WELL-POSEDNESS ANALYSIS VIA GENERALIZED FRACTIONAL DERIVATIVES

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Abstract. In this manuscript, we investigate the existence and uniqueness of solutions for nonlinear initial value problems of fractional differential equations within the framework of Ψ -Caputo sense. We utilize two fixed point theorems: the Schauder fixed point theorem (SFPT) and the Banach fixed point theorem (BFPT). Furthermore, we establish the notion of continuation. To validate the credibility of our key findings, we provide an illustrative example.
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1. INTRODUCTION

Fractional calculus is a field within mathematical analysis that broadens the notions of differentiation and integration to include non-integer orders. Unlike traditional calculus, which deals exclusively with whole numbers, fractional calculus allows us to work with fractional or non-integer orders, opening up a rich field of mathematical exploration. Its applications span a wide range of scientific disciplines, from physics and engineering to biology and finance. Fractional calculus has proven to be a powerful tool for modeling complex systems with memory effects, non-local behavior, and fractal geometry. It has been employed in solving differential equations in physics, optimizing control systems, understanding anomalous transport phenomena, and even in designing novel financial models to better capture the dynamics of markets. In recent years, interest in fractional calculus and its applications has surged, as researchers and engineers recognize its potential to address a variety of real-world challenges and phenomena with greater accuracy and insight. Boundary value problems (BVPs) are widely used in various fields of science and engineering, including physics, heat transfer, fluid mechanics, quantum mechanics, and more. They often arise when studying physical systems where the behavior of the system is influenced by external conditions at its boundaries, for nonlinear fractional differential equations see [1-3], for predictor-corrector approach [4-6], for the existence and uniqueness see [7-14] and references therein.

In this work, we focus on investigating the following nonlinear Ψ -Caputo fractional value problem (Ψ -CFVP)

$$\begin{cases} D_{a^+}^{\alpha,\Psi} x(t) = \mathbf{F}(t, x(t)), \ 0 < a < t, \\ x(a^+) = x_a, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$. Let Ψ be increasing function via $\Psi'(t)\neq 0$, $\forall t$. $\mathbf{F}(t,x)$ from $[a,+\infty)\times\Omega$ to $\mathbb{R}$.

Our paper is organized as follows: In Section 2, we give a concise review of fundamental definitions and essential preliminary information that will serve as a foundation for the subsequent sections. In Section 3, we demonstrate the existence and uniqueness of the solution, employing both the SFPT and the BFPT and we also delve into the discussion of the continuation theorem. Finally, in the last section, we provide an illustrative example to showcase the practical application of the results we have derived.

2. ESSENTIAL PRELIMINARIES

Within this section, we introduce fundamental definitions and initial facts that will serve as a foundational framework for the subsequent sections.

Definition 2.1 [15]. *The fractional integral*

$$\Im_{0^+}^{\alpha}\mathbf{F}(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\mathbf{F}(s)}{(t-s)^{1-\alpha}}ds,$$

where $\alpha>0$ *, is called Riemann-Liouville fractional integral of order* α *for a function* $\mathbf{F}:(0,+\infty)\rightarrow\mathrm{R}$ *and* $\Gamma(.)$ *is the gamma function defined by*

$$\Gamma(\alpha)=\int_0^{+\infty}t^{\alpha-1}e^{-s}ds.$$

Definition 2.2 [15]. *The Riemann-Liouville fractional derivative of order* $\alpha>0$ *, for a continuous function* $\mathbf{F}:(0,+\infty)\rightarrow\mathrm{R}$ *is given by*

$${}^{RL}D_{0^+}^{\alpha}\mathbf{F}(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}\mathbf{F}(s)ds.$$

$\Gamma(.)$ *is the gamma function, provided that the right side is point-wise defined on* $(0,+\infty)$ *and* $n=[\alpha]+1,$ $[\alpha]$ *stands for the greatest integer less than* α*. where* $n-1<\alpha<n.$

Definition 2.3 [16]. *The Caputo fractional derivative of order* $\alpha>0$ *, for a continuous function* $\mathbf{F}:(0,+\infty)\rightarrow\mathrm{R}$ *is intended by*

$${}^{C}D_{0^+}^{\alpha}\mathbf{F}(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}\mathbf{F}^{(n)}(s)ds,\ n-1<\alpha<n.$$

Definition 2.4 [19, 20]. *The Hadamard fractional integral of order* $\alpha>0$ *, for a continuous function* $f:[1,+\infty)\rightarrow\mathrm{R}$ *is given by*

$${}^{H}\Im_1^{\alpha}\mathbf{F}(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathbf{F}(s)\frac{ds}{s}.$$

