ON QUASI TOTALLY *m*-CLASS A_k^* OPERATORS

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ABSTRACT. In this present paper, we will introduce a new class of operators that we call Quasi Totally m-Class A_k^* operator in Hilbert spaces. It is a generalization of some previous studies in the field of classes of operators, especially for this especially for a Quasi m-Class A_k^*. We will study some properties , provide example and discuss tensor product of this class of operators . example and discuss tensor product of this class of operators .

Key words : Class A_k^* operator; Quasi M-Class A_k^* operator ;Class A_k operator.

1. INTRODUCTION

The spectral properties of linear operators on Hilbert's space are one of the important tools in quantum mechanics. In fact, linear operators and quantum mechanics have interrelationships. The first goal of this article is to extend operator to a new class of operators and present some properties. In 1998, T.Furuta and m.Itho have defined the well- known class A operator as $(\Gamma^{å}\Gamma)^2 \leq \Gamma^{å2}\Gamma^2$ see [**1**].In 2011,Young min Han and Ju Hee Son have defined quasi-m-hyponormal as $\Gamma^{å}(m^2 |\Gamma-\eta|)\Gamma \geq \Gamma^{å} |(\Gamma-\eta)^{å}|\Gamma$ for all $\eta \in \mathbb{C}$ and some positive integer m and have studied some proprieties of this class see[**2**].In 2012,S. Panayappan, N. Jayanthi has defined class A_k operator as $|\Gamma|^2 \leq \left|\Gamma^{k+1}\right|^{\frac{2}{k+1}}$ for some positive integer k see[**6**]. In2013,S.Panayappan and Jayanthi introduced and some proprieties of the class A_k^* operator as $|\Gamma^{*}|^2 \leq \left|\Gamma^{k+1}\right|^{\frac{2}{k+1}}$ some positive integer k and studied some proprietes and tensor product of this class see[**7**].In 2019, P.Shanmugapriya and P.maheswari Naik has defined m-Class A_k^* operators as $|\Gamma^{*}|^2 \leq m\left|\Gamma^{k+1}\right|^{\frac{2}{k+1}}$ for some m and k are positives integers and studied some spectral properties and tensor Product of this class see[**10**].

Throughout this note we assume that H is an infinite dimensional separable Hilbert space. Let $B(H)$ know the algebra of bounded linear operators that act on H , If $\Gamma \in B(H)$ the nul space of Γ we will refer to it $N(\Gamma)$ and the range space of Γ we will refer to it $R(\Gamma)$.

Definition 1.1. [7] Let $\Gamma \in B(H)$.an operator Γ is said to be class A_k^* if there is a positive integer k such that

$$\left(\Gamma^{*k}\Gamma^{k}\right)^{\frac{1}{k}} \geq \Gamma\Gamma^{*} \qquad \text{i.e} \qquad \left|\Gamma^{k}\right|^{\frac{2}{k}} \geq \left|\Gamma^{*}\right|^{2}$$

Definition 1.2. Let $\Gamma \in B(H)$. an operator Γ is said to be Totally m-Class A_k^* if there are two positive integers k and m such that

$$m\left[\left(\Gamma^{k}-\eta\right)^{*}\left(\Gamma^{k}-\eta\right)\right]^{\frac{1}{k}} \geq (\Gamma-\eta)(\Gamma-\eta)^{*}; \qquad \forall \eta \in \mathbb{C}$$

$$\text{i.e:} \qquad m\left|\Gamma^{k}-\eta\right|^{\frac{2}{k}} \geq \left|(\Gamma-\eta)^{*}\right|^{2} \qquad \forall \eta \in \mathbb{C}$$

Definition 1.3. Let $\Gamma \in B(H)$. an operator Γ is said to be Quasi Totally m-Class A_k^* if there a re two positive integers k and m such that

$$\Gamma^{*}\left(m\left[\left(\Gamma^{k}-\eta\right)^{*}\left(\Gamma^{k}-\eta\right)\right]^{\frac{1}{k}}\right)\Gamma \geq \Gamma^{*}(\Gamma-\eta)(\Gamma-\eta)^{*}\Gamma \qquad \forall \eta \in \mathbb{C}$$

$$\text{i.e} \qquad \Gamma^{*}\left(m\left|\Gamma^{k}-\eta\right|^{\frac{2}{k}} - \left|(\Gamma-\eta)^{*}\right|^{2}\right)\Gamma \geq 0; \qquad \forall \eta \in \mathbb{C}$$

In particular (choose $\eta = 0$) an operator Γ is called Quasi m-Class A_k^* [**9**] if

$$\Gamma^*\left(m\left(\Gamma^{*k}\Gamma^k\right)^{\frac{1}{k}}\right)\Gamma \geq \Gamma^*\left(\Gamma\Gamma^*\right)\Gamma$$

In general, the following implication holds:

Hyponormal operator [**11,12**] $\Rightarrow$ Class A_k^* operator [**7,8**]

$\Rightarrow$ Totally m-Class A_k^* operator

$\Rightarrow$ Quasi Totally m-Class A_k^* operator.

