IMPROVED RATIO-TYPE ESTIMATORS FOR POPULATION MEAN UNDER SIMPLE RANDOM SAMPLING

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ABSTRACT
In this study, sample size and known population parameters were used to enhance the estimation of the population mean for the main variable under investigation. A new modified generalized ratio-type estimator for the population mean, along with its variants, were proposed, and their efficiencies were evaluated using two real data sets. The large-sample properties, biases, and mean squared errors (MSE) of the newly proposed estimators were approximated to the first order using Taylor's series of expansion. The optimal values of the scalar parameters that minimize MSE were identified, and the corresponding minimum MSEs were computed. A theoretical comparison between the proposed estimators and existing estimators of the population mea-n using auxiliary information was conducted. The conditions under which the proposed estimators outperform existing ones were established. Additionally, a numerical analysis was performed to assess the performance of the proposed estimators against other related estimators, confirming the conditions under which the new estimators show superiority. The results demonstrated that the proposed estimators exhibit lower bias, MSE, as well as higher percent relative efficiency, making them preferable for practical applications.
Keywords: Estimation, Bias, Mean squared error, Efficiency, Sample size.

1. INTRODUCTION

Sampling theory plays a vital role in statistics, underpinning a wide range of practical applications across various fields. It involves selecting a subset of a population, studying specific properties of that subset, and using the results to infer characteristics of the entire population. Though sampling is an intuitive concept, frequently applied in everyday situations such as assessing the quality of food or products, its theoretical underpinnings are far more complex. The primary objective of sampling is to make accurate inferences about the population, while saving time and resources compared to a full census. In the context of statistical research, the goal is to estimate population parameters (such as the mean, variance, or median) using data from a sample. These sample-derived measures, known as statistics, serve as proxies for the population parameters, and their accuracy depends on the choice of sampling method. Simple random sampling without replacement (SRSWOR) is commonly employed for estimating the population mean, as it is unbiased and produces estimates with minimal variance. An important aspect of sampling theory is the use of auxiliary information (additional data) related to the population that can improve the precision of estimators. Auxiliary variables, such as demographic or economic data, can enhance the accuracy of estimates when incorporated at various stages of sampling design and estimation. For example, when estimating average income, data on housing size or car ownership might serve as auxiliary variables. Incorporating this information not only reduces cost and effort but also helps in minimizing errors. Several techniques leverage auxiliary information to increase the precision of estimators. Among the most commonly used are the ratio, product, and regression estimation methods. These techniques optimize the use of auxiliary variables to enhance the efficiency of the estimators, reducing the sampling error, although such errors cannot be entirely eliminated. Over time, researchers have developed more sophisticated estimators that maximize efficiency and precision. The aim of this paper is to develop some estimators for estimation of a population mean using information from the auxiliary variable(s) and known value from some population parameters under simple random sampling without a replacement sampling scheme. (Singh, 2003)

2. LITERATURE REVIEW OF EXISTING ESTIMATORS

Consider a problem of estimation population mean ($\bar{Y}$) from a given population. Suppose that the population to be investigated (studied) consist of N units. Assuming that no information other than the study variable is known (available at hand), we begin by selecting a sample of size n from the population using *SRSWOR* sampling scheme. Let $y = \{y_1, y_2, y_3, ..., y_n\}$ be the values of the individual unit selected and $Y = \{Y_1, Y_2, Y_3, ..., Y_N\}$ be the values of the individual unit in the population. To estimate the population mean we use the sample mean as an estimator given by;

$$t_0 = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

The estimator in (1) is unbiased and has minimum variance. Its variance is;

$$V(t_0) = \theta \bar{Y}^2 C_y^2 \tag{2}$$

Where; $\theta = \frac{1-f}{n}$, $f = \frac{n}{N}$ and $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$

The ratio method of estimation is used to estimate the population mean ($\bar{Y}$) of the study variable (Y) when there is a high positive correlation between the study variable (Y) and the auxiliary variable (X). Cochran (1940) was first to give his contribution by defining a ratio estimator for estimation of the population mean given as;

$$t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) = R\bar{X} \tag{3}$$

The estimator in (3) is biased but has small mean squared error compared to the usual sample mean (t_0). The correct expression for the bias and mean squared error up to the first order of approximations were;

$$B(t_R) = \theta \bar{Y}\left[C_x^2 - \rho C_x C_y\right] \tag{4}$$

$$MSE(t_R) = \theta \bar{Y}^2\left[C_y^2 + C_x^2 - 2\rho C_x C_y\right] \tag{5}$$

Numerous modified ratio estimators have been developed by different researchers in the field of sampling survey. Sisodia and Dwivedi (1981) have utilized the known coefficient of variation of the auxiliary variable and proposed a ratio estimator. Their estimator as well as its bias and mean square error are;

$$t_1 = \bar{y}\left(\frac{\bar{X} + C_x}{\bar{x} + C_x}\right) \tag{6}$$

$$B(t_1) = \theta \overline{Y} \left[\psi_1^2 C_x^2 - \psi_1 \rho C_x C_y \right] \tag{7}$$

$$MSE(t_1) = \theta \overline{Y}^2 \left[C_y^2 + \psi_1^2 C_x^2 - 2\psi_1 \rho C_x C_y \right] \tag{8}$$

Where; $\psi_1 = \left(\dfrac{\overline{X}}{\overline{x} + C_x} \right)$

Upadhyaya and Singh (1999) examined the influence of the coefficient of variation and coefficient of kurtosis in estimating the population mean, and as a result, proposed several ratio estimators. The estimators, along with the expressions for their bias and mean squared error, are presented as follows:

$$t_2 = \overline{y} \left(\frac{\overline{X} C_x + \beta_2}{\overline{x} C_x + \beta_2} \right) \tag{9}$$

$$t_3 = \overline{y} \left(\frac{\overline{X} \beta_2 + C_x}{\overline{x} \beta_2 + C_x} \right) \tag{10}$$

$$B(t_2) = \theta \overline{Y} \left[\psi_2^2 C_x^2 - \psi_2 \rho C_x C_y \right] \tag{11}$$

$$B(t_3) = \theta \overline{Y} \left[\psi_3^2 C_x^2 - \psi_3 \rho C_x C_y \right] \tag{12}$$

$$MSE(t_2) = \theta \overline{Y}^2 \left[C_y^2 + \psi_2^2 C_x^2 - 2\psi_2 \rho C_x C_y \right] \tag{13}$$

$$MSE(t_3) = \theta \overline{Y}^2 \left[C_y^2 + \psi_3^2 C_x^2 - 2\psi_3 \rho C_x C_y \right] \tag{14}$$

Where; $\psi_2 = \left(\dfrac{\overline{X} C_x}{\overline{X} C_x + \beta_2} \right)$ and $\psi_3 = \left(\dfrac{\overline{X} C_x}{\overline{X} C_x + \beta_2} \right)$

Singh and Tailor (2003) explored the significance of the correlation coefficient in estimating the population mean and proposed a corresponding estimator. The estimator, along with the expressions for its bias and mean squared error, is presented as follows:

$$t_4 = \overline{y} \left(\frac{\overline{X} + \rho}{\overline{x} + \rho} \right) \tag{15}$$

$$B(t_4) = \theta \overline{Y} \left[\psi_4^2 C_x^2 - \psi_4 \rho C_x C_y \right] \tag{16}$$

$$MSE(t_4) = \theta \overline{Y}^2 \left[C_y^2 + \psi_4^2 C_x^2 - 2\gamma_4 \rho C_x C_y \right] \tag{17}$$