Definition 2.5 [19, 20]. *The Caputo-Hadamar fractional derivative fractional integral of order* $\alpha>0$ *, for a continuous function* $\mathbf{F}:[1,+\infty)\rightarrow\mathrm{R}$ *is given by*

$${}^{CH}D_1^{\alpha}\mathbf{F}(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n\mathbf{F}(s)\frac{ds}{s},$$

where $\delta^n=\left(t\frac{d}{dt}\right)^n,\quad n\in\mathrm{N}.$

Definition 2.6 [17, 21]. *The* $\Psi-$ *Riemann-Liouville fractional integral of order* $\alpha>0$ *for a continued function* $f:[a,t]\rightarrow\mathrm{R}$ *is referred to as*

$$\Im_a^{\alpha,\Psi}\mathbf{F}(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s)ds,$$

Definition 2.7 [17, 21]. *The $\Psi-$ Caputo fractional derivative of order $\alpha>0$, for a continuous function $\mathbf{F}:[a,t]\to\mathrm{R}$ stands for of*

$$D_a^{\alpha,\Psi}\mathbf{F}(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(\Psi(t)-\Psi(s))^{n-\alpha-1}\Psi'(s)\partial_\Psi^n\mathbf{F}(s)ds,\ t>a,\ n-1<\alpha<n,$$

where $\partial_\Psi^n=\left(\frac{1}{\Psi'(s)}\frac{d}{dt}\right)^n,\quad n\in\mathrm{N}.$

Lemma 2.8 [17, 21]. *Let $\alpha,\beta\geq 0$, $\mathbf{F}\in C([a,b],\mathrm{R})$. Then $\forall t\in[a,b]$, and by assuming $F_a(t)=\Psi(t)-\Psi(s)$, we have*

1. $\Im_a^{\alpha,\Psi}\Im_a^{\beta,\Psi}\mathbf{F}(t)=\Im_a^{\alpha+\beta,\Psi}\mathbf{F}(t),$
2. $D_a^{\alpha,\Psi}\Im_a^{\alpha,\Psi}\mathbf{F}(t)=\mathbf{F}(t),$
3. $\Im_a^{\alpha,\Psi}(F_a(t))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(F_a(t))^{\beta+\alpha-1},$
4. $D_a^{\alpha,\Psi}(F_a(t))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(F_a(t))^{\beta-\alpha-1},$
5. $D_a^{\alpha,\Psi}(F_a(t))^k=0,$ *for* $k\in\{0,\dots,n-1\},\quad n\in\mathrm{N},\quad \alpha\in(n-1,n].$

Lemma 2.9 [17, 21]. *Let $n-1<\alpha_1\leq n$, $\alpha_2>0$, $a>0$, $\mathbf{F}\in L(a,T)$, $D_a^{\alpha_1,\Psi}\mathbf{F}\in L(a,T)$. Then the fractional differential equation*

$$D_a^{\alpha_1,\Psi}\mathbf{F}=0,$$

has the unique solution

$$\mathbf{F}(t)=\sigma_0+\sigma_1(\Psi(t)-\Psi(s))+\sigma_2(\Psi(t)-\Psi(s))^2+$$

$$\cdots+\sigma_{n-1}(\Psi(t)-\Psi(s))^{n-1},$$

and

$$\Im_a^{\alpha_1,\Psi}D_a^{\alpha_1,\Psi}\mathbf{F}(t)=\mathbf{F}(t)+\sigma_0+\sigma_1(\Psi(t)-\Psi(s))+\sigma_2(\Psi(t)-\Psi(s))^2+$$

$$\cdots+\sigma_{n-1}(\Psi(t)-\Psi(s))^{n-1},$$

with $\sigma_i\in\mathrm{R},\quad i=0,1,\dots,n-1.$

Furthermore,

$$D_a^{\alpha_1,\Psi}\Im_a^{\alpha_1,\Psi}\mathbf{F}(t)=\mathbf{F}(t),$$

and

$$\Im_a^{\alpha_1,\Psi}\Im_a^{\alpha_2,\Psi}\mathbf{F}(t)=\Im_a^{\alpha_2,\Psi}\Im_a^{\alpha_1,\Psi}\mathbf{F}(t)=\Im_a^{\alpha_1+\alpha_2,\Psi}\mathbf{F}(t).$$

Lemma 2.10 [18]. *Assuming the continuity of $\mathbf{F}(t,u)$, the initial value problem for the nonlinear $\Psi-$ Caputo fractional value problem (1.1) can be expressed as an equivalent Volterra integral equation in the following manner*

$$x(t)=x_a+\frac{1}{\Gamma(\alpha)}\int_a^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds. \tag{2.1}$$

Lemma 2.11 [22]. *Consider the subset* $\mathbf{M}$ *within the space* $C\langle a, T\rangle$. *The subset* $\mathbf{M}$ *is precompact if, and only if, the following conditions are verified:*

(i) $\{v(t)\ :\ x \in \mathbf{M}\}$ *is uniformly bounded,*

(ii) $\{v(t)\ :\ x \in \mathbf{M}\}$ *is equicontinuous on* $[a,T]$.