Example1.4. Let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B\left(\mathbb{C}^2\right)$

Then

$$I_2^*\left(m\left|I_2^k - \eta\right|^{\frac{2}{k}} - \left|\left(I_2 - \eta\right)^*\right|^2\right)I_2 = \begin{pmatrix} m\left|1\text{-}\eta\right|^{\frac{2}{k}} - \left|1\text{-}\eta\right|^2 & 0 \\ 0 & m\left|1\text{-}\eta\right|^{\frac{2}{k}} - \left|1\text{-}\eta\right|^2 \end{pmatrix} \quad ;\forall \eta \in \mathbb{C}.$$

for $k = 1$ and for $m \geq 1$ then I_2 is Quasi Totally m-Class A_k^* operator.

2. MAIN RESULTS

The following example shows that Γ_1 and Γ_2 are a Quasi Totally m-Class A_k^* operator but the sum $\Gamma_1 + \Gamma_2$ isn't a Quasi Totally m-Class A_k^* operator.

Example2.1: Let $\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in B\left(\mathbb{C}^3\right)$ then

$$\Gamma_1^*\left(m\left|\Gamma_1^k - \eta\right|^{\frac{2}{k}} - \left|\left(\Gamma_1 - \eta\right)^*\right|^2\right)\Gamma_1 = \begin{pmatrix} m\left|1\text{-}\eta\right|^{\frac{2}{k}} - \left|1\text{-}\eta\right|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

for $m\left|1\text{-}\eta\right|^{\frac{2}{k}} \geq \left|1\text{-}\eta\right|^2$ then Γ_1 is a Quasi Totally m-Class A_k^* operator.

and let $\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in B\left(\mathbb{C}^3\right)$

Then

$$\Gamma_2^*\left(m\left|\Gamma_2^k-\eta\right|^{\frac{2}{k}}-\left|\left(\Gamma_2-\eta\right)^*\right|^2\right)\Gamma_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&m|\eta|^{\frac{2}{k}}-|\eta|^2-1\end{pmatrix};\quad \forall\eta\in\mathbb{C}.$$

for $m|\eta|^{\frac{2}{k}}\geq|\eta|^2+1$ then Γ_2 is Quasi Totally m-Class A_k^* operator.

and $\Gamma_1+\Gamma_2=\begin{pmatrix}1&0&0\\0&0&1\\0&0&0\end{pmatrix}\in B\left(\mathbb{C}^3\right)$ then

$$\left(\Gamma_1+\Gamma_2\right)^*\left(m\left|\left(\Gamma_1+\Gamma_2\right)^k-\eta\right|^{\frac{2}{k}}-\left|\left(\Gamma_1+\Gamma_2-\eta\right)^*\right|^2\right)\left(\Gamma_1+\Gamma_2\right)=\begin{pmatrix}m|1-\eta|^{\frac{2}{k}}-|1-\eta|^2&0&0\\0&0&0\\0&0&m|\eta|^{\frac{2}{k}}-|\eta|^2-1\end{pmatrix};\quad \forall\eta\in\mathbb{C}.$$

for $\eta=0$

$$\left(\Gamma_1+\Gamma_2\right)^*\left(m\left|\left(\Gamma_1+\Gamma_2\right)^k\right|^{\frac{2}{k}}-\left|\left(\Gamma_1+\Gamma_2\right)^*\right|^2\right)\left(\Gamma_1+\Gamma_2\right)=\begin{pmatrix}m-1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

so $\Gamma_1+\Gamma_2$ isn't Quasi Totally m-Class A_k^* operator.

Proposition 2.2. Let $\Gamma_1,\Gamma_2\in B(H)$ are Quasi Totally m-Class A_k^* operator such that $\Gamma_1^k\left(\Gamma_2-\eta\right)=\Gamma_2^k\left(\Gamma_1-\eta\right)=\left(\Gamma_2-\eta\right)^*\Gamma_1^k=\left(\Gamma_1-\eta\right)^*\Gamma_2^k=0$, for $k\in\mathbb{N}^*$ and for all $\eta\in\mathbb{C}$ then $\Gamma_1+\Gamma_2$ is Quasi Totally m-class A_k^* operator.

Proof. Let $\Gamma_1,\Gamma_2\in B(H)$ are Quasi Totally m- Class A_k^* operator then

$$\Gamma_1^*\left(m\left|\Gamma_1^k-\eta\right|^{\frac{2}{k}}-\left|\left(\Gamma_1-\eta\right)^*\right|^2\right)\Gamma_1\geq0 \text{ and } \Gamma_2^*\left(m\left|\Gamma_2^k-\eta\right|^{\frac{2}{k}}-\left|\left(\Gamma_2-\eta\right)^*\right|^2\right)\Gamma_2\geq0;\quad \forall\eta\in\mathbb{C}$$

so

$$(\Gamma_1+\Gamma_2)^*\left(m'\left((\Gamma_1+\Gamma_2)^k-\eta'\right)^{\frac{2}{k}}-\left|\left(\Gamma_1+\Gamma_2-\eta'\right)^*\right|^2\right)(\Gamma_1+\Gamma_2)$$

$$=(\Gamma_1+\Gamma_2)^*\left(m'\left|\Gamma_1^k+\Gamma_2^k+\sum_{i=1}^{k-1}\binom{i}{k-1}\Gamma_1^{k-1}\Gamma_2^{k-1-i}-\eta-\eta\right|^{\frac{2}{k}}-\left|\Gamma_1^*+\Gamma_2^*-\overline{\eta}-\overline{\eta}\right|^2\right)(\Gamma_1+\Gamma_2)$$

