MAXIMAL-MINIMAL FUNCTIONS IN A TOPO-SPACE

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Abstract: In this chapter we introduce minimal $\widehat{D_\alpha}$-open sets, maximal $\widehat{D_\alpha}$-closed sets, maximal $\widehat{D_\alpha}$-open sets, minimal $\widehat{D_\alpha}$-closed sets, minimal $\widehat{D_\alpha}$-continous function and maximal $\widehat{D_\alpha}$-continous functions in topological spae as follows. The notions of this chapter are minimal $\widehat{D_\alpha}$-closed set, maximal $\widehat{D_\alpha}$-open sets, minimal $\widehat{D_\alpha}$-open set, maximal $\widehat{D_\alpha}$-closed set,minimal $\widehat{D_\alpha}$-continous, maximal $\widehat{D_\alpha}$-continuous,minimal $\widehat{D_\alpha}$-irresolute,maximal $\widehat{D_\alpha}$-irresolute, minimal-maximal $\widehat{D_\alpha}$-continuous and maimal-minimal $\widehat{D_\alpha}$-continuous and their basic properties are studied.

1.Introduction:

This section presents an overview of strong and weak forms of minimal closed sets, maximal open and minimal continuous map contributed by various topologists. Nakaoka and Oda [6] have introduced the concepts of minimal closed, maximal open and minimal continuous. Andrijevic [1] gave some properties of α - closure of a set A is denoted by αCl(A), and defined as intersection of all α - closed sets containing the set A. Dr. Haji M. Hasan[2] introduced maximal α-open set.

The author of this paper have introduced $\widehat{D_\alpha}$-open sets [4] and $\widehat{D_\alpha}$-closed sets [3] in topological spaces. Further we have introduced $\widehat{D_\alpha}$ – continuous function[5] in a topological space. In this paper we introduce minimal $\widehat{D_\alpha}$-open sets, maximal $\widehat{D_\alpha}$-closed sets, maximal $\widehat{D_\alpha}$-open sets, minimal $\widehat{D_\alpha}$-closed sets, minimal $\widehat{D_\alpha}$-continous function and maximal $\widehat{D_\alpha}$-continous functions in topological spae as follows. The notions of this chapter are minimal $\widehat{D_\alpha}$-closed set, maximal $\widehat{D_\alpha}$-open sets, minimal $\widehat{D_\alpha}$-open set, maximal $\widehat{D_\alpha}$-closed set, minimal $\widehat{D_\alpha}$-continous, maximal $\widehat{D_\alpha}$-continuous, minimal $\widehat{D_\alpha}$-irresolute,maximal $\widehat{D_\alpha}$-irresolute, minimal-maximal $\widehat{D_\alpha}$-continuous and maimal-minimal $\widehat{D_\alpha}$-continuous and their basic properties are studied.

2.Preliminaries:

Definition 2.1. [6] A proper nonempty open subset U of X is said to be a minimal open set if any open set contained in U is φ or U.

Definition 2.2. [6] A proper nonempty open subset U of X is said to be a maximal open set if any open set containing U is X or U.

Definition 2.3. [6] A proper nonempty closed subset F of X is said to be a minimal closed set if any closed set contained in F is φ or F.

Definition 2.4. [6] A proper nonempty closed subset F of X is said to be a maximal closed set if any closed set containing F is X or F.

Theorem 2.5. [6] Let X be a topological space and $F \subset X$. F is a minimal closed set iff $X - F$ is a maximal open set.

Theorem 2.6. [6] Let X be a topological space and $U \subset X$. U is a minimal open set iff $X - U$ is a maximal closed set.

Definition 2.7. [6] Let X and Y be the topological spaces. A function $f : X \to Y$ is called a

1. minimal continuous (briefly min-continuous) if $f^{-1}(U)$ is an open set in X for every minimal open set U in Y .

2. maximal continuous (briefly max-continuous) if $f^{-1}(U)$ is an open set in X for every maximal

open set U in Y .

3. minimal irresolute (briefly min-irresolute) if $f^{-1}(U)$ is minimal open set in X for every minimal open set U in Y .

4. maximal irresolute (briefly max-irresolute) if $f^{-1}(U)$ is maximal open set in X for every maximal open set U in Y .

5. minimal-maximal continuous (briefly min-max-continuous) if $f^{-1}(U)$ is maximal open set in X for every minimal open set U in Y .

6. maximal-minimal continuous (briefly max-min-continuous) if $f^{-1}(U)$ is minimal open set in X for every maximal open set U in Y .

Definition 2.8: $\widehat{D_\alpha}$-closed set [3] if $cl(A) \subset U$ whenever $A \subset U$ and U is D-open in (X, τ).

Definition 2.9: A function f : (X, τ) → (Y, σ) is said to be $\widehat{D_\alpha}$-continuous if $f^{-1}(H)$ is $\widehat{D_\alpha}$-closed in (X, τ) for every closed set H in Y .