Lemma 2.12 [22, 23]. *U Give a closed, bounded, and convex subset* $(\mathbf{SFPT})$ *within the Banach space* X, *if we assume that the mapping* $P\ :\ U \to U$ *is completely continuous, then it follows that* P *possesses a fixed point within the set* U.

Lemma 2.13 [22, 23]. *of a U empty closed subset -: Consider a non* $(\mathbf{BFPT})$ *Banach space* X. *Additionally, suppose that for every natural number* n, *we have* $b_n \geq 0$, *and that the series* $\sum_{n=0}^{\infty} b_n$ *is convergent. Furthermore, suppose that the mapping* $\mathbf{A}\ :\ U \to U$ *verifies the following conditions:*

$$\left\|\mathbf{A}^n x - \mathbf{A}^n v\right\| \leq b_n \|x - v\|,\ x, v \in U.$$

Then $\mathbf{A}$ *possesses a uniquely determined fixed point denoted as* x^*. *Moreover, if* x_0 *belongs to* U, *then* $\left(\mathbf{A}^n x_0\right)_{n=1}^{\infty}$ *tends to* x^*.

3. WELL-POSEDNESS

Now, let's examine the local existence and uniqueness of the solution to $\Psi-$ CFVP (1.1). We make the following assumption:

$(\mathbf{H}_1)$: $\mathbf{F}(t,x) \in C([a,+\infty) \times \Omega, \mathrm{R})$, where $\Omega \subset \mathrm{R}$, and that $\mathbf{F}(t,x)$ is a bounded continuous map defined on $[a,T] \times \overline{\Omega_0}$, where Ω_0 is a supposed to be a bounded subset of ✋.

Theorem 3.1. *If assumption* $(\mathbf{H}_1)$ *is satisfied, then* $\Psi-$ *CFVP (1.1) possesses at least one solution* $x(t) \in C[a,h]$ *for a certain* $h \in [a,T]$.

Proof. First we start by setting

$$S = \left\{x \in C[a,T]\ :\ \|x - x_a\|_{C[a,T]} = \max_{t \in [a,T]} |x - x_a| \leq \chi\right\}$$

where $\chi > 0$ is a constant. From the continuity of $\mathbf{F}(t,x)$, one can find constant $M > 0$ verifying

$$\max\{|\mathbf{F}(t,x)\ :\ t \in [a,T],\ x \in N|\} \leq M.$$

Once more, allow

$$S_h = \left\{x \in C[a,h]\ :\ \|x - x_a\|_{C[a,h]} = \max_{t \in [a,h]} |x - x_a| \leq \chi\right\}$$

where $\Psi(h) = \min\left\{\Psi(a) + \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \chi\right)^{\frac{1}{\alpha}}, T\right\}$.

Clearly, the non-empty set S_h is a closed bounded convex subset of $C[a,h]$. Notice that for $h \leq T$, S_h and $C[a,h]$ can be respectively regarded as restrictions of S and $C[a,T]$.

Now let's define an operator

$$\mathbf{A}x(t) = x_a + \frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s, x(s))ds,\ t \in [a, h]. \qquad (3.1)$$

For $x \in C[a,h]$, we get

$$|\mathbf{A}x(t) - x_a| \le \frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)|\mathbf{F}(s, x(s))|ds,$$

$$\le \frac{M}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)ds,$$

$$\le \frac{M}{\Gamma(\alpha+1)}(\Psi(t) - \Psi(a))^{\alpha}.$$

Therefore, we have

$$\|\mathbf{A}x(t) - x_a\|_{C[a,h]} \le \chi.$$

The outcome demonstrates that, $\mathbf{A}S_h \subset S_h$.

Next we will prove the continuity of $\mathbf{A}$ is . Take $x_n, x \in S_h$ such that $\|x_n - x\|_{C[a,h]} \to 0$ as $n \to \infty$. Then

$$|\mathbf{A}x_n(t) - \mathbf{A}x(t)| \le \frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)|\mathbf{F}(s, x_n(s) - \mathbf{F}(s, x(s)))|ds,$$

$$\le \frac{1}{\Gamma(\alpha+1)}\|\mathbf{F}(s, x_n(s) - \mathbf{F}(s, x(s)))\|_{C[a,h]}(\Psi(t) - \Psi(s))^{\alpha}.$$

Then,

$$\|\mathbf{A}x_n(t) - \mathbf{A}x(t)\| \le \frac{1}{\Gamma(\alpha+1)}\|\mathbf{F}(s, x_n(s) - \mathbf{F}(s, x(s)))\|_{C[a,h]}(\Psi(t) - \Psi(s))^{\alpha}.$$

Using the fact that F is continuous, we get $\|\mathbf{F}(s, x_n(s) - \mathbf{F}(s, x(s)))\|_{C[a,h]} \to 0$ when $n \to \infty$. So $\|\mathbf{A}x_n(t) - \mathbf{A}x(t)\| \to 0$ as $n \to \infty$. Hence proving the continuity of the operator $\mathbf{A}$.