Where; $\psi_4 = \left(\dfrac{\overline{X}}{\overline{X} + \rho} \right)$

Singh *et al*. (2004) inspired by Singh and Tailor (2003) utilizes the coefficient of kurtosis to propose an estimator for the population mean. The estimator proposed as well its expression for bias and mean square error are given as;

$$t_5 = \overline{y} \left(\frac{\overline{X} + \beta_2}{\overline{x} + \beta_2} \right) \tag{18}$$

$$B(t_5) = \theta \overline{Y} \left[\psi_5^2 C_x^2 - \psi_5 \rho C_x C_y \right] \tag{19}$$

$$MSE\left(t_5\right)=\theta\overline{Y}^2\left[C_y^2+\psi_5^2C_x^2-2\psi_5\rho C_xC_y\right] \tag{20}$$

Where; $\psi_5=\left(\dfrac{\overline{X}}{\overline{X}+\beta_2}\right)$

Al-Omari *et al.* (2009) utilizes the known value of third Quartile information of the auxiliary variable and proposed a ratio estimator of population mean. The proposed estimator with their bias and mean square error expressions up to the first order of approximations are given as;

$$t_6=\overline{y}\left(\frac{\overline{X}+Q_3}{\overline{x}+Q_3}\right) \tag{21}$$

$$B\left(t_6\right)=\theta\overline{Y}\left[\psi_6^2C_x^2-\psi_6\rho C_xC_y\right] \tag{22}$$

$$MSE\left(t_6\right)=\theta\overline{Y}^2\left[C_y^2+\psi_6^2C_x^2-2\psi_6\rho C_xC_y\right] \tag{23}$$

Where; $\psi_6=\dfrac{\overline{X}}{\overline{X}+Q_3}$

Yan and Tian (2010) utilize the information on the known coefficient of skewness kurtosis of the auxiliary information and proposed some estimators of population mean. The proposed estimators together with the expression for their bias and mean squared error are given respectively as;

$$t_7=\overline{y}\left(\frac{\overline{X}+\beta_1}{\overline{x}+\beta_1}\right) \tag{24}$$

$$t_8=\overline{y}\left(\frac{\overline{X}\beta_2+\beta_1}{\overline{x}\beta_2+\beta_1}\right) \tag{25}$$

$$t_9=\overline{y}\left(\frac{\overline{X}\beta_1+\beta_2}{\overline{x}\beta_1+\beta_2}\right) \tag{26}$$

$$t_{10}=\overline{y}\left(\frac{\overline{X}C_x+\beta_1}{\overline{x}C_x+\beta_1}\right) \tag{27}$$

$$B\left(t_7\right)=\theta\overline{Y}\left[\psi_7^2C_x^2-\psi_7\rho C_xC_y\right] \tag{28}$$

$$B\left(t_8\right)=\theta\overline{Y}\left[\psi_8^2C_x^2-\psi_8\rho C_xC_y\right] \tag{29}$$

$$B\left(t_9\right)=\theta\overline{Y}\left[\psi_9^2C_x^2-\psi_9\rho C_xC_y\right] \tag{30}$$

$$B\left(t_{10}\right)=\theta\overline{Y}\left[\psi_{10}^2C_x^2-\psi_{10}\rho C_xC_y\right] \tag{31}$$

$$MSE\left(t_7\right)=\theta\overline{Y}^2\left[C_y^2+\psi_7^2C_x^2-2\psi_7\rho C_xC_y\right] \tag{32}$$

$$MSE\left(t_8\right)=\theta\overline{Y}^2\left[C_y^2+\psi_8^2C_x^2-2\psi_8\rho C_xC_y\right] \tag{33}$$

$$MSE\left(t_9\right)=\theta\overline{Y}^2\left[C_y^2+\psi_9^2C_x^2-2\psi_9\rho C_xC_y\right] \tag{34}$$

$$MSE\left(t_{10}\right)=\theta\overline{Y}^2\left[C_y^2+\psi_{10}^2C_x^2-2\psi_{10}\rho C_xC_y\right] \tag{35}$$

Where; $\psi_7 = \left(\dfrac{\overline{X}}{\overline{X} + \beta_1} \right)$, $\psi_8 = \left(\dfrac{\overline{X}\beta_2}{\overline{X}\beta_2 + \beta_1} \right)$, $\psi_9 = \left(\dfrac{\overline{X}\beta_1}{\overline{X}\beta_1 + \beta_2} \right)$ and $\psi_{10} = \left(\dfrac{\overline{X}C_x}{\overline{X}C_x + \beta_2} \right)$

Subramani and Kumarapandiyan (2012a) utilizes the known information on coefficient of variation and median and proposed an estimator of population mean. The proposed estimator with its bias and mean square error up to the first order of approximations are given respectively as;

$$t_{11} = \overline{y}\left(\frac{\overline{X}C_x + Md}{\overline{x}C_x + Md} \right) \tag{36}$$

$$B\left(t_{11}\right) = \theta\overline{Y}\left[\psi_{11}^2 C_x^2 - \psi_{11}\rho C_x C_y \right] \tag{37}$$

$$MSE\left(t_{11}\right) = \theta\overline{Y}^2\left[C_y^2 + \psi_{11}^2 C_x^2 - 2\psi_{11}\rho C_x C_y \right] \tag{38}$$

Where; $\psi_{11} = \left(\dfrac{\overline{X}C_x}{\overline{X}C_x + Md} \right)$

Subramani (2012b) use the information on the given median together with the coefficient of skewness and kurtosis of the auxiliary variable to propose some estimators of the population mean as;

$$t_{12} = \overline{y}\left(\frac{\overline{X}\beta_1 + Md}{\overline{x}\beta_1 + Md} \right) \tag{39}$$

$$t_{13} = \overline{y}\left(\frac{\overline{X}\beta_2 + Md}{\overline{x}\beta_2 + Md} \right) \tag{40}$$

$$B\left(t_{12}\right) = \theta\overline{Y}\left[\psi_{12}^2 C_x^2 - \psi_{12}\rho C_x C_y \right] \tag{41}$$

$$B\left(t_{13}\right) = \theta\overline{Y}\left[\psi_{13}^2 C_x^2 - \psi_{13}\rho C_x C_y \right] \tag{42}$$

$$MSE\left(t_{12}\right) = \theta\overline{Y}^2\left[C_y^2 + \psi_{12}^2 C_x^2 - 2\psi_{12}\rho C_x C_y \right] \tag{43}$$

$$MSE\left(t_{13}\right) = \theta\overline{Y}^2\left[C_y^2 + \psi_{13}^2 C_x^2 - 2\psi_{13}\rho C_x C_y \right] \tag{44}$$

Where $\psi_{12} = \left(\dfrac{\overline{X}\beta_1}{\overline{X}\beta_1 + Md} \right)$ and $\psi_{13} = \left(\dfrac{\overline{X}\beta_2}{\overline{X}\beta_2 + Md} \right)$

Subramani and Kumarapandiyan (2012c) utilized the information on known third Quartile of auxiliary variable and suggested an estimator of the population mean. The estimator proposed with their bias and mean square error are given by;

$$t_{14} = \overline{y}\left(\frac{\overline{X} + Q_3}{\overline{x} + Q_3} \right) \tag{45}$$

$$B\left(t_{14}\right) = \theta\overline{Y}\left[\psi_{14}^2 C_x^2 - \psi_{14}\rho C_x C_y \right] \tag{46}$$

$$MSE\left(t_{14}\right) = \theta\overline{Y}^2\left[C_y^2 + \psi_{14}^2 C_x^2 - 2\psi_{14}\rho C_x C_y \right] \tag{47}$$

Where; $\psi_{14} = \dfrac{\overline{X}}{\overline{X} + Q_3}$

Subramani and Kumarapandiyan (2013a) use the known median information of the auxiliary variable to propose a ratio estimator of the population mean. The proposed estimator as well as the expression for its bias and mean square error up to the first order of approximation is given by;

$$t_{15} = \bar{y}\left(\frac{\bar{X}+Md}{\bar{x}+Md}\right) \tag{48}$$

$$B\left(t_{15}\right) = \theta\bar{Y}\left[\psi_{15}^2 C_x^2 - \psi_{15}\rho C_x C_y\right] \tag{49}$$

$$MSE\left(t_{15}\right) = \theta\bar{Y}^2\left[C_y^2 + \psi_{15}^2 C_x^2 - 2\psi_{15}\rho C_x C_y\right] \tag{50}$$