$$=(\Gamma_1+\Gamma_2)^*\left(m'\left[\left|\Gamma_1^k-\eta\right|^2+\left|\Gamma_2^k-\eta\right|^2\right]^{\frac{1}{k}}-\left|\Gamma_1^*-\overline{\eta}\right|^2-\left|\Gamma_2^*-\overline{\eta}\right|^2\right)(\Gamma_1+\Gamma_2)$$

$$=(\Gamma_1+\Gamma_2)^*\left(m\left|\Gamma_1^k-\eta\right|^{\frac{2}{k}}+m\left|\Gamma_2^k-\eta\right|^{\frac{2}{k}}-\left|\Gamma_1^*-\overline{\eta}\right|^2-\left|\Gamma_2^*-\overline{\eta}\right|^2\right)(\Gamma_1+\Gamma_2)$$

$$=\Gamma_1^*\left(m\left|\Gamma_1^k-\eta\right|^{\frac{2}{k}}-\left|\Gamma_1^*-\overline{\eta}\right|^2\right)\Gamma_1+\Gamma_2^*\left(m\left|\Gamma_2^k-\eta\right|^{\frac{2}{k}}-\left|\Gamma_2^*-\overline{\eta}\right|^2\right)\Gamma_2$$

$$\geq 0.$$

Then $\Gamma_1+\Gamma_2$ is Quasi Totally m-Class A_k^* operator.

Proposition 2.3. Let $\Gamma_1,\Gamma_2\in B(H)$. If Γ_2 is a Quasi Totally m- Class A_k^* operator and Γ_1 is unitary equivalent to Γ_2, then Γ_1 is Quasi Totally m-Class A_k^* operator.

Proof. Γ_2 is Quasi Totally m- Class A_k^* operator then

$$\Gamma_2^*\left(m\left|\Gamma_2^k-\eta\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta)^*\right|^2\right)\Gamma_2\geq 0 \qquad \Rightarrow\left(U\Gamma_1U^*\right)^*\left(m\left|\left(U\Gamma_1U^*\right)^k-\eta\right|^{\frac{2}{k}}-\left|(U\Gamma_1U^*-\eta)^*\right|^2\right)U\Gamma_1U^*\geq 0$$

$$\Rightarrow U\Gamma_1U^*\left(m\left|U\left(\Gamma_1\right)^kU^*-\eta UU^*\right|^{\frac{2}{k}}-\left|(U\Gamma_1U^*-\eta UU^*)^*\right|^2\right)U\Gamma_1U^*\geq 0$$

$$\Rightarrow U\Gamma_1U^*\left(mU\left|\left(\Gamma_1\right)^k-\eta\right|^{\frac{2}{k}}U^*-U\left|(\Gamma_1-\eta)^*\right|^2U^*\right)U\Gamma_1U^*\geq 0$$

$$\Rightarrow U\Gamma_1\left(m\left|\left(\Gamma_1\right)^k-\eta\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta)^*\right|^2\right)\Gamma_1U^*\geq 0$$

$$\Rightarrow \Gamma_1\left(m\left|\left(\Gamma_1\right)^k-\eta\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta)^*\right|^2\right)\Gamma_1\geq 0; \qquad \forall\eta\in\mathbb{C}$$

Then Γ_1 is a Quasi Totally m- Class A_k^* operator.

The following example shows that Γ_1 and Γ_2 are Quasi m- Class A_k^* operator but the product $\Gamma_1\Gamma_2$ isn't Quasi m- Class A_k^* operator.

Example2.4.

Let $\Gamma_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \in B(\mathbb{C}^3)$

So

$$\Gamma_1^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_1 = \begin{pmatrix} 8m-8 & 0 & -8m+8 \\ 0 & 0 & 0 \\ -8m+8 & 0 & 8m-8 \end{pmatrix}$$

for all $m \geq 1$. Then

$$\Gamma_1^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_1 = \begin{pmatrix} 8m-8 & 0 & -8m+8 \\ 0 & 0 & 0 \\ -8m+8 & 0 & 8m-8 \end{pmatrix} \geq 0$$

Hence Γ_1 is Quasi m-Class A_k^* operator.

and let $\Gamma_1 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in B(\mathbb{C}^3)$

So

$$\Gamma_2^*\left(m\left|\Gamma_2^k\right|^{\frac{2}{k}} - \left|\Gamma_2^*\right|^2\right)\Gamma_2 = \begin{pmatrix} \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \\ 0 & 0 & 0 \\ \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \end{pmatrix}$$

for $m2^{2/k}-16 \geq 0$.

Then

$$\Gamma_2^*\left(m\left|\Gamma_2^k\right|^{\frac{2}{k}} - \left|\Gamma_2^*\right|^2\right)\Gamma_2 = \begin{pmatrix} \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \\ 0 & 0 & 0 \\ \frac{m2^{2/k}-16}{2} & 0 & \frac{m2^{2/k}-16}{2} \end{pmatrix} \geq 0$$

Hence Γ_2 is Quasi m-Class A_k^* operator.

And $\Gamma_1\Gamma_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in B(\mathbb{C}^3)$

$$\left(\Gamma_1\Gamma_2\right)^*\left(m\left|\left(\Gamma_1\Gamma_2\right)^k\right|^{\frac{2}{k}} - \left|\left(\Gamma_1\Gamma_2\right)^*\right|^2\right)\left(\Gamma_1\Gamma_2\right) = \begin{pmatrix} -8 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & -8 \end{pmatrix} \geq 0.$$

Then $\Gamma_1\Gamma_2$ is not Quasi m-Class A_k^* operator.

