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Some properties of the second kind degenerate q-Euler polynomials associated with the p-adic integral on $\mathbb{Z}_p$

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Abstract : In this paper, we introduce the second kind degenerate q-Euler numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$. We also obtain some explicit formulas for the second kind degenerate q-Euler numbers and polynomials.

Key words : Euler numbers and polynomials, the second kind Euler numbers and polynomials, the second kind degenerate Euler numbers and polynomials, the second kind degenerate q-Euler numbers and polynomials, p-adic integral on $\mathbb{Z}_p$.

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1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. Let p be a fixed odd prime number. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x)-g(y)}{x-y}$$

have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For $g \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \le x < p^N} g(x)(-1)^x, \text{ (see [3]).} \tag{1}$$

If we take $g_1(x) = g(x+1)$ in (1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [6])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x - \lambda k) \tag{3}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. Note that $(x|\lambda)$ is a homogeneous polynomials in λ and x of degree n, so if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n = x^n$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n \frac{t^n}{n!}. \tag{5}$$

For $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, if we take $g(x) = q^x e^{(2x+1)t}$ in (2), then we easily see that

$$I_{-1}(q^x e^{(2x+1)t}) = \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{qe^{2t}+1}.$$

Let us define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows(see [5]):

$$\int_{\mathbb{Z}_p} q^y e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{6}$$

$$\int_{\mathbb{Z}_p} q^y e^{(x+2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{7}$$

Recently, many mathematicians have studied in the area of the degenerate Bernoulli umbers and polynomials, degenerate Euler numbers and polynomials, degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 6]). Our aim in this paper is to define the second kind degenerate q-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$. We investigate some properties which are related to the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$.

2. Some properties of the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$

In this section, we introduce the second kind degenerate q-Euler numbers and polynomials, and we obtain explicit formulas for them. For $t,\lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = q^x(1+\lambda t)^{(2x+1)/\lambda}$ in (2), then we easily see that

$$\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{(2x+1)/\lambda} d\mu_{-1}(x) = \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}. \tag{8}$$

Let us define the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$ as follows:

$$\int_{\mathbb{Z}_p} q^y(1+\lambda t)^{(2y+1)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!}, \tag{9}$$

$$\int_{\mathbb{Z}_p} q^y(1+\lambda t)^{(2y+1+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!}. \tag{10}$$

Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$. From (7) and (10), we note that

$$\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0}\frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.$$

Thus, we have

$$\lim_{\lambda\to 0}\mathcal{E}_{n,q}(x,\lambda) = E_{n,q}(x), (n \geq 0).$$

From (5) and (9), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \left(\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{11}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}.$$

By (8), (9), and (10), we obtain the following Witt's formula.

Theorem 2. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^x (2x+1|\lambda)_n d\mu_{-1}(x) = \mathcal{E}_{n,q}(\lambda),$$
$$\int_{\mathbb{Z}_p} q^y (x+2y+1|\lambda)_n d\mu_{-1}(y) = \mathcal{E}_{n,q}(x,\lambda).$$

By (5) and (9), we can derive the following recurrence relation:

$$\begin{aligned}\sum_{n=0}^{\infty} 2(1|\lambda)_n \frac{t^n}{n!} &= 2(1+\lambda t)^{1/\lambda} = (q(1+\lambda t)^{2/\lambda}+1)\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= q(1+\lambda t)^{2/\lambda}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= \left(\sum_{l=0}^{\infty} q(2|\lambda)_l \frac{t^l}{l!}\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!}\right) + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q(2|\lambda)_l \mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda)\right)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (12), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$, we have

$$q\sum_{l=0}^{n}\binom{n}{l}(2|\lambda)_l \mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda) = 2(1|\lambda)_n.$$

By (5), (9), and (10), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} q\mathcal{E}_{n,q}(x+2,\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} \\ &= \frac{2q(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+2)/\lambda} + \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= 2(1+\lambda t)^{(x+1)/\lambda} = 2\sum_{n=0}^{\infty}(x+1|\lambda)_n\frac{t^n}{n!}.\end{aligned} \tag{13}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (13), we have the following theorem.