3.Minimal $\widehat{D_\alpha}$-open sets and maximal $\widehat{D_\alpha}$-closed sets in Topo-space

Definition 3.1. A proper nonempty $\widehat{D_\alpha}$-open subset U of X is said to be a minimal $\widehat{D_\alpha}$-open set if any $\widehat{D_\alpha}$-open set contained in U is φ or U.

Remark 3.2. Minimal open set and minimal $\widehat{D_\alpha}$-open are independent. It show by the following example.

Example 3.3. Let X = {a, b, c} and τ = {φ, {a, b}, X}. Since {a} is minimal $\widehat{D_\alpha}$-open set but not minimal open set and {a, b} is minimal open set but not minimal $\widehat{D_\alpha}$-open set.

Theorem 3.4. Every minimal open set is $\widehat{D_\alpha}$-open set but not conversely.

Example 3.5. Let X = {a, b, c} and τ = {φ, {b}, X}. Then the subset {a, b} is $\widehat{D_\alpha}$-open set but not minimal open set.

Theorem 3.6.

i) Let U be a minimal $\widehat{D_\alpha}$-open set and W be a $\widehat{D_\alpha}$-open set. Then U ∩ W = φ or U ⊂ W.

ii) Let U and V be minimal $\widehat{D_\alpha}$-open sets. Then U ∩ V = φ or U = V .

Proof.

i) Let U be a minimal $\widehat{D_\alpha}$-open set and W be a $\widehat{D_\alpha}$-open set. If U ∩ W = φ, then there is nothing to prove. If U ∩ W ≠ φ. Then U ∩ W ⊂ U. Since U is a minimal $\widehat{D_\alpha}$-open set, we have U ∩ W = U. Therefore U ⊂ W.

ii) Let U and V be minimal $\widehat{D_\alpha}$-open sets. If U ∩ V ≠φ, then U ⊂ V and V ⊂ U by

(i). Therefore U = V .

Theorem 3.7. Let V be a nonempty finite $\widehat{D_\alpha}$-open set. Then there exists atleast one (finite) minimal $\widehat{D_\alpha}$-open set U such that U ⊂ V .

Proof. Let V be a nonempty finite $\widehat{D_\alpha}$-open set.

If V is a minimal $\widehat{D_\alpha}$-open set, we may set U = V .

If V is not a minimal $\widehat{D_\alpha}$-open set, then there exists (finite) $\widehat{D_\alpha}$ open set V1 such that φ ≠ V1 ⊂ V .

If V1 is a minimal $\widehat{D_\alpha}$-open set, we may set U = V1.

If V1 is not a minimal $\widehat{D_\alpha}$-open set, then there exists (finite) $\widehat{D_\alpha}$-open set V2 such that φ ≠ V2 ⊂ V1.

Continuing this process, we have a sequence of $\widehat{D_\alpha}$-open sets V ⊃ V1 ⊃ V2 ⊃ V3 ⊃ · · · ⊃ Vk ⊃ · · ·

Since V is a finite set, this process repeats only finitely.

Then finally we get a minimal $\widehat{D_\alpha}$-open set U = Vn for some positive integer n.

We now introduce Maximal $\widehat{D_\alpha}$-closed sets in topological spaces as follows.

Definition 3.8. A proper nonempty $\widehat{D_\alpha}$-closed set F ⊂ X is said to be maximal $\widehat{D_\alpha}$-closed set if any $\widehat{D_\alpha}$-closed set containing F is either X or F.

Theorem 3.9. A proper nonempty subset F of X is maximal $\widehat{D_\alpha}$-closed set iff X −F is a minimal $\widehat{D_\alpha}$-open set.

Proof. Let F be a proper maximal $\widehat{D_\alpha}$-closed set.

Suppose X − F is not a minimal $\widehat{D_\alpha}$-open set.

Then there exists $\widehat{D_\alpha}$-open set U ≠ X − F such that φ ≠ U ⊂ X − F.

That is F ⊂ X − U and X − U is a $\widehat{D_\alpha}$-closed set which is a contradiction for F is a maximal $\widehat{D_\alpha}$-closed set.

Conversely let X − F be a minimal $\widehat{D_\alpha}$-open set.

Suppose F is not a maximal $\widehat{D_\alpha}$ closed set, then there exists $\widehat{D_\alpha}$-closed set E ≠ F such that F ⊂ E ≠ X.

That is φ ≠ X − E ⊂ X − F and X − E is a $\widehat{D_\alpha}$-open set which is a contradiction for X − F is a minimal $\widehat{D_\alpha}$-open set. Therefore F is a maximal $\widehat{D_\alpha}$-closed set.

4.Minimal $\widehat{D_\alpha}$-closed sets and maximal $\widehat{D_\alpha}$-open sets in Topo space

Definition 4.1. A proper nonempty $\widehat{D_\alpha}$-closed subset F of X is said to be a minimal $\widehat{D_\alpha}$-closed set if any $\widehat{D_\alpha}$-closed set contained in F is φ or F.

Remark 4.2. Minimal closed set and minimal $\widehat{D_\alpha}$-closed set are independent. It shown by the following example.

Example 4.3. Let X = {a, b, c} and τ = {φ