We say that an operator Γ_2 doubly commutes with Γ_1 if Γ_2 commutes with Γ_1 and Γ_1^* .

Proposition 2.5. Let $\Gamma_1, \Gamma_2 \in B(H)$ are doubly commutes. If Γ_2 is normal operator and Γ_1 is Quasi m-Class A_k^* operator, then $\Gamma_1\Gamma_2$ is Quasi m-Class A_k^* operator.

Proof. Let Γ_1 is a Quasi m-Class A_k^* operator and Γ_1 is a normal operator such that $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$ and $\Gamma_1\Gamma_2^* = \Gamma_2^*\Gamma_1$ .we have that

$$\Gamma_1^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_1 \geq 0 \quad \text{and} \quad \Gamma_2\Gamma_2^* = \Gamma_2^*\Gamma_2$$

and so

$$\begin{aligned}
\langle(\Gamma_1\Gamma_2)^*\left(m\left|(\Gamma_1\Gamma_2)^k\right|^{\frac{2}{k}} - \left|(\Gamma_1\Gamma_2)^*\right|^2\right)(\Gamma_1\Gamma_2)\mu;\mu\rangle &= \langle\Gamma_1^*\Gamma_2^*\left(m\left|\Gamma_1^k\Gamma_2^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\Gamma_2^*\right|^2\right)(\Gamma_1\Gamma_2)\mu;\mu\rangle \\
&= \langle\Gamma_1^*\Gamma_2^*\left(m\left(\Gamma_2^{*k}\right)^{\frac{1}{k}}\left|\Gamma_1^k\right|^{\frac{2}{k}}\left(\Gamma_2^k\right)^{\frac{1}{k}} - \Gamma_2^*\left|\Gamma_1^*\right|^2\Gamma_2\right)(\Gamma_1\Gamma_2)\mu;\mu\rangle \\
&= \langle\Gamma_1^*\Gamma_2^*\Gamma_2^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_2\Gamma_1\Gamma_2\mu;\mu\rangle \\
&= \langle\Gamma_2^{*2}\Gamma_1^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_1\Gamma_2^2\mu;\mu\rangle \\
&= \langle\Gamma_1^*\left(m\left|\Gamma_1^k\right|^{\frac{2}{k}} - \left|\Gamma_1^*\right|^2\right)\Gamma_1\Gamma_2^2\mu;\Gamma_2^2\mu\rangle \geq 0
\end{aligned}$$

then $\Gamma_1\Gamma_2$ is Quasi m-Class A_k^* operator.

Proposition 2.6. If Γ is Quasi m-Class A_k^* operator and Γ is normal operator, then Γ^* is Quasi m-Class A_k^* operator.

Proof. Γ is Quasi m-Class A_k^* operator and Γ is normal operator then

$$\begin{aligned}\Gamma\left(m\left(\Gamma^k\Gamma^{*k}\right)^{\frac{1}{k}}-\left(\Gamma^*\Gamma\right)^2\right)\Gamma^* &= \Gamma\left(m\left(\Gamma^{*k}\Gamma^k\right)^{\frac{1}{k}}-\left(\Gamma\Gamma^*\right)^2\right)\Gamma^* \\ &= \Gamma^*\left(m\left(\Gamma^{*k}\Gamma^k\right)^{\frac{1}{k}}-\left(\Gamma\Gamma^*\right)^2\right)\Gamma \\ &\geq 0\end{aligned}$$

Then Γ^* is Quasi m-Class A_k^* operator.

Lemme.2.7: [**5**] (Holder-mcCarthy's inequality) Let $\Gamma \geq 0$. Then

1. $\langle \Gamma^r x, x\rangle \leq \langle \Gamma x, x^r\rangle \|x\|^{2(1\text{-}r)}$, for $r > 1$ and all $x \in H$

2. $\langle \Gamma^r x, x\rangle \leq \langle \Gamma x, x\rangle^r \|x\|^{2(1\text{-}r)}$, for $0 \leq r \leq 1$ and all $x \in H$

Proposition 2.8. if Γ is Totally m- class A_k^* operator .then

$$m\left\|\left(\Gamma^k-\eta\right)\mu\right\|^{\frac{1}{k}}\|\mu\|^{1\text{-}\frac{1}{k}} \geq \left\|\left(\Gamma-\eta\right)^*\mu\right\|; \qquad \forall \eta \in \mathbb{C}, \forall \mu \in H.$$

Proof. Suppose that Γ is Totally m-Class A_k^*. we have

$$m\left(\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right)^{\frac{1}{k}}-\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^* \geq 0; \qquad \forall \eta \in \mathbb{C}.$$

Let $\mu \in H$. then

$$\begin{aligned}& m\left\|\left(\Gamma^k-\eta\right)\mu\right\|^{\frac{2}{k}}\|\mu\|^{2\text{-}\frac{2}{k}}-\left\|\left(\Gamma-\eta\right)^*\mu\right\|^2 \\ &= m\langle\left(\Gamma^k-\eta\right)\mu,\left(\Gamma^k-\eta\right)\mu\rangle^{\frac{2}{k}}\|\mu\|^{2\text{-}\frac{2}{k}}-\langle\left(\Gamma-\eta\right)^*\mu,\left(\Gamma-\eta\right)^*\mu\rangle \\ &= m\langle\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\mu,\mu\rangle^{\frac{1}{k}}\|\mu\|^{2\text{-}\frac{2}{k}}-\langle\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^*\mu,\mu\rangle \\ &\geq \langle m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}\mu,\mu\rangle-\langle\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^*\mu,\mu\rangle \\ &= \langle\left(m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}-\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^*\right)\mu,\mu\rangle \\ &\geq 0\end{aligned}$$