Where $\psi_{15} = \left(\frac{\bar{X}}{\bar{X}+Md}\right)$

Subramani (2013b) utilizes the information on correlation coefficient and coefficient of skewness of the auxiliary variable and proposed some estimators of population mean. The proposed estimators with their bias and mean square error are given respectively by;

$$t_{16} = \bar{y}\left(\frac{\bar{X}\rho+\beta_1}{\bar{x}\rho+\beta_1}\right) \tag{51}$$

$$t_{17} = \bar{y}\left(\frac{\bar{X}\beta_1+\rho}{\bar{x}\beta_1+\rho}\right) \tag{52}$$

$$B\left(t_{16}\right) = \theta\bar{Y}\left[\psi_{16}^2 C_x^2 - \psi_{16}\rho C_x C_y\right] \tag{53}$$

$$B\left(t_{17}\right) = \theta\bar{Y}\left[\psi_{17}^2 C_x^2 - \psi_{17}\rho C_x C_y\right] \tag{54}$$

$$MSE\left(t_{16}\right) = \theta\bar{Y}^2\left[C_y^2 + \psi_{16}^2 C_x^2 - 2\psi_{16}\rho C_x C_y\right] \tag{55}$$

$$MSE\left(t_{17}\right) = \theta\bar{Y}^2\left[C_y^2 + \psi_{17}^2 C_x^2 - 2\psi_{17}\rho C_x C_y\right] \tag{56}$$

Where; $\psi_{16} = \frac{\bar{X}\rho}{\bar{X}\rho+\beta_1}$ and $\psi_{17} = \frac{\bar{X}\beta_1}{\bar{X}\beta_1+\rho}$

Yadav, S.K *et al* (2014) motivated by Prasad (1989) and Jeelani *et al* (2013a) proposed a ratio type estimator for population mean. The estimator proposed together with the expression for its bias and mean square error are given by;

$$t_{18} = \delta_1\bar{y}\left(\frac{\bar{X}\beta_1+QD}{\bar{x}\beta_1+QD}\right) \tag{57}$$

$$B\left(t_{18}\right) = \theta\bar{Y}\left[\delta_1 y\psi_{18}^2 C_x^2 - \delta\psi_{18}\rho C_x C_y\right] + \bar{Y}\left(\delta_1 - 1\right) \tag{58}$$

$$MSE\left(t_{18}\right) = \theta\bar{Y}^2\left[\delta_1^2 C_y^2 + \left(3\delta_1^2 - 2\delta_1\right)\psi_{18}^2 C_x^2 - 2\left(2\delta_1^2 - \delta_1\right)\psi_{18}\rho C_x C_y + \left(\delta_1 - 1\right)^2\right] \tag{59}$$

Where; $\psi_{18} = \left(\frac{\bar{X}\beta_1}{\bar{X}\beta_1+QD}\right)$ and $\delta_1 = \frac{\psi_{18}^2 C_x^2 - \psi_{18}\rho C_x C_y + 1}{C_y^2 + 3\psi_{18}^2 C_x^2 - 4\psi_{18}\rho C_x C_y + 1}$

Kumarapandiyan and Subramani (2016) proposed some ratio estimators for population mean using known value of coefficient of skewness combined with functions of Quartiles. The proposed estimators together with the expressions for their bias and mean square error respectively are given by;

$$t_{19} = \bar{y}\left(\frac{\bar{X}\beta_1 + Q_1}{\bar{x}\beta_1 + Q_1}\right) \tag{60}$$

$$t_{20} = \bar{y}\left(\frac{\bar{X}\beta_1 + Q_3}{\bar{x}\beta_1 + Q_3}\right) \tag{61}$$

$$B(t_{19}) = \theta\bar{Y}\left[\psi_{19}^2 C_x^2 - \psi_{19}\rho C_x C_y\right] \tag{62}$$

$$B(t_{20}) = \theta\bar{Y}\left[\psi_{20}^2 C_x^2 - \psi_{20}\rho C_x C_y\right] \tag{63}$$

$$MSE(t_{19}) = \theta\bar{Y}^2\left[C_y^2 + \psi_{19}^2 C_x^2 - 2\psi_{19}\rho C_x C_y\right] \tag{64}$$

$$MSE(t_{20}) = \theta\bar{Y}^2\left[C_y^2 + \psi_{20}^2 C_x^2 - 2\psi_{20}\rho C_x C_y\right] \tag{65}$$

Where:

$$\psi_{19} = \frac{\bar{X}\beta_1}{\bar{X}\beta_1 + Q_1},\ \psi_{20} = \frac{\bar{X}\beta_1}{\bar{X}\beta_1 + Q_3}$$

Jerajuddin and Kishun (2016) utilizes the known sample size information of the auxiliary variable and proposed an estimator of the population mean. The estimator proposed together with its bias and mean squared error are given as;

$$t_{21} = \bar{y}\left(\frac{\bar{X} + n}{\bar{x} + n}\right) \tag{66}$$

$$B(t_{21}) = \theta\bar{Y}\left[\psi_{21}^2 C_x^2 - \psi_{21}\rho C_x C_y\right] \tag{67}$$

$$MSE(t_{21}) = \theta\bar{Y}^2\left[C_y^2 + \psi_{21}^2 C_x^2 - 2\psi_{21}\rho C_x C_y\right] \tag{68}$$

Where; $\psi_{21} = \left(\frac{\bar{X}}{\bar{X} + n}\right)$

Yadav (2019b) proposed some ratio estimators using some combinations of population parameters. The proposed estimators with their bias and mean square error are given by;

$$t_{22} = \bar{y}\left(\frac{n\bar{X} + \rho}{n\bar{x} + \rho}\right) \tag{69}$$

$$t_{23} = \bar{y}\left(\frac{n\bar{X} + C_x}{n\bar{x} + C_x}\right) \tag{70}$$

$$t_{24} = \bar{y}\left(\frac{n\bar{X} + \rho C_x}{n\bar{x} + \rho C_x}\right) \tag{71}$$

$$t_{25} = \bar{y}\left(\frac{n\rho\bar{X} + C_x}{n\rho\bar{x} + C_x}\right) \tag{72}$$

$$t_{26} = \bar{y}\left(\frac{nC_x\bar{X} + \rho}{nC_x\bar{x} + \rho}\right) \tag{73}$$

$$B\left(t_{22}\right)=\theta\overline{Y}\left[\psi_{22}^{2}C_{x}^{2}-\psi_{22}\rho C_{x}C_{y}\right] \tag{74}$$

$$B\left(t_{23}\right)=\theta\overline{Y}\left[\psi_{23}^{2}C_{x}^{2}-\psi_{23}\rho C_{x}C_{y}\right] \tag{75}$$

$$B\left(t_{24}\right)=\theta\overline{Y}\left[\psi_{24}^{2}C_{x}^{2}-\psi_{24}\rho C_{x}C_{y}\right] \tag{76}$$

$$B\left(t_{25}\right)=\theta\overline{Y}\left[\psi_{25}^{2}C_{x}^{2}-\psi_{25}\rho C_{x}C_{y}\right] \tag{77}$$

$$B\left(t_{26}\right)=\theta\overline{Y}\left[\psi_{26}^{2}C_{x}^{2}-\psi_{26}\rho C_{x}C_{y}\right] \tag{78}$$

$$MSE\left(t_{22}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{22}^{2}C_{x}^{2}-2\psi_{22}\rho C_{x}C_{y}\right] \tag{79}$$

$$MSE\left(t_{23}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{23}^{2}C_{x}^{2}-2\psi_{23}\rho C_{x}C_{y}\right] \tag{80}$$

$$MSE\left(t_{24}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{24}^{2}C_{x}^{2}-2\psi_{24}\rho C_{x}C_{y}\right] \tag{81}$$

$$MSE\left(t_{25}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{25}^{2}C_{x}^{2}-2\psi_{25}\rho C_{x}C_{y}\right] \tag{82}$$

$$MSE\left(t_{26}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{26}^{2}C_{x}^{2}-2\psi_{26}\rho C_{x}C_{y}\right] \tag{83}$$