Theorem 4. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$q\mathcal{E}_{n,q}(x+2,\lambda) + \mathcal{E}_{n,q}(x,\lambda) = 2(x+1|\lambda)_n.$$

By (1) and (5), we have

$$\begin{aligned}&\sum_{m=0}^{\infty}(q^n\mathcal{E}_{m,q}(2n,\lambda) + \mathcal{E}_{m,q}(\lambda))\frac{t^m}{m!} \\ &= \int_{\mathbb{Z}_p} q^{x+n}(1+\lambda t)^{(2x+2n+1)/\lambda}d\mu_{-1}(x) + (-1)^n\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{(2x+1)/\lambda}d\mu_{-1}(x) \\ &= 2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(1+\lambda t)^{(2l+1)/\lambda} = \sum_{m=0}^{\infty}\left(2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(2l+1|\lambda)_m\right)\frac{t^m}{m!}.\end{aligned} \tag{14}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (14), we have the following theorem.

Theorem 5. For $m \in \mathbb{Z}_+$, we have

$$q^n \mathcal{E}_{m,q}(2n, \lambda) + \mathcal{E}_{m,q}(\lambda) = 2\sum_{l=0}^{n-1}(-1)^{n-1-l} q^l (2l+1|\lambda)_m.$$

By (10), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{E}_{n,q^{-1}}(-x,-\lambda)\frac{t^n}{n!} = \frac{2(1-\lambda t)^{-1/\lambda}}{q^{-1}(1-\lambda t)^{-2/\lambda}+1}(1-\lambda t)^{x/\lambda} \\
&= \frac{2q}{(1-\lambda t)^{2/\lambda}+1}(1-\lambda t)^{(x+1)/\lambda} = \sum_{n=0}^{\infty}(-1)^n q \mathcal{E}_{n,q}(x+1,\lambda)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (15), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q^{-1}}(-x,-\lambda) = (-1)^n q \mathcal{E}_{n,q}(x+1,\lambda), \quad \mathcal{E}_{n,q^{-1}}(-\lambda) = (-1)^n q \mathcal{E}_{n,q}(1|\lambda).$$

For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\
&= \frac{2(1+\lambda t)^{1/\lambda}}{q^d(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l q^l (1+\lambda t)^{2l/\lambda} \\
&= \sum_{n=0}^{\infty}\left(d^n \sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d}, \frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing coefficients of $\dfrac{t^n}{n!}$ in the above equation, we have the following theorem:

Theorem 7. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = d^n \sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d}, \frac{\lambda}{d}\right).$$

In particular,

$$\mathcal{E}_{n,q}(\lambda) = d^n \sum_{l=0}^{d-1}(-1)^l q^l \mathcal{E}_{n,q^d}\left(\frac{2l+1-d}{d}, \frac{\lambda}{d}\right).$$

From (10), we derive

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x+y,\lambda)\frac{t^n}{n!} = \frac{2(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda} \\
&= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda} \\
&= \left(\sum_{n=0}^{\infty}\mathcal{E}_{m,q}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Therefore, by (16), we have the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x+y,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}.$$

From Theorem 8, we note that $\mathcal{E}_{n,q}(x,\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (10), we obtain

$$\begin{aligned}\frac{2e^t}{qe^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!} = \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{17}$$

Thus, by (17), we have the following theorem.

Theorem 9. For $n\in\mathbb{Z}_+$, we have

$$E_{m,q}(x) = \sum_{n=0}^{m}\lambda^{m-n}\mathcal{E}_{n,q}(x,\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (7), we have

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} = \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{m=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^m}{m!}, \tag{18}$$

and

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \tag{19}$$

Thus, by (18) and (19), we have the following theorem.

Theorem 10. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = \sum_{n=0}^{m}\lambda^{m-n}E_{n,q}(x)S_1(m,n).$$

Letting $q\to 1$ in Theorem 10 gives the theorem

$$\mathcal{E}_{n}(x,\lambda) = \sum_{n=0}^{m}\lambda^{m-n}E_{n}(x)S_1(m,n).$$

which was proved by Ryoo [4].

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