Therefore

$$m'\left\|\left(\Gamma^k-\eta\right)\mu\right\|^{\frac{1}{k}}\|\mu\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\mu\right\|;\qquad \forall\eta\in\mathbb{C},\forall\mu\in\mathbb{C}.$$

Proposition 2.9. if Γ is Quasi Totally m- class A_k^* operator .then

$$m\left\|\left(\Gamma^k-\eta\right)\Gamma\mu\right\|^{\frac{1}{k}}\|\Gamma\mu\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\Gamma\mu\right\|;\qquad \forall\eta\in\mathbb{C},\forall\mu\in H$$

Proof. Suppose that Γ is Quasi Totally m-Class A_k^*. we have

$$\Gamma^*\left(m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}\right)\Gamma\geq\Gamma^*\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^*\Gamma$$

Let $\mu\in H$. Then

$$\begin{aligned}
m\left\|\left(\Gamma^k-\eta\right)\Gamma\mu\right\|^{\frac{2}{k}}\|\Gamma\mu\|^{2-\frac{2}{k}} &= m\langle\left(\Gamma^k-\eta\right)\Gamma\mu,\left(\Gamma^k-\eta\right)\Gamma\mu\rangle^{\frac{1}{k}}\|\Gamma\mu\|^{2-\frac{2}{k}}\\
&= m\langle\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\Gamma\mu,\Gamma\mu\rangle^{\frac{1}{k}}\|\Gamma\mu\|^{2-\frac{2}{k}}\\
&\geq m\langle\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}\Gamma\mu,\Gamma\mu\rangle\\
&\geq\langle\Gamma^*m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}\Gamma\mu,\mu\rangle\\
&\geq\langle\Gamma^*\left(\Gamma^k-\eta\right)\left(\Gamma^k-\eta\right)^*\Gamma\mu,\mu\rangle\\
&=\left\|\left(\Gamma-\eta\right)^*\Gamma\mu\right\|^2.
\end{aligned}$$

Therefore

$$m'\left\|\left(\Gamma^k-\eta\right)\Gamma\mu\right\|^{\frac{1}{k}}\|\Gamma\mu\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\Gamma\mu\right\|;\qquad \forall\eta\in\mathbb{C},\forall\mu\in H$$

Proposition 2.10. Let Γ is a Quasi Totally m-Class A_k^*. then $N\left(\Gamma-\alpha\right)\subset N\left(\left(\Gamma-\alpha^k\right)^*\right)$ for each $\alpha\neq0$.

Proof. Suppose Γ is Quasi Totally m-Class A_k^* .it follows from proposition **2.9** that

$$m\left\|\left(\Gamma^k-\eta\right)\Gamma\mu\right\|^{\frac{1}{k}}\|\Gamma\mu\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\Gamma\mu\right\|;\qquad \forall\eta\in\mathbb{C},\forall\mu\in H$$

and we have $\mu\in N\left(\Gamma-\alpha\right)$ then $\Gamma\mu=\alpha\mu$. In particular,

$$m\left\|\left(\Gamma^k-\alpha^k\right)\Gamma\mu\right\|^{\frac{1}{k}}\left\|\Gamma\mu\right\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\alpha^k\right)^*\Gamma\mu\right\|$$

Since

$$0\geq\left\|\left(\Gamma-\alpha^k\right)^*\Gamma\mu\right\|.$$

and we have $\alpha\neq0$, then $\left\|\left(\Gamma-\alpha^k\right)^*\mu\right\|=0$.

Therefore $\mu\in N\left(\left(\Gamma-\alpha^k\right)^*\right)$.

Hence $N\left(\Gamma-\alpha\right)\subset N\left(\Gamma-\alpha^k\right)^*$ for each $\alpha\neq0$.

Proposition 2.11. Suppose Γ is Quasi Totally m-Class A_k^* and it has dense range. then Γ is Totally m-Class A_k^*.

Proof. Let Γ is Quasi Totally m-Class A_k^* then

$$m\left\|\left(\Gamma^k-\eta\right)\Gamma\mu\right\|^{\frac{1}{k}}\left\|\Gamma\mu\right\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\Gamma\mu\right\|;\qquad\forall\eta\in\mathbb{C},\forall\mu\in H$$

Since Γ has dense range ,then $\overline{\Gamma\left(H\right)}=H$.

Let $\mu\in H$. Then there exists a sequence $\left(x_n\right)$ in H such that $\Gamma\left(x_n\right)\to\mu$ as $n\to\infty$.