Where;

$$\psi_{22}=\frac{n\overline{X}}{n\overline{X}+\rho C_{x}},\ \psi_{23}=\frac{n\overline{X}}{n\overline{X}+C_{x}},\ \psi_{24}=\frac{n\overline{X}}{n\overline{X}+\rho C_{x}},\psi_{25}=\frac{n\rho\overline{X}}{n\rho\overline{X}+C_{x}}\ \&\psi_{26}=\frac{nC_{x}\overline{X}}{nC_{x}\overline{X}+\rho}$$

Suleiman and Adewara (2021), inspired by the works of Sisodia and Dwivedi (1981), Upadhyaya and Singh (1999), Jerajuddin & Kishun (2016), and Gupta and Yadav (2018), proposed an estimator for the population mean using information on sample size. The proposed estimator, along with its bias and mean squared error, are presented as:

$$t_{27}=\overline{y}\left[\delta_{2}+(1-\delta_{2})\left(\frac{\overline{X}+C_{x}n}{\overline{x}+C_{x}n}\right)\right] \tag{84}$$

$$B\left(t_{27}\right)=\theta\overline{Y}\left[\psi_{27}^{2}C_{x}^{2}-\psi_{27}\rho C_{x}C_{y}+\delta_{2}\psi_{27}\rho C_{x}C_{y}-\delta_{2}\psi_{27}^{2}C_{x}^{2}\right] \tag{85}$$

$$MSE\left(t_{27}\right)=\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{27}^{2}C_{x}^{2}-2\psi_{27}\rho C_{x}C_{y}+\delta_{2}^{2}\psi_{27}^{2}C_{x}^{2}+2\delta_{2}\psi_{27}\rho C_{x}C_{y}-2\delta_{2}\psi_{27}^{2}C_{x}^{2}\right] \tag{86}$$

Where;

$$\psi_{27}=\left(\frac{\overline{X}}{\overline{X}+C_{x}n}\right)\ \text{and}\ \ \delta_{2}=\frac{\psi_{27}^{2}C_{x}^{2}-\psi_{27}\rho C_{x}C_{y}}{\psi_{27}^{2}C_{x}^{2}}$$

The summary of the existing proposed ratio estimators in the literature is presented in table 1

Table 1: Existing Ratio Estimators in the Literature

S/N	Estimators	Bias	MSE
1	$\overline{y}\left(\frac{\overline{X}+C_{x}}{\overline{x}+C_{x}}\right)$ Sisodia & Dwivedi 1981	$\theta\overline{Y}\left[\psi_{1}^{2}C_{x}^{2}-\psi_{1}\rho C_{x}C_{y}\right]$	$\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{1}^{2}C_{x}^{2}-2\psi_{1}\rho C_{x}C_{y}\right]$
2	$\overline{y}\left(\frac{\overline{X}C_{x}+\beta_{2}}{\overline{x}C_{x}+\beta_{2}}\right)$	$\theta\overline{Y}\left[\psi_{2}^{2}C_{x}^{2}-\psi_{2}\rho C_{x}C_{y}\right]$	$\theta\overline{Y}^{2}\left[C_{y}^{2}+\psi_{2}^{2}C_{x}^{2}-2\psi_{2}\rho C_{x}C_{y}\right]$

	Upadhyaya & Singh 1999		
3	$\bar{y}\left(\frac{\bar{X}\beta_2 + C_x}{\bar{x}\beta_2 + C_x}\right)$ Upadhyaya & Singh 1999	$\theta\bar{Y}\left[\psi_3^2 C_x^2 - \psi_3 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_3^2 C_x^2 - 2\psi_3 \rho C_x C_y\right]$
4	$\bar{y}\left(\frac{\bar{X} + \rho}{\bar{x} + \rho}\right)$ Singh & Tailor 2003	$\theta\bar{Y}\left[\psi_4^2 C_x^2 - \psi_4 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_4^2 C_x^2 - 2\psi_4 \rho C_x C_y\right]$
5	$\bar{y}\left(\frac{\bar{X} + \beta_2}{\bar{x} + \beta_2}\right)$ Singh *et al* 2004	$\theta\bar{Y}\left[\psi_5^2 C_x^2 - \psi_5 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_5^2 C_x^2 - 2\psi_5 \rho C_x C_y\right]$
6	$\bar{y}\left(\frac{\bar{X} + Q_3}{\bar{x} + Q_3}\right)$ Al-Omari *et al* 2009	$\theta\bar{Y}\left[\psi_6^2 C_x^2 - \psi_6 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_6^2 C_x^2 - 2\psi_6 \rho C_x C_y\right]$
7	$\bar{y}\left(\frac{\bar{X} + \beta_1}{\bar{x} + \beta_1}\right)$ Yan and Tian 2010	$\theta\bar{Y}\left[\psi_7^2 C_x^2 - \psi_7 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_7^2 C_x^2 - 2\psi_7 \rho C_x C_y\right]$
8	$\bar{y}\left(\frac{\bar{X}\beta_2 + \beta_1}{\bar{x}\beta_2 + \beta_1}\right)$ Yan and Tian 2010	$\theta\bar{Y}\left[\psi_8^2 C_x^2 - \psi_8 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_8^2 C_x^2 - 2\psi_8 \rho C_x C_y\right]$
9	$\bar{y}\left(\frac{\bar{X}\beta_1 + \beta_2}{\bar{x}\beta_1 + \beta_2}\right)$ Yan and Tian 2010	$\theta\bar{Y}\left[\psi_9^2 C_x^2 - \psi_9 \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_9^2 C_x^2 - 2\psi_9 \rho C_x C_y\right]$
10	$\bar{y}\left(\frac{\bar{X}C_x + \beta_1}{\bar{x}C_x + \beta_1}\right)$ Yan and Tian 2010	$\theta\bar{Y}\left[\psi_{10}^2 C_x^2 - \psi_{10} \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_{10}^2 C_x^2 - 2\psi_{10} \rho C_x C_y\right]$
11	$\bar{y}\left(\frac{\bar{X}C_x + Md}{\bar{x}C_x + Md}\right)$	$\theta\bar{Y}\left[\psi_{11}^2 C_x^2 - \psi_{11} \rho C_x C_y\right]$	$\theta\bar{Y}^2\left[C_y^2 + \psi_{11}^2 C_x^2 - 2\psi_{11} \rho C_x C_y\right]$

