}t - 1$ pebbles on v_6 and one pebble each on all the vertices of $\tilde{\mu}_i$ we cannot move t pebbles to α. Therefore, $\pi_{\mu t}(S_n) \geq 2^{3^{n-1}}t + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$. From Theorem 3.3 the result is true for $n = 1$. Assume that the statement is true for $2 \leq t' \leq t - 1$. Clearly, the graph S_n has at least $2^{3^{n-1+1}}t + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$ pebbles. Hence, we can move a pebble to the target with a maximum expense of $2^{3^{n-1}} + 3^{n-1}$ pebbles. Then the remaining pebbles on $V(S_n)$ is $t2^{3^{n-1}} + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor - 2^{3^{n-1}} = (t-1)2^{3^{n-1}} + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor = \pi_{(t-1)\mu}(S_n)$ and hence $t - 1$ extra pebbles can be shifted to α. Thus, $\pi_{\mu t}(S_n) \leq 2^{3^{n-1}}t + 3^{n-1} - \left\lfloor \frac{3^{n-1}}{2} \right\rfloor$.

5. Application

Fractals are used to simulate many real-world features, including mountains, forests, clouds, and coastlines. The graph S_n is used in digital image processing. Let S_2 be the fixed point in this process. We would want to present the mathematical version of the image processing. Let V be the function which takes S_n to $V(S_2)$. So we observe that $V(S_n) = S_2$. A sequence of sets S_n that is endless can be obtained by carrying out this method indefinitely. The sequence $\{S_n\}$ converges to S_2. S_2 cannot be distinguished from S_5. The computer software uses S_5 rather than S_2 for improved resolution. In addition, the application could quickly determine certain attributes of a digital image by using S_2 in instead of S_5.

5. Conclusion

The monophonic pebbling number $\pi_\mu(G)$ is a specialized variant of the pebbling number that incorporates the monophonic path constraint. It combines elements of traditional graph theory, path constraints, and optimization, making it a rich area for study in combinatorial and graph-theoretic research.

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