In particular,

$$m\left\|\left(\Gamma^k-\eta\right)\Gamma x_n\right\|^{\frac{1}{k}}\left\|\Gamma\mu\right\|^{1-\frac{1}{k}}\geq\left\|\left(\Gamma-\eta\right)^*\Gamma x_n\right\|$$

Therefore

$$\begin{aligned}m\left\|\left(\Gamma^k-\eta\right)\mu\right\|^{\frac{1}{k}}\left\|\mu\right\|^{1-\frac{1}{k}} &= m\left\|\lim_{n\to\infty}\left(\Gamma^k-\eta\right)\Gamma x_n\right\|^{\frac{1}{k}}\left\|\lim_{n\to\infty}\Gamma x_n\right\|^{1-\frac{1}{k}}\\ &=\lim_{n\to\infty}m\left\|\left(\Gamma^k-\eta\right)\Gamma x_n\right\|^{\frac{1}{k}}\left\|\Gamma x_n\right\|^{1-\frac{1}{k}}\\ &\geq\lim_{n\to\infty}\left\|\left(\Gamma-\eta\right)^*\Gamma x_n\right\|\\ &=\left\|\lim_{n\to\infty}\left(\Gamma-\eta\right)^*\Gamma x_n\right\|\\ &=\left\|\left(\Gamma-\eta\right)^*\mu\right\|\end{aligned}$$

Hence Γ is Totally m-Class A_k^*

Theorem 2.12. *Let $\Gamma \in B(H)$ such that $H \neq \overline{R(\Gamma)}$,*
if Γ is a Quasi Totally m-Class A_k^* operator then $\Gamma_1 = \Gamma_{\overline{R(\Gamma)}}$ is a Totally m -Class A_k^* operator and $\Gamma_3 = 0$.

Proof. Let $\Gamma \in B(H)$ such that $H \neq \overline{R(\Gamma)}$.
the matrix representation of Γ such that $\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ 0 & \Gamma_3 \end{pmatrix}$ on $H = \overline{R(\Gamma)} \oplus N(\Gamma^*)$ see[**3**]
and let $P_{\overline{R(\Gamma)}}$ be the projection onto $\overline{R(\Gamma)}$.
Then $\begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} = \Gamma P_{\overline{R(\Gamma)}} = P_{\overline{R(\Gamma)}} \Gamma P_{\overline{R(\Gamma)}}$.
Since Γ is an Quasi Totally m -Class A_k^* operator, we have

$$\Gamma^* \left(m\left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) \right]^{\frac{1}{k}} \right) \Gamma \geq \Gamma^* \left((\Gamma - \eta)(\Gamma - \eta)^* \right) \Gamma$$

then

$$P_{\overline{R(\Gamma)}} \left(m\left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) \right]^{\frac{1}{k}} \right) P_{\overline{R(\Gamma)}} \geq P_{\overline{R(\Gamma)}} \left((\Gamma - \eta)(\Gamma - \eta)^* \right) P_{\overline{R(\Gamma)}}$$

Therefore

$$\begin{aligned} P_{\overline{R(\Gamma)}} \left(m\left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) \right]^{\frac{1}{k}} \right) P_{\overline{R(\Gamma)}} &\leq m\left[P_{\overline{R(\Gamma)}} \left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) P_{\overline{R(\Gamma)}} \right]^{\frac{1}{k}} \\ &= m\begin{pmatrix} \left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) \right]^{\frac{1}{k}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

On other hand

$$P_{\overline{R(\Gamma)}} \left((\Gamma - \eta)(\Gamma - \eta)^* \right) P_{\overline{R(\Gamma)}} = \begin{pmatrix} (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* + \Gamma_2 \Gamma_2^* & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$m\left[\left(\Gamma^k - \eta\right)^* \left(\Gamma^k - \eta\right) \right]^{\frac{1}{k}} \geq (\Gamma_1 - \eta)(\Gamma_1 - \eta)^* + \Gamma_2 \Gamma_2^*$$

Then

$$m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}} \geq \left(\Gamma_1-\eta\right)\left(\Gamma_1-\eta\right)^*$$

Hence Γ_1 is a Totally m-Class A_k^*.

In addition, let $\mu=\mu_1+\mu_2 \in H=\overline{R(\Gamma)} \oplus N\left(\Gamma^*\right)$. a simple computation shows that

$$\begin{aligned}\langle \Gamma_3\mu_2;\mu_2\rangle &= \langle \Gamma(I-P_{\overline{R(\Gamma)}})\mu;(I-P_{\overline{R(\Gamma)}})\mu\rangle \\ &= \langle (I-P_{\overline{R(\Gamma)}})\mu;\Gamma^*(I-P_{\overline{R(\Gamma)}})\mu\rangle \\ &= 0\end{aligned}$$

So, $\Gamma_3=0$.

Theorem 2.13. *Let Γ is Quasi Totally m-Class A_k^* such that $H \neq \overline{R(\Gamma)}$. Then*

$$\Gamma=\begin{pmatrix}\Gamma_1 & \Gamma_2 \\ 0 & 0\end{pmatrix} \quad \text{on} \quad H=\overline{\Gamma(H)} \oplus N\left(\Gamma^*\right)$$

where $\Gamma_1=\Gamma_{\overline{\Gamma(H)}}$ is a Totally m-Class A_k^* operator.