	Subramani *et al* 2012		
12	$\bar{y}\left(\frac{\bar{X}\beta_1+Md}{\bar{x}\beta_1+Md}\right)$ Subramani 2012	$\theta\bar{Y}\left[\psi_{12}^2C_x^2-\psi_{12}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{12}^2C_x^2-2\psi_{12}\rho C_xC_y\right]$
13	$\bar{y}\left(\frac{\bar{X}\beta_2+Md}{\bar{x}\beta_2+Md}\right)$ Subramani 2012	$\theta\bar{Y}\left[\psi_{13}^2C_x^2-\psi_{13}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{13}^2C_x^2-2\psi_{13}\rho C_xC_y\right]$
14	$\bar{y}\left(\frac{\bar{X}+Q_3}{\bar{x}+Q_3}\right)$ Subramani *et al* 2012	$\theta\bar{Y}\left[\psi_{14}^2C_x^2-\psi_{14}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{14}^2C_x^2-2\psi_{14}\rho C_xC_y\right]$
15	$\bar{y}\left(\frac{\bar{X}+Md}{\bar{x}+Md}\right)$ Subramani *et al* 2013	$\theta\bar{Y}\left[\psi_{15}^2C_x^2-\psi_{15}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{15}^2C_x^2-2\psi_{15}\rho C_xC_y\right]$
16	$\bar{y}\left(\frac{\bar{X}\rho+\beta_1}{\bar{x}\rho+\beta_1}\right)$ Subramani 2013	$\theta\bar{Y}\left[\psi_{16}^2C_x^2-\psi_{16}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{16}^2C_x^2-2\psi_{16}\rho C_xC_y\right]$
17	$\bar{y}\left(\frac{\bar{X}\beta_1+\rho}{\bar{x}\beta_1+\rho}\right)$ Subramani 2013	$\theta\bar{Y}\left[\psi_{17}^2C_x^2-\psi_{17}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{17}^2C_x^2-2\psi_{17}\rho C_xC_y\right]$
18	$\delta_1\bar{y}\left(\frac{\bar{X}\beta_1+QD}{\bar{x}\beta_1+QD}\right)$ Yadav, S.K *et al* 2014	$\theta\bar{Y}\left[\delta_1\psi_{18}^2C_x^2-\delta_1\psi_{18}\rho C_xC_y\right]$ $+\bar{Y}\left(\delta_1-1\right)$	$=\theta\bar{Y}^2\begin{bmatrix}\delta_1^2C_y^2+\left(3\delta_1^2-2\delta_1\right)\psi_{18}^2C_x^2-2\\ \left(2\delta_1^2-\delta_1\right)\psi_{18}\rho C_xC_y+\left(\delta_1-1\right)^2\end{bmatrix}$
19	$\bar{y}\left(\frac{\bar{X}\beta_1+Q_1}{\bar{x}\beta_1+Q_1}\right)$ Kumarapandiyan *et al* 2016	$\theta\bar{Y}\left[\psi_{19}^2C_x^2-\psi_{19}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{19}^2C_x^2-2\psi_{19}\rho C_xC_y\right]$
20	$\bar{y}\left(\frac{\bar{X}\beta_1+Q_3}{\bar{x}\beta_1+Q_3}\right)$ Kumarapandiyan *et al* 2016	$\theta\bar{Y}\left[\psi_{20}^2C_x^2-\psi_{20}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{20}^2C_x^2-2\psi_{20}\rho C_xC_y\right]$
21	$\bar{y}\left(\frac{\bar{X}+n}{\bar{x}+n}\right)$ Jerajuddin *et al* 2016	$\theta\bar{Y}\left[\psi_{21}^2C_x^2-\psi_{21}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{21}^2C_x^2-2\psi_{21}\rho C_xC_y\right]$

22	$\bar{y}\left(\frac{n\bar{X}+\rho}{n\bar{x}+\rho}\right)$ Yadav 2019	$\theta\bar{Y}\left[\psi_{22}^2C_x^2-\psi_{22}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{22}^2C_x^2-2\psi_{22}\rho C_xC_y\right]$
23	$\bar{y}\left(\frac{n\bar{X}+C_x}{n\bar{x}+C_x}\right)$ Yadav 2019	$\theta\bar{Y}\left[\psi_{23}^2C_x^2-\psi_{23}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{23}^2C_x^2-2\psi_{23}\rho C_xC_y\right]$
24	$\bar{y}\left(\frac{n\bar{X}+\rho C_x}{n\bar{x}+\rho C_x}\right)$ Yadav 2019	$\theta\bar{Y}\left[\psi_{24}^2C_x^2-\psi_{24}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{24}^2C_x^2-2\psi_{24}\rho C_xC_y\right]$
25	$\bar{y}\left(\frac{n\rho\bar{X}+C_x}{n\rho\bar{x}+C_x}\right)$ Yadav 2019	$\theta\bar{Y}\left[\psi_{25}^2C_x^2-\psi_{25}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{25}^2C_x^2-2\psi_{25}\rho C_xC_y\right]$
26	$\bar{y}\left(\frac{nC_x\bar{X}+\rho}{nC_x\bar{x}+\rho}\right)$ Yadav 2019	$\theta\bar{Y}\left[\psi_{26}^2C_x^2-\psi_{26}\rho C_xC_y\right]$	$\theta\bar{Y}^2\left[C_y^2+\psi_{26}^2C_x^2-2\psi_{26}\rho C_xC_y\right]$
27	$\bar{y}\left[\begin{matrix}\delta_2+(1-\delta_2)\\ \left(\frac{\bar{X}+C_xn}{\bar{x}+C_xn}\right)\end{matrix}\right]$ Suleiman *et al* 2021	$\theta\bar{Y}\left[\begin{matrix}\psi_{27}^2C_x^2-\psi_{27}\rho C_xC_y+\\ \delta_2\psi_{27}\rho C_xC_y-\delta_2\psi_{27}^2C_x^2\end{matrix}\right]$	$\theta\bar{Y}^2\left[\begin{matrix}C_y^2+\psi_{27}^2C_x^2-2\psi_{27}\rho C_xC_y+\\ \delta_2^2\psi_{27}^2C_x^2+2\delta_2\psi_{27}\rho C_xC_y-2\delta_8\psi_{27}^2C_x^2\end{matrix}\right]$

3. PROPOSED ESTIMATOR

After studying the works of Sisodia and Dwivedi (1981), Jerajuddin and Kishun (2016), Sabo *et al*. (2020), Suleiman and Adewara (2021) motivated by Yadav *et al* (2024), a generalized estimator of the population mean using information on the size of the sample was proposed as,

$$t_{pri}=\bar{y}\left[\alpha_i+(1-\alpha_i)\left(\frac{ab\bar{X}+cd}{ab\bar{x}+cd}\right)\right] \tag{87}$$

Where, α_i is a scalar to be optimized such that, the mean square error $MSE_{\min}\left(t_{pri}\right)$, is minimum.

a,b,c and d are parameters of the auxiliary variables which are known as $C_{x,}\ \beta_{1,}\ \beta_{2,}\ \rho,n,M_d$. The generalized estimator will generate various ratio estimators for different a,b,c and d combination of parameters when $\rho\in[0.5,1]$

3.1 FAMILY OF THE GENERALIZED PROPOSED ESTIMATOR

Some family members of the proposed generalized estimator in (87) are;

$$t_{pr1} = \bar{y}\left[\alpha_1 + (1-\alpha_1)\left(\frac{\rho M_d \bar{X} + C_x n}{\rho M_d \bar{x} + C_x n}\right)\right] \quad (88)$$

$$t_{pr2} = \bar{y}\left[\alpha_2 + (1-\alpha_2)\left(\frac{\beta_1 C_x \bar{X} + \beta_2 n}{\beta_1 C_x \bar{x} + \beta_2 n}\right)\right] \quad (89)$$

$$t_{pr3} = \bar{y}\left[\alpha_3 + (1-\alpha_3)\left(\frac{M_d C_x \bar{X} + \rho n}{M_d C_x \bar{x} + \rho n}\right)\right] \quad (90)$$

$$t_{pr4} = \bar{y}\left[\alpha_4 + (1-\alpha_4)\left(\frac{\beta_2 C_x \bar{X} + \beta_1 n}{\beta_2 C_x \bar{x} + \beta_1 n}\right)\right] \quad (91)$$

$$t_{pr5} = \bar{y}\left[\alpha_5 + (1-\alpha_5)\left(\frac{\beta_2 C_x \bar{X} + M_d n}{\beta_2 C_x \bar{x} + M_d n}\right)\right] \quad (92)$$

$$t_{pr6} = \bar{y}\left[\alpha_6 + (1-\alpha_6)\left(\frac{\beta_1 \beta_2 \bar{X} + M_d n}{\beta_1 \beta_2 \bar{x} + M_d n}\right)\right] \quad (93)$$

$$t_{pr7} = \bar{y}\left[\alpha_7 + (1-\alpha_7)\left(\frac{\beta_1 \rho \bar{X} + C_x n}{\beta_1 \rho \bar{x} + C_x n}\right)\right] \quad (94)$$

$$t_{pr8} = \bar{y}\left[\alpha_8 + (1-\alpha_8)\left(\frac{\beta_1 \beta_2 \bar{X} + \rho n}{\beta_1 \beta_2 \bar{x} + \rho n}\right)\right] \quad (95)$$

$$t_{pr9} = \bar{y}\left[\alpha_9 + (1-\alpha_9)\left(\frac{\beta_1 C_x \bar{X} + M_d n}{\beta_1 C_x \bar{x} + M_d n}\right)\right] \quad (96)$$

$$t_{pr10} = \bar{y}\left[\alpha_{10} + (1-\alpha_{10})\left(\frac{\beta_1 M_d \bar{X} + \beta_2 n}{\beta_1 M_d \bar{X} + \beta_2 n}\right)\right] \quad (97)$$