The next step is to show that $\mathbf{A}S_h$ is equicontinuous. It is obvious that

$$\frac{1}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)ds = \frac{1}{\Gamma(\alpha+1)}(\Psi(t) - \Psi(a))^{\alpha} \to 0 \text{ as } t \to a^+.$$

Furthermore, one can find a certain $\tilde{h} \in (a, h)$ with the property that for $t \in \left[a, \tilde{h}\right]$ we have

$$\frac{2M}{\Gamma(\alpha)}\int_a^t (\Psi(t) - \Psi(s))^{\alpha-1}\Psi'(s)ds < \varepsilon.$$

Let $x \in S_h$ and $t_1 < t_2$, for $t_1, t_2 \in \left[a, \tilde{h}\right]$, we get

$$\begin{aligned}\left|\mathbf{A}x(t_1)-\mathbf{A}x(t_2)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}\Psi'(s)\left|\mathbf{F}(s,x(s))\right|ds\right.\\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\Psi'(s)\left|\mathbf{F}(s,x(s))\right|ds\right|,\\ &\leq \frac{M}{\Gamma(\alpha)}\int_a^{t_1}\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}\Psi'(s)ds\\ &\quad+\frac{M}{\Gamma(\alpha)}\int_a^{t_2}\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\Psi'(s)ds<\varepsilon.\end{aligned} \tag{3.2}$$

For $t_1,t_2\in\left[\frac{a+\tilde{h}}{2},h\right]$, one gets for $t_1,t_2\in\left[\frac{a+\tilde{h}}{2},h\right]$, one gets

$$\begin{aligned}\left|\mathrm{A}x(t_1)-\mathrm{A}x(t_2)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}\Psi'(s)\mathrm{F}(s,x(s))ds\right.\\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\Psi'(s)\mathrm{F}(s,x(s))ds\right|,\\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|\left[\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}-\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\right]\mathrm{F}(s,x(s))\right|\Psi'(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\mathrm{F}(s,x(s))\right|\Psi'(s)ds,\\ &\leq \frac{M}{\Gamma(\alpha)}\int_a^{\frac{a+\tilde{h}}{2}}\left|\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}-\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\right|\Psi'(s)ds\\ &\quad+\frac{M}{\Gamma(\alpha)}\int_{\frac{a+\tilde{h}}{2}}^{t_1}\left|\left(\Psi(t_1)-\Psi(s)\right)^{\alpha-1}-\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\right|\Psi'(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\left(\Psi(t_2)-\Psi(s)\right)^{\alpha-1}\right|\Psi'(s)ds,\\ &\leq \frac{2M}{\Gamma(\alpha)}\int_a^{\frac{a+\tilde{h}}{2}}\left(\Psi\left(\frac{a+\tilde{h}}{2}\right)-\Psi(s)\right)^{\alpha-1}\Psi'(s)ds\\ &\quad+\frac{M}{\Gamma(\alpha+1)}\left[\left(\Psi(t_1)-\Psi\left(\frac{a+\tilde{h}}{2}\right)\right)^{\alpha}-\left(\Psi(t_2)-\Psi\left(\frac{a+\tilde{h}}{2}\right)\right)^{\alpha}\right.\\ &\quad\left.+\left(\Psi(t_2)-\Psi(t_1)\right)^{\alpha}\right]+\frac{M}{\Gamma(\alpha+1)}\left(\Psi(t_2)-\Psi(t_1)\right)^{\alpha},\\ &\leq \frac{\varepsilon}{2}+\frac{M}{\Gamma(\alpha+1)}\left[\left(\Psi(t_1)-\Psi\left(\frac{a+\tilde{h}}{2}\right)\right)^{\alpha}-\left(\Psi(t_2)-\Psi\left(\frac{a+\tilde{h}}{2}\right)\right)^{\alpha}\right.\\ &\quad\left.+\left(\Psi(t_2)-\Psi(t_1)\right)^{\alpha}\right]+\frac{M}{\Gamma(\alpha+1)}\left(\Psi(t_2)-\Psi(t_1)\right)^{\alpha}.\end{aligned}$$

Then, there exists $\tilde{h}_1\in\left(0,\frac{\tilde{h}-a}{2}\right)$ such that for $t_1,t_2\in\left[\frac{a+\tilde{h}}{2},h\right]$ and $\left|t_1-t_2\right|<h,$

$$\left|\mathbf{A}x(t_1)-\mathbf{A}x(t_2)\right|<2\varepsilon.$$

Thus, from *(3.2)* and *(3.3)* one has $\{\mathbf{A}x(t) : x\in S_h\}$ is equicontinuous.
Furthermore, it is evident that $\{\mathbf{A}x(t) : x\in S_h\}$ is uniformly bounded due to $\mathbf{A}S_h\subset S_h$.
By using Lemma 2.11, we see that $\mathbf{A}S_h$ is precompact and hence $\mathbf{A}$ is completely continuous. Consequently from Lemma 2.12, $\Psi-$ CFVP (1.1) has at least a local solution, and hence concluding our proof.

To ensure the existence of a unique solution to $\Psi-$ CFVP (1.1), we need the following assumptions:
$(\mathbf{H}_2)$: $\mathbf{F}(t,x)$: $[a,+\infty)\times\Omega\to\mathrm{R}$, in $\Psi-$ CFVP (1.1) meet the Lipschitz condition in relation to the second variable, *i.e*,

$$\left|\mathbf{F}(t,x)-\mathbf{F}\left(t,\tilde{x}\right)\right|\le L\left|x-\tilde{x}\right|,\ L>0.$$

Theorem 3.2. *When f assumption* $(\mathbf{H}_2)$ *is satisfied, the* $\Psi-$ *CFVP (1.1) possesses a unique solution* $x(t)\in C[a,h]$, *with* $h\in[a,T]$.

Proof. As $\Psi-$ CFVP (1.1) is equivalent to equation (2.1), demonstrating the existence of a unique solution for (2.1) suffices.
We know that x the unique solution to integral equation (2.1) if and only if $\mathbf{A}x=x$, $(x$ is a fixed point of $\mathbf{A})$. So we only need to show that the operator $\mathbf{A}$ defined by (3.1) has a unique fixed point.
Let $x\in S_h$, we have

$$\begin{aligned}\left|\mathbf{A}x(t)-x_a\right| &\le \frac{1}{\Gamma(\alpha)}\int_a^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\left|\mathbf{F}(s,x(s))\right|ds,\\ &\le \frac{\|\mathbf{F}\|_{C[a,h]}}{\Gamma(\alpha+1)}(\Psi(t)-\Psi(a))^{\alpha},\\ &\le \frac{\|\mathbf{F}\|_{C[a,h]}}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)}{\|f\|_{C[a,h]}}\chi=\chi.\end{aligned}$$

So, $\mathbf{A}x\in S_h$ if $x\in S_h$.
For any $a<t_1<t_2<h$,

$$\left|\mathbf{A}x(t_1)-\mathbf{A}x(t_2)\right| = \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\Psi(t_1)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds\right.$$

$$\left.-\frac{1}{\Gamma(\alpha)}\int_a^{t_2}(\Psi(t_2)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds\right|,$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,x(s))\right|\Psi'(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(\Psi(t_2)-\Psi(s))^{\alpha-1}\mathbf{F}(s,x(s))\right|\Psi'(s)ds,$$

$$\leq \frac{\|\mathbf{F}\|_{C[a,h]}}{\Gamma(\alpha)}\int_a^{t_1}\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\Psi'(s)ds$$

$$+\frac{\|\mathbf{F}\|_{C[a,h]}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\Psi(t_2)-\Psi(s))^{\alpha-1}\Psi'(s)ds,$$

$$\leq \frac{\|\mathbf{F}\|_{C[a,h]}}{\Gamma(\alpha+1)}\left[2(\Psi(t_2)-\Psi(t_1))^{\alpha}-(\Psi(t_2)-\Psi(a))^{\alpha}+(\Psi(t_1)-\Psi(a))^{\alpha}\right].$$

This outcome suggests the continuity of Ax .
Furthermore, for arbitrary $n \in \mathrm{N}$ and $t \in [a,h]$, one have t

$$\left\|\mathbf{A}^n x-\mathbf{A}^n \tilde{x}\right\|_{C[a,t]} \leq \frac{L^n}{\Gamma(1+n\alpha)}(\Psi(t)-\Psi(a))^{n\alpha}\left\|x-\tilde{x}\right\|_{C[a,t]}.$$

The mentioned equation is evidently valid when $n=0$. Using induction, assume that the case for $n-1$ is correct, we observe that

$$\left\|\mathbf{A}^n x-\mathbf{A}^n \tilde{x}\right\|_{C[a,t]} = \left\|\mathbf{A}\left(\mathbf{A}^{n-1}x\right)-\mathbf{A}\left(\mathbf{A}^{n-1}\tilde{x}\right)\right\|_{C[a,t]}. \tag{3.4}$$

$$=\frac{1}{\Gamma(\alpha)}\max_{a<\tau<t}\int_a^{\tau}(\Psi(\tau)-\Psi(s))^{\alpha-1}\Psi'(s)\left|\mathbf{F}\left(s,\mathbf{A}^{n-1}x(s)\right)-\mathbf{F}\left(s,\mathbf{A}^{n-1}\tilde{x}(s)\right)\right|ds.$$

Through the Lipschitz condition and the assumption induction, we deduce that

$$\left\|\mathbf{A}^n x-\mathbf{A}^n \tilde{x}\right\|_{C[a,t]}$$
$$\leq \frac{L}{\Gamma(\alpha)} \max_{a<\tau<t} \int_a^\tau (\Psi(\tau)-\Psi(s))^{\alpha-1} \Psi'(s) \left|\mathbf{A}^{n-1} x(s)-\mathbf{A}^{n-1} \tilde{x}(s)\right| ds,$$
$$\leq \frac{L^n}{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}$$
$$\max_{a<\tau<t} \int_a^\tau \left[(\Psi(\tau)-\Psi(s))^{\alpha-1}\right]\left[(\Psi(\tau)-\Psi(a))^{(n-1)\alpha}\left\|x-\tilde{x}\right\|_{C[a,\tau]}\right]\Psi'(s)ds,$$
$$\leq \frac{L^n \left\|u-\tilde{u}\right\|_{C[a,t]}}{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)} \max_{a<\tau<t}\left[(\Psi(\tau)-\Psi(a))^{(n-1)\alpha}\right]\int_a^\tau \left[(\Psi(\tau)-\Psi(s))^{\alpha-1}\right]\Psi'(s)ds,$$
$$\leq \frac{\left(L\left[(\Psi(\tau)-\Psi(a))^{\alpha}\right]\right)^n}{\Gamma(1+n\alpha)}\left\|x-\tilde{x}\right\|_{C[a,t]}.$$

Consequently, we attain the desired outcome (3.4) $\cdot$ This allows us to obtain the required result within $[a,h]$

$$\left\|\mathbf{A}^n x-\mathbf{A}^n \tilde{x}\right\|_{C[a,h]} \leq \frac{\left(L(\Psi(\tau)-\Psi(a))^{\alpha}\right)^n}{\Gamma(1+n\alpha)}\left\|x-\tilde{x}\right\|_{C[a,h]}.$$

Referring to Lemma *2.13*, it is necessary to confirm the convergence of the series $\sum_{n=0}^{\infty} b_n$. Notice that

$$\sum_{n=0}^{\infty} \frac{\left(L\left[(\Psi(\tau)-\Psi(a))^{\alpha}\right]\right)^n}{\Gamma(1+n\alpha)} = E_\alpha\left[L(\Psi(\tau)-\Psi(a))^{\alpha}\right],$$

where E_α is the Mittag-Leffler function. Hence, we can employ Lemma *2.13* to infer the uniqueness of $\Psi-$ CFVP (1.1).

In the upcoming discussion, our focus lies on finding a continuous solution to $\Psi-$ CFVP (1.1) $\cdot$

To achieve this objective, we put forth and substantiate the following continuation result, drawing upon the fundamental concept found in [11].

Theorem 3.3. *Assuming the fulfillment of hypothesis* $(\mathbf{H}_1)$ *, the function* $x=x(t)$, $t\in[a,\beta)$ *, is non-continuable if and only if, for some* $\xi\in\left(a,\frac{a+\beta}{2}\right)$ *, and for every closed bounded subset* $V\subset[\xi,+\infty)\times\mathrm{R}$ *, there exists* $t^*\in[\xi,\beta)$ *such that* $\left(t^*,x(t^*)\right)\notin V$.

Proof. (The first part ⇐) Let's assume the continuity of the solution $x=x(t)$ for $\Psi-$ CFVP (1.1). Then we have existence of $\tilde{u}(t)$ a solution of $\Psi-$ CFVP (1.1) defined on $\left[a,\tilde{\beta}\right)$, $\left(\beta<\tilde{\beta}\right)$ verifying $x(t)=\tilde{x}(t)$ for $t\in[a,\beta)$, which implies that

$\lim_{t\to\beta^-} x(t)=\tilde{x}(\beta)$. Now let $K\subset[\xi,+\infty)\times\mathrm{R},$ be the compact subset given by $K=\{(t,x(t))\ :\ t\in[\xi,\beta]\}$. Note that there is no $t^*\in[\xi,\beta)$ such that $(t,x(t))\notin K.$ Hence we get a contradiction, and thus $x(t)$ is non-continuable.

(The second $\Rightarrow$) Assume the existence of a compact subset $V\subset[\xi,+\infty)\times\mathrm{R}$ such that

$$\{(t,x(t))\ :\ t\in[\xi,\beta)\}\subset V.$$

The compact nature of V implies that β is finite. According to hypothesis $(\mathbf{H}_1)$, one can find a $K>0$ verifying $\max_{(t,x)\in V}|\mathbf{F}(t,x)|\le K.$

Our theorem is demonstrated in the subsequent discussion through two distinct steps.

Step 1. First we start by demonstrating that $\lim_{t\to\beta^-} x(t)$ exists.

Let

$$G(t)=\int_a^{\xi}(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)ds,\ t\in[2\xi,\beta).$$

It is evident to see the uniform continuity of $G(t)$ on $[2\xi,\beta).$ For $t_1,t_2\in[2\xi,\beta)$ and $t_1<t_2,$ we have

$$\begin{aligned}
|x(t_1)-x(t_2)| &= \\
&\left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\Psi(t_1)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds\right.\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^{t_2}(\Psi(t_2)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds\right|,\\
&\le\frac{1}{\Gamma(\alpha)}\int_a^{\xi}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,x(s))\right|\Psi'(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi}^{t_1}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,x(s))\right|\Psi'(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(\Psi(t_2)-\Psi(s))^{\alpha-1}\mathbf{F}(s,x(s))\right|\Psi'(s)ds,\\
&\le\frac{M}{\Gamma(\alpha)}\int_a^{\xi}\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\Psi'(s)ds\\
&+\frac{K}{\Gamma(\alpha)}\int_{\xi}^{t_1}\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\Psi'(s)ds\\
&+\frac{K}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\Psi(t_2)-\Psi(s))^{\alpha-1}\Psi'(s)ds,\\
&\le\frac{M}{\Gamma(\alpha)}|G(t_1)-G(t_2)|+\frac{K}{\Gamma(\alpha+1)}\Big[2(\Psi(t_2)-\Psi(t_1))^{\alpha}\\
&+(\Psi(t_1)-\Psi(\xi))^{\alpha}-(\Psi(t_2)-\Psi(\xi))^{\alpha}\Big].
\end{aligned}$$

By the continuity of $G(t)$, we see that the Cauchy sequence convergence criterion applies, and thus proving the following desired convergence

$$\lim_{t\to\beta^-} x(t) = x^*.$$

Step 2. Now we demonstrate that $x(t)$ can be extended to a continuous function. The fact that V is closed implies that $(\beta, x^*)\in V$. Define $x(\beta)=x^*$. Clearly $x(t)\in C[a,\beta]$.

Now we introduce the following operator B :

$$By(t)=x(\beta)+\frac{1}{\Gamma(\alpha)}\int_\beta^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,y(s))ds,$$

Let

$$Q_b=\{(t,y)\ :\ \beta\le t\le\beta+1,\ |y|\le b\},$$

and $M=\max_{(t,y)\in Q_b}|\mathbf{F}(t,y)|$ which exists by the continuity of $\mathbf{F}$ on Q_b . Also we set

$$I_h=\left\{y\in C[\beta,\beta+h]\ :\ \max_{t\in[\beta,\beta+h]}|y(t)|\le b,\ y(\beta)=x(\beta)\right\},$$

where $\Psi(\beta+h)=\min\left\{1,\Psi(\beta)+\left(\frac{\Gamma(\alpha+1)}{M}b\right)^{\frac{1}{\alpha}}\right\}.$

We assert that B is completely continuous on Q_b. Set $\{y_n\}\subseteq C[\beta,\beta+h]$, $\|y_n-y\|_{C[\beta,\beta+h]}\to 0$ as $n\to+\infty$. Then we get

$$|By_n(t)-By(t)|=\left|\frac{1}{\Gamma(\alpha)}\int_\beta^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,y_n(s)-\mathbf{F}(s,y(s)))ds\right|,$$

$$\le\frac{1}{\Gamma(\alpha+1)}\|\mathbf{F}(s,y_n(s)-\mathbf{F}(s,y(s)))\|_{C[\beta,\beta+h]}(\Psi(\beta+h)-\Psi(\beta))^\alpha.$$

Due to the continuity of $\mathbf{F}$, it follows that $\|\mathbf{F}(s,x_n(s)-\mathbf{F}(s,x(s)))\|_{C[\beta,\beta+h]}$ approaches 0 as n approaches infinity. This demonstrates the continuity of the operator B .