Proof. Let Γ is Quasi Totally m-Class A_k^* then

$$\Gamma^*\left(m\left[\left(\Gamma^k-\eta\right)^*\left(\Gamma^k-\eta\right)\right]^{\frac{1}{k}}-\left(\Gamma-\eta\right)\left(\Gamma-\eta\right)^*\right)\Gamma \geq 0 \qquad \forall \eta \in \mathbb{C}$$

And Γ doesn't have dense range. we can represent Γ as the 2×2 operator matrix as follows:

$$\Gamma=\begin{pmatrix}\Gamma_1 & \Gamma_2 \\ 0 & 0\end{pmatrix} \quad \text{on} \quad H=\overline{\Gamma(H)} \oplus N\left(\Gamma^*\right)$$

Therefore

$$\Gamma^*\left(m\left[\left(\Gamma_1^k-\eta\right)^*\left(\Gamma_1^k-\eta\right)\right]^{\frac{1}{k}}-\left(\Gamma_1-\eta\right)\left(\Gamma_1-\eta\right)^*\right)\Gamma \geq 0 \qquad \forall \eta \in \mathbb{C}$$

Hence

$$\left\langle\left(m\left[\left(\Gamma_1^k-\eta\right)^*\left(\Gamma_1^k-\eta\right)\right]^{\frac{1}{k}}-\left(\Gamma_1-\eta\right)\left(\Gamma_1-\eta\right)^*\right)\Gamma\mu;\Gamma\mu\right\rangle \geq 0\,; \qquad \forall \eta \in \mathbb{C}, \forall \mu \in H$$

So

$$\langle\left(m\left[\left(\Gamma_1^{\,k}-\eta\right)^*\left(\Gamma_1^{\,k}-\eta\right)\right]^{\frac{1}{k}}-\left(\Gamma_1-\eta\right)\left(\Gamma_1-\eta\right)^*\right)\mathrm{u};\mathrm{u}\rangle\geq 0 \;;\qquad \forall\eta\in\mathbb{C},\forall\mu\in H.$$

Then Γ_1 is a Totally m-Class A_k^* operator.

3.Tensor Product of Quasi Totally m-Class A_k^* operators

The tensor product of operators is a mathematical operation that combines two operators into a single operator acting on a tensor product space. If you have two operators Γ_1 and Γ_2 acting on spaces H and K respectively, the tensor product operator $\Gamma_1\otimes\Gamma_2$ acts on the tensor product space $H\otimes K$ see **[4,6]** *.It is defined as:*

$$\left(\Gamma_1\otimes\Gamma_2\right)\left(\mu_1\otimes\mu_2\right)=\left(\Gamma_1\mu_1\right)\otimes\left(\Gamma_2\mu_2\right)$$

These are some of the properties we need in the following theorem

1. $\left(\Gamma_1\otimes\Gamma_2\right)\left(\Gamma_1\otimes\Gamma_2\right)^*=\left(\Gamma_1\Gamma^*{}_1\right)\otimes\left(\Gamma_2\Gamma^*{}_2\right)$
2. $\left|\Gamma_1\otimes\Gamma_2\right|^\rho=\left|\Gamma_1\right|^\rho\otimes\left|\Gamma_2\right|^\rho \qquad \forall\rho\in\mathbb{R}^+.$

In the sequel, we present an important result related to the tensor product of elements of the considered class of operators

Theorem 3.1. *Let $\Gamma_1\in B(H)$ and $\Gamma_2\in B(K)$ such that $\Gamma_1\otimes I_K=-I_H\otimes\Gamma_2$ if Γ_1 and Γ_2 are a Quasi Totally m-Class A_k^* operator then $\Gamma_1\otimes\Gamma_2$ is a Quasi Totally m-Class A_k^**

Proof. Let $\Gamma_1\in B(H)$ and $\Gamma_2\in B(K)$.

let Γ_1 and Γ_2 are a Quasi Totally m-Class A_k^* operator. we have

$$\langle\Gamma_1^*\left(m\left|\left(\Gamma_1^k-\eta I_H\right)\right|^{\frac{2}{k}}-\left|\left(\Gamma_1-\eta I_H\right)^*\right|^2\right)\Gamma_1\mu;\mu\rangle\geq 0;\qquad\forall\eta\in\mathbb{C},\forall\mu\in H$$

And

$$\langle\Gamma_2^*\left(m\left|\left(\Gamma_2^k-\eta I_H\right)\right|^{\frac{2}{k}}-\left|\left(\Gamma_2-\eta I_H\right)^*\right|^2\right)\Gamma_2\mu;\mu\rangle\geq 0;\qquad\forall\eta\in\mathbb{C},\forall\mu\in H$$

Then

$$(\Gamma_1\otimes\Gamma_2)^*\left(m\left|(\Gamma_1\otimes\Gamma_2)^k-\eta'\right|^{\frac{2}{k}}-\left|\left((\Gamma_1\otimes\Gamma_2)-\eta'\right)^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=(\Gamma_1\otimes\Gamma_2)^*\left(m\left|\Gamma_1^{\,k}\otimes\Gamma_2^{\,k}-\eta\Gamma_1^{\,k}\otimes I_k-\eta I_H\otimes\Gamma_2^{\,k}+\eta^2 I_H\otimes I_K\right|^{\frac{2}{k}}\right.$$

$$\left.-\left|\left(\Gamma_1\otimes\Gamma_2-\eta\Gamma_1\otimes I_K-\eta I_H\otimes\Gamma_2+\eta^2 I_H\otimes I_K\right)^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=(\Gamma_1\otimes\Gamma_2)^*\left(m\left|\Gamma_1^{\,k}\otimes(\Gamma_2^{\,k}-\eta I_K)-\eta I_H\otimes(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|\Gamma_1^{\,*}\otimes(\Gamma_2^{\,*}-\eta I_K)-\eta I_H\otimes(\Gamma_2^{\,*}-\eta I_K)\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=(\Gamma_1\otimes\Gamma_2)^*\left(m'\left|(\Gamma_1^{\,k}-\eta I_H)\otimes(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\otimes(\Gamma_2-\eta I_K)^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=\left(\Gamma_1^{\,*}\left[m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\right]\otimes\Gamma_2^{\,*}\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}\right.$$