3.2 SAMPLING PROPERTIES OF THE PROPOSED ESTIMATORS

To study the large sample properties (Bias and Mean Squared Error) of the proposed modified ratio estimators, we have used the following approximations as:

$$e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}; \qquad e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$$

So that;

$$\bar{y} = \bar{Y}\left(1 + e_0\right); \quad \bar{x} = \bar{X}\left(1 + e_1\right) \quad (98)$$

Such that;

$$E(e_0) = E(e_1) = 0$$
$$E(e_0^2) = \theta C_y^2; \quad E(e_1^2) = \theta C_x^2; \quad E(e_0 e_1) = \theta\rho C_x C_y \tag{99}$$

Where;

$$\theta = \frac{N-n}{nN} = \frac{1-f}{n}; \quad f = \frac{n}{N}$$

Expressing equation (87) in terms of the definitions in (98) we get;

$$t_{pri} = \overline{Y}(1+e_0)\left[\alpha_i + (1-\alpha_i)(1+\gamma_i e_1)^{-1}\right] \tag{100}$$

Where; $\gamma_i = \dfrac{\overline{X}\rho M_d}{\overline{X}\rho M_d + C_x n}$

We assume that $|e_1| < 1$, so that $(1+\gamma_i e_1)^{-1}$ may be expanded. Now expanding the right-hand side of equation (100), we have;

$$t_{pri} = \overline{Y}\left[1 + e_0 - \gamma_i e_1 + \gamma_i^2 e_1^2 + \alpha_i\gamma_i e_1 - \alpha_i\gamma_i^2 e_1^2 - \gamma e_0 e_1 + \alpha_i\gamma_i e_0 e_1\right] \tag{101}$$

Subtracting $\overline{Y}$ from both sides of equation (101) we get;

$$t_{pri} - \overline{Y} = \overline{Y}\left[e_0 - \gamma_i e_1 - \gamma_i e_0 e_1 + \gamma_i^2 e_1^2 + \alpha_i\gamma_i e_1 + \alpha_i\gamma_i e_0 e_1 - \alpha_i\gamma_i^2 e_1^2\right] \tag{102}$$

Taking expectation on both sides of equation (102) and substituting the values of different expectations in (99), we get the bias of t_{pri} as;

$$B(t_{pri}) = \gamma_i\theta\overline{Y}\left[(1-\alpha)\gamma_i C_x^2 - (1-\alpha)\rho C_x C_y\right] \tag{103}$$

Squaring both sides of equation (102), retaining the terms up to the first order of approximation and taking the expectation, we have;

$$MSE(t_{pri}) = \theta\overline{Y}^2\left[C_y^2 + (1-\alpha)^2\gamma_i^2 C_x^2 - 2(1-\alpha)\gamma_i\rho C_x C_y\right] \tag{104}$$

The MSE is minimum for optimal value of α_i which can be obtain by differentiating equation (104) with respect to α_i and equating the result to zero which yield;

$$\frac{\partial MSE(t_{pri})}{\partial\alpha_i} = 0$$

$$\alpha_i = \frac{\gamma C_x - \rho C_y}{\gamma C_x} = \frac{A_i}{B_i}$$

Where $A_i = \gamma_i C_x - \rho C_y$ and $B_i = \gamma_i C_x$

Thus, the minimum mean squared error of t_{pri} is;

$$MSE_{\min}(t_i) = \theta\overline{Y}^2\left[C_y^2 + \gamma_i^2 C_x^2 - 2\rho\gamma_i C_x C_y - \frac{A_i^2}{B_i}\right] \tag{105}$$

$$MSE_{\min}(t_i) = \theta\overline{Y}^2\left[C_y^2\left(1-\rho^2\right)\right] \tag{106}$$

Similarly, the biases and mean squared errors of the proposed family of the generalized ratio estimators can be obtained.

4. THEORETICAL EFFICIENCY COMPARISON

The efficiency of our proposed modified ratio estimators with that of the other competing existing ratio estimators in the literature is compared using their variances and mean squared errors (MSE).

i. **Proposed Estimator Vs. Usual Sample Mean**

The proposed estimators are better than the usual sample mean estimator if;

$$V(t_0) - MSE_{\min}(t_{pri}) = \theta\bar{Y}^2\left[\gamma_{pi}^2 C_x^2 - 2\gamma_{pi}\rho C_x C_y - \frac{A_i^2}{B_i}\right] > 0$$

Or $\gamma_{pi}^2 C_x^2 - 2\gamma_{pi}\rho C_x C_y > \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$

ii. **Proposed Estimators Vs. Cochran Ratio Estimator**

The proposed ratio estimators performs better than the Cochran ratio estimator if;

$$MSE(t_R) - MSE(t_{pri}) > 0$$

Or

$$\rho C_x C_y(2 + \gamma_{pi}) - C_x^2(1 - \gamma_{pi}^2) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$$

iii. **Proposed Estimators Vs. Modified Ratio Estimator by Sisodia and Dwivedi**

The proposed estimators performs better than the modified ratio estimator by Sisodia and Dwivedi if:

$$MSE(t_1) - MSE(t_{pri}) > 0$$

Or

If $2\rho C_x C_y(\psi_1 - \gamma_{pi}) - C_x^2(\psi_1 - \gamma_{pi}^2) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$

iv. **Proposed Estimators Vs. Modified Ratio Estimator by Upadhyaya and Singh (a)**

The proposed estimators have better efficiency compared to the modified ratio estimator by Upadhyaya and Singh (a) if;

$$MSE(t_2) - MSE(t_{pri}) > 0$$

Or

$$2\rho C_x C_y(\psi_2 - \gamma_{pi}) - C_x^2(\psi_2 - \gamma_{pi}^2) < \frac{A_i^2}{B_i}$$

v. **Proposed Estimators Vs. Modified Ratio Estimator by Upadhyaya and Singh (b)**

The proposed estimators perform well compared to the modified ratio estimator by Upadhyaya and Singh (b) if;

$$MSE(t_3) - MSE(t_{pri}) > 0$$

Or

$$2\rho C_x C_y \left(\psi_3 - \gamma_{pi}\right) - C_x^2 \left(\psi_3 - \gamma_{pi}^2\right) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$$

vi. **Proposed Estimators Vs. Modified Ratio Estimator by Singh and Tailor**

The proposed estimators perform well compared to the modified ratio estimator by Singh and Tailor if;

$$MSE\left(t_4\right) - MSE\left(t_{pri}\right) > 0$$

Or

$$2\rho C_x C_y \left(\psi_4 - \gamma_{pi}\right) - C_x^2 \left(\psi_4 - \gamma_{pi}^2\right) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$$

vii. **Proposed Estimators Vs. Modified Ratio Estimator by Singh et al (d)**

The proposed estimators are better than the modified ratio estimator by Singh et al. (d) if;

$$MSE\left(t_5\right) - MSE\left(t_{pri}\right) > 0$$

Or

$$2\rho C_x C_y \left(\psi_5 - \gamma_{pi}\right) - C_x^2 \left(\psi_5 - \gamma_{pi}^2\right) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$$

viii. **Proposed Estimators Vs. Modified Ratio Estimator by Al Omari et al (b)**

The proposed estimators are better than the modified ratio estimator by Al Omari et al. (b) if;

$$MSE\left(t_6\right) - MSE\left(t_{pri}\right) > 0$$

Or

$$2\rho C_x C_y \left(\psi_6 - \gamma_{pi}\right) - C_x^2 \left(\psi_6 - \gamma_{pi}^2\right) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10)$$

ix. **Proposed Estimators Vs. Modified Ratio Estimator by Yan and Tian (a, b, c, d)**

The proposed estimators perform well more than the modified ratio estimator by Yan and Tian (a, b, c, d) if;

$$MSE\left(t_j\right) - MSE\left(t_{pri}\right) > 0$$

Or

$$2\rho C_x C_y \left(\psi_j - \gamma_{pi}\right) - C_x^2 \left(\psi_j - \gamma_{pi}^2\right) < \frac{A_i^2}{B_i} \qquad (i = 1, 2, ..., 10,\ j = 7, 8, 9, 10)$$

x. **Proposed Estimators Vs. Modified Ratio Estimator by Subramani et al (a, b, c, d, h)**

The proposed estimators are better than the modified ratio estimator by Subramani et al. (a, b, c, d, h) if;

$$MSE\left(t_j\right) - MSE\left(t_{pri}\right) > 0$$

Or

$$2\rho C_x C_y\left(\psi_j-\gamma_{pi}\right)-C_x^2\left(\psi_j-\gamma_{pi}^2\right)<\frac{A_i^2}{B_i}\qquad (i=1,2,\dots,10,\ j=11,12,13,14,15)$$

xi. **Proposed Estimators Vs. Modified Ratio Estimator by Subramani (a, b, c, d)**

The proposed estimators perform well compared to the modified ratio estimator by Subramani (a, b, c, d) if;

$$MSE\left(t_j\right)-MSE\left(t_{pri}\right)>0$$

Or

$$2\rho C_x C_y\left(\psi_j-\gamma_{pi}\right)-C_x^2\left(\psi_j-\gamma_{pi}^2\right)<\frac{A_i^2}{B_i}\qquad (i=1,2,\dots,10,\ j=16,17,19,20)$$

xii. **Proposed Estimators Vs. Modified Ratio Estimator by Yadav et al**

The proposed estimators are better than the modified ratio estimator by Yadav et al if;

$$MSE\left(t_{18}\right)-MSE\left(t_{pri}\right)>0$$

Or

$$2\rho C_x C_y\left(\psi_{18}-\gamma_{pi}\right)-C_x^2\left(\psi_{18}-\gamma_{pi}^2\right)<\frac{A_i^2}{B_i}\qquad (i=1,2,\dots,10)$$

xiii. **Proposed Estimators Vs. Modified Ratio Estimator by Jerajuddin et al**

The proposed estimators are better than the modified ratio estimator by Jerajuddin et al if;

$$MSE\left(t_{21}\right)-MSE\left(t_{pri}\right)>0$$

Or

$$2\rho C_x C_y\left(\psi_{21}-\gamma_{pi}\right)-C_x^2\left(\psi_{21}-\gamma_{pi}^2\right)<\frac{A_i^2}{B_i}\qquad (i=1,2,\dots,10)$$

xiv. **Proposed Estimators Vs. Modified Ratio Estimator by Yadav et al**

The proposed estimators have better efficiency compared to the modified ratio estimator by Baghel if;

$$MSE\left(t_i\right)-MSE\left(t_{pri}\right)>0$$

Or

$$2\rho C_x C_y\left(\psi_j-\gamma_{pi}\right)-C_x^2\left(\psi_j-\gamma_{pi}^2\right)<\frac{A_i^2}{B_i}\qquad (i=22,\dots,26,\ j=22,\dots,26)$$

xv. **Proposed Estimators Vs. Modified Ratio Estimator by Suleiman et al**

The proposed estimators are better than the modified ratio estimator by Sulaiman *et al*

$$MSE\left(t_{27}\right)-MSE\left(t_{pri}\right)>0$$

Or

$$C_x^2\left(\psi_{27}^2-\gamma_{pri}\right)-2\left(\psi_{27}-\gamma_{pri}\right)>\left(\frac{\tau_2^2}{\lambda_2}-\frac{A_i^2}{B_i}\right)\ (i=1,2,\dots,10)$$

5. EMPIRICAL STUDY

Here, we compare the performance of the proposed estimators numerically using two real data sets. The data sets used in this study is hypothetical, consisting of population on peppermint yield and field area from both the Siddhan and Banikodar blocks in the Barabanki district of Uttar Pradesh, India as adopted from (Yadav et al 2019) and (Yadav et al., 2024).

Population I: [Yadav *et al* (2024)]

Y_i = The production (Yield) of peppermint oil in kilogram

X_i = The area of the field in Siddhan

Table 2: Population Statistics for the real data set 1(Population I)

$$N=150,\quad n=40,\quad \bar{Y}=49.58,\quad \bar{X}=6.5833,\quad \rho_{xy}=0.9363,\quad C_x=0.6617$$
$$C_y=0.78133,\quad Q_1=4,\quad Q_3=10,\quad M_d=5\quad D_1=2,\quad D_3=3,\ D_4=5$$
$$D_6=6,\quad D_7=8,\quad D_8=10,\quad D_9=13,\quad QD=3\quad C_y^2=0.6105,\ C_x^2=0.4378$$
$$S_y^2=3866.165,\quad S_x^2=18.9779,\quad \beta_1=1.4984,\quad \beta_2=5.408,\quad f=0.2667$$
$$\theta=0.0183,\quad \alpha_1=-1.0548,\quad \alpha_2=-36.7371,\quad \alpha_3=-2.0069,\quad \alpha_4=-2.9181$$
$$\alpha_5=-9.3525,\quad \alpha_6=-4.2500,\ \alpha_7=-3.2739,\quad \alpha_8=-0.8819,\quad \alpha_9=-33.9779$$
$$\alpha_{10}=-4.9541$$

Population II: [Yadav *et al* (2019)]

Y_i = The production (Yield) of peppermint oil in kilogram

X_i = The area of the field in Barabanki

Table 3: Population Statistics for the real data set 2 (Population II)

$$N=150,\quad n=40,\quad \bar{Y}=33,462,\quad \bar{X}=4.2047,\quad \rho_{xy}=0.9112,\quad C_x=0.7326$$
$$C_y=0.7662,\quad Q_1=2,\quad Q_3=5,\quad M_d=3,\quad QD=1.5,\quad C_y^2=0.5809,$$
$$S_x^2=9.48877,\quad S_x=3.0804,\quad S_y^2=650.4112,\quad S_y=25.5032,\quad \beta_1=2.8014,$$
$$\theta=0.0183,\quad \alpha_1=-2.3649,\quad \alpha_2=-72.2063,\quad \alpha_3=-3.6862,\quad \alpha_4=-2.3421$$
$$\alpha_5=-2.1941,\quad \alpha_6=-0.5355,\quad \alpha_7=-2.5361,\quad \alpha_8=-0.1265,\quad \alpha_9=-13.1285$$
$$\alpha_{10}=-17.5523,\quad \beta_2=16.4402,\quad f=0.2667,\quad C_x^2=0,5367$$

Table 4: Numerical comparison of the proposed and competing estimators using population I

S/N	ESTIMATORS	MSE	PRE
1	t_0	70.75296	100
2	t_R	9.29369	761.30105
3	t_1	10.69546	661.52330
4	t_2	30.79215	229.77597
5	t_3	9.50562	744.32767
6	t_4	11.41492	619.82879

7	t_5	24.44910	289.38881
8	t_6	34.20737	206.83543
9	t_7	13.02541	543.19181
10	t_8	9.80963	721.26023
11	t_9	19.45193	363.73234
12	t_{10}	15.35980	460.63725
13	t_{11}	29.51302	239.73473
14	t_{12}	18.64205	397.53423
15	t_{13}	11.38232	621.60403
16	t_{14}	34.20737	206.83543
17	t_{15}	23.37780	302.65021
18	t_{16}	13.32889	530.82410
19	t_{17}	10.60460	667.19122
20	t_{18}	43.02620	164.44157
21	t_{19}	16.60978	425.97169
22	t_{20}	27.54833	256.83212
23	t_{21}	55.91249	126.54232
24	t_{22}	9.33183	758.18955
25	t_{23}	9.32081	759.08596
26	t_{24}	9.31861	759.26517
27	t_{25}	9.32300	758.90765
28	t_{26}	9.35193	756.55998
29	t_{27}	9.08935	778.41606
30	t_{pr1}	**8.72733**	**810.70568**
31	t_{pr2}	**8.72733**	**810.70568**
32	t_{pr3}	**8.72733**	**810.70568**
33	t_{pr4}	**8.72733**	**810.70568**
34	t_{pr5}	**8.72733**	**810.70568**
35	t_{pr6}	**8.72733**	**810.70568**
36	t_{pr7}	**8.72733**	**810.70568**
37	t_{pr8}	**8.72733**	**810.70568**

38	t_{pr9}	**8.72733**	**810.70568**
39	t_{pr10}	**8.72733**	**810.70568**

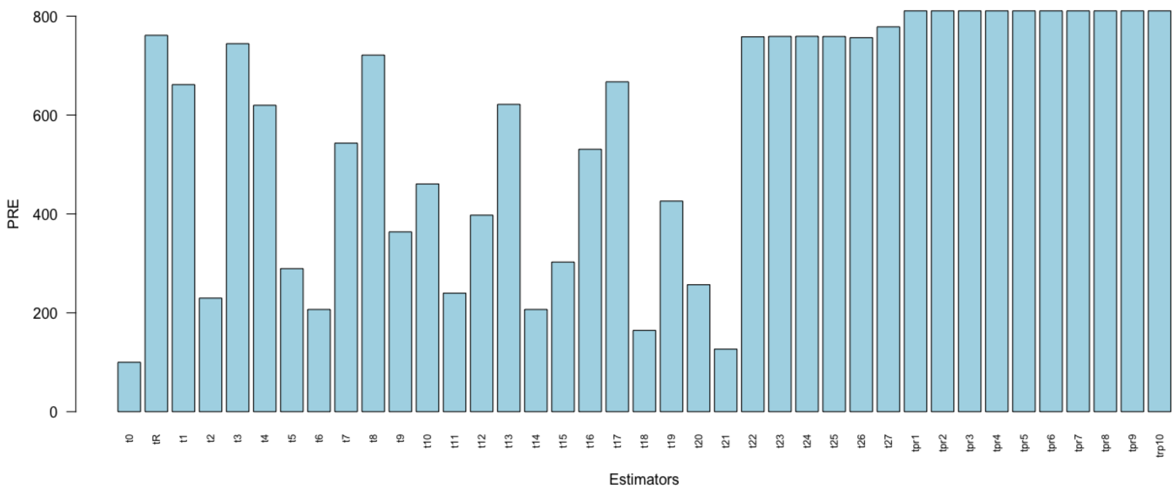

Fig. 1: Percentage Relative Efficiencies of the Proposed and Competing Estimators using Pop. I.

Table 5: Numerical comparison of the proposed and competing estimators using population II

S/N	ESTIMATORS	MSE	PRE
1	t_0	11.90310	100
2	t_R	2.04894	580.93941
3	t_1	2.12147	561.07793
4	t_2	8.88649	133.94602
5	t_3	2.03815	584.01492
6	t_4	2.19416	542.49006
7	t_5	8.11189	563.62311
8	t_6	4.67286	254.72837
9	t_7	3.35044	355.26976
10	t_8	2.02111	588.93875
11	t_9	5.11562	232.68147
12	t_{10}	3.99928	297.63107
13	t_{11}	4.16208	285.98922
14	t_{12}	2.27005	524.35409
15	t_{13}	2.02041	589.14281

16	t_{14}	4.67286	254.72837
17	t_{15}	3.47970	342.07259
18	t_{16}	3.52819	337.37129
19	t_{17}	2.02590	587.54628
20	t_{18}	117.23405	10.15328
21	t_{19}	2.11459	562.90345
22	t_{20}	2.68486	443.34155
23	t_{21}	10.01912	118.80385
24	t_{22}	9.33183	127.5538
25	t_{23}	2.04413	582.30641
26	t_{24}	2.04455	582.18679
27	t_{25}	2.04371	582.42608
28	t_{26}	2.04118	583.14798
29	t_{27}	2.03923	583.70561
30	t_{pr1}	**2.01923**	**589.48708**
31	t_{pr2}	**2.01923**	**589.48708**
32	t_{pr3}	**2.01923**	**589.48708**
33	t_{pr4}	**2.01923**	**589.48708**
34	t_{pr5}	**2.01923**	**589.48708**
35	t_{pr6}	**2.01923**	**589.48708**
36	t_{pr7}	**2.01923**	**589.48708**
37	t_{pr8}	**2.01923**	**589.48708**
38	t_{pr9}	**2.01923**	**589.48708**
39	t_{pr10}	**2.01923**	**589.48708**

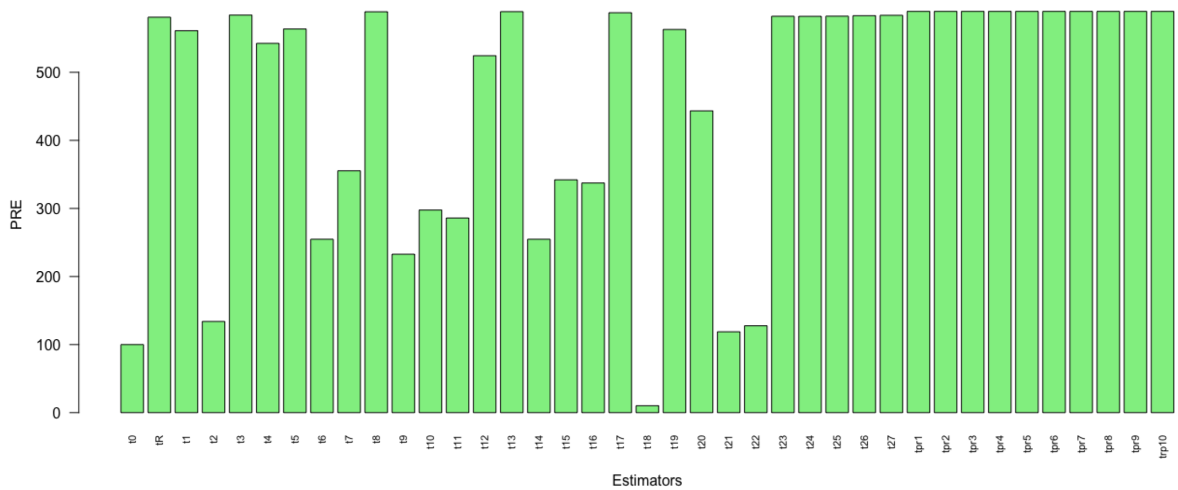

Fig. 2: Percentage Relative Efficiencies of the Proposed and Competing Estimators using Pop. II.

Results from table 4 and 5 indicates that the proposed estimators perform better compared with other competitors as they demonstrate lower MSEs and higher PREs values. This is further justified by figure 1 and 2. The PREs were calculated using;

$$PRE\left(t_j, t_0\right) = \frac{V\left(t_0\right)}{MSE\left(t_j\right)} \times 100 \tag{107}$$

6. CONCLUSION

In this study, we proposed a new generalized ratio-type estimator for the population mean under simple random sampling without replacement, incorporating auxiliary variable information. Several family members of this estimator were introduced, and their statistical properties were rigorously derived and analyzed. Both theoretical and empirical comparisons demonstrate that the proposed estimators consistently outperform existing alternatives. Their ability to minimize error and maximize efficiency across various data scenarios makes them particularly valuable, especially when auxiliary parameter information is readily available.

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