We aim to establish the equicontinuity of BI_h . For any $y\in I_h$, it holds that $By(\beta)=x(\beta)$, and

$$\begin{aligned}|By(t)| &=\left|\frac{1}{\Gamma(\alpha)}\int_\beta^t(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,y(s))ds\right|\\ &\le\frac{M}{\Gamma(\alpha+1)}(\Psi(t)-\Psi(\beta))^\alpha,\\ &\le\frac{M}{\Gamma(\alpha+1)}(\Psi(\beta+h)-\Psi(\beta))^\alpha\le b.\end{aligned}$$

Therefore $BI_h\subset I_h$, indicating that B maps the set I_h to itself.

Take $L(t)=\frac{1}{\Gamma(\alpha)}\int_a^\beta(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,x(s))ds.$ Verifying the continuity of $L(t)$ on $[\beta,\beta+h]$ is straightforward . For all $y\in I_h$, $\beta\le t_1\le t_2\le\beta+h,$ we get

$$
\begin{aligned}
\left|By(t_1)-By(t_2)\right| \le & \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\Psi(t_1)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,y(s))ds\right.\\
& \left.-\frac{1}{\Gamma(\alpha)}\int_a^{t_2}(\Psi(t_2)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}(s,y(s))ds\right|,\\
& \le \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,y(s))\right|\Psi'(s)ds\\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(\Psi(t_2)-\Psi(s))^{\alpha-1}\mathbf{F}(s,y(s))\right|\Psi'(s)ds,\\
& \le \frac{1}{\Gamma(\alpha)}\int_a^{\beta}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,y(s))\right|\Psi'(s)ds\\
& +\frac{1}{\Gamma(\alpha)}\int_{\beta}^{t_1}\left|\left[(\Psi(t_1)-\Psi(s))^{\alpha-1}-(\Psi(t_2)-\Psi(s))^{\alpha-1}\right]\mathbf{F}(s,y(s))\right|\Psi'(s)ds\\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(\Psi(t_2)-\Psi(s))^{\alpha-1}\mathbf{F}(s,y(s))\right|\Psi'(s)ds,\\
& \le \left|L(t_1)-L(t_2)\right|+\frac{M}{\Gamma(\alpha+1)}\left[2(\Psi(t_2)-\Psi(t_1))^{\alpha}\right.\\
& \left.+(\Psi(t_1)-\Psi(\beta))^{\alpha}-(\Psi(t_2)-\Psi(\beta))^{\alpha}\right]. \qquad (3.5)
\end{aligned}
$$

Considering the uniform continuity of $L(t)$ on $[\beta,\beta+h]$ and (3.5), it a results that $\{By(t)\ :\ y\in I_h\}$ is equicontinuous. Therefore B is completely continuous.

By Lemma *2.12*, the operator B has a fixed point $\tilde{x}(t)\in I_h$, *i.e*,

$$
\begin{aligned}
\tilde{x}(t) &= x(\beta)+\frac{1}{\Gamma(\alpha)}\int_{\beta}^{t}(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}\left(s,\tilde{x}(s)\right)ds,\\
&= x_a+\frac{1}{\Gamma(\alpha)}\int_a^{t}(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}\left(s,\hat{u}(s)\right)ds,
\end{aligned}
$$

where,

$$
\hat{x}(t)=\begin{cases} x(t), & t\in[a,\beta],\\ \tilde{x}(t), & t\in[\beta,\beta+h].\end{cases}
$$

This implies that, $\hat{x}(t)\in C[a,\beta+h]$ and

$$
\hat{x}(t)=x_a+\frac{1}{\Gamma(\alpha)}\int_a^{t}(\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathbf{F}\left(s,\hat{x}(s)\right)ds.
$$

Hence, in accordance with *Lemma 2.12*, $\hat{x}(t)$ constitutes a solution to equation (1.1) on the interval $[a,\beta+h]$, which contradicts the assumption that $x(t)$ is non-continuable. Thus, the proving the desired result.

4. EXAMPLE

Consider the following IVP

$$D^{\Psi(t)}x(t)+x^2(t)=\frac{1}{\Gamma(2-\alpha)}(\Psi(t))^{1+\alpha}+(\Psi(t))^2, 0<\alpha<1, t>1,$$

$$u(1^+)=0.$$

Where $\Psi(t)=\log t.$

The exact solution of this equation is $x(t)=\Psi(t)=\log t$.

5. CONCLUTION

In this work, we examined the existence and uniqueness of solution for nonlinear initial value problems of fractional differential equations incorporating $\Psi-$ Caputo derivative. We employed both the Schauder fixed point theorem and Banach contraction theorem. Additionally we delved into the continuation theorem, To provide a tangible demonstration of our primary findings, we included an illustrative example.

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