$$\left.+\Gamma_1^{\,*}\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\Gamma_2^{\,*}\left[m\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta I_K)^*\right|^2\right]\right)(\Gamma_1\otimes\Gamma_2)$$

$$=\Gamma_1^{\,*}\left[m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\right]\Gamma_1\otimes\Gamma_2^{\,*}\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}\Gamma_2$$

$$+\Gamma_1^{\,*}\left|(\Gamma_1-\eta I_H)^*\right|^2\Gamma_1\otimes\Gamma_2^{\,*}\left[m\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta I_K)^*\right|^2\right]\Gamma_2$$

$$=(\Gamma_1\otimes\Gamma_2)^*\left(m'\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}\otimes\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\left|(\Gamma_2-\eta I_K)^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=(\Gamma_1^{\,*}\otimes\Gamma_2^{\,*})\left(m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}\otimes\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}\right.$$

$$\left.+\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\left|(\Gamma_2-\eta I_K)^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)$$

$$=(\Gamma_1^{\,*}\otimes\Gamma_2^{\,*})\left(\left[m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\right]\otimes\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}\right.$$

$$\left.+\left|(\Gamma_1-\eta I_H)^*\right|^2\otimes\left[m\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta I_K)^*\right|^2\right]\right)(\Gamma_1\otimes\Gamma_2)$$

Therefore for ever $\mu\in H$ and $u\in K$

$$\langle\Gamma_1^{\,*}\left[m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\right]\Gamma_1\mu;\mu\rangle\langle\Gamma_2^{\,*}\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}\Gamma_2\,\mathrm{u};\mathrm{u}\rangle$$

$$+\langle\Gamma_1^{\,*}\left|(\Gamma_1-\eta I_H)^*\right|^2\Gamma_1\mu;\mu\rangle\langle\Gamma_2^{\,*}\left[m\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta I_K)^*\right|^2\right]\Gamma_2\,\mathrm{u};\mathrm{u}\rangle$$

$$=\langle\Gamma_1^{\,*}\left[m\left|(\Gamma_1^{\,k}-\eta I_H)\right|^{\frac{2}{k}}-\left|(\Gamma_1-\eta I_H)^*\right|^2\right]\Gamma_1\mu;\mu\rangle\left\|(\Gamma_2^{\,k}-\eta I_K)^{\frac{1}{k}}\Gamma_2 u\right\|$$

$$+\left\|(\Gamma_1-\eta I_H)^*\Gamma_1\mu\right\|\langle\Gamma_2^{\,*}\left[m\left|(\Gamma_2^{\,k}-\eta I_K)\right|^{\frac{2}{k}}-\left|(\Gamma_2-\eta I_K)^*\right|^2\right]\Gamma_2\,\mathrm{u};\mathrm{u}\rangle$$

$$\geq 0$$

Then

$$(\Gamma_1\otimes\Gamma_2)^*\left(m\left|(\Gamma_1\otimes\Gamma_2)^k-\eta'\right|^{\frac{2}{k}}-\left|(\Gamma_1\otimes\Gamma_2-\eta')^*\right|^2\right)(\Gamma_1\otimes\Gamma_2)\geq 0$$

Hence $\Gamma_1\otimes\Gamma_2$ is a Quasi Totally m-Class A_k^* operator.

REFERENCES

1. T. Furuta, M. Ito and T. Yamazaki. ”A subclass of paranormal operators including class of log-hyponormal and several related classes.” Sci. math 1.3 (1998): 389-403.

2. Y. Han and J.H. Son. ” On quasi-M-hyponormal operators ”, Filomat, 25(1) (2011), 37-52

3. J. Han, H. Lee and W.Y. Lee. ”Invertible completions of 2 × 2 upper triangular operator matrices.” Proc. Amer. Math. Soc., 128.1 (2000): 119-123.
4. H. Jinchuan. ”On the tensor products of operators.” Acta Math. Sinica, 9(2) (1993): 195-202
5. C.A. McCarthy. ”Cp”, Isr. J. Math. 5 (1967): 249-271.
6. S. Panayappan, N. Jayanthi and D. Sumathi. ”Weyl’s theorem and tensor product for Class A_k operators.” Pure Mathematical Sciences 1.1 (2012): 13-23.
7. S. Panayappan and N. Jayanthi. ”Weyl and Weyl type theorems for class A_k^* k and quasi class A_k^* operators.” Int. J. math. Anal.(Russia) 7 (2013): 683-698.
8. P. Shanmugapriya and P.M. Naik. “A study on Spectral properties of extended class A_k^* operator on a complex Hilbert Space” in National Conference “COMIST 19”,during 15.02.2019-16.02.2019 at Bannari Amman Institute of Technology, Sathyamangalam
9. P. Shanmugapriya, P.M. Naik and A. Benali. ”Kronecker Product and spectral properties of Quasi M-Class A_k^* Operators”, Theory , Quantum PhysicsarXiv:quant-ph/0104019v1.s
10. P. Shanmugapriya and P.M. Naik. ”Spectral properties of m- class A_k^* operator”, Int. J. Math. Appl., 7 (2) (2019), 171–176.
11. I.H. Sheth. ” Quasi-hyponormal operators”, Rev. Roumaine Math. Pures Appl.19 (1974), 1049–1053.
12. J.G. Stampfli. ”Hyponormal operators.” (1962): 1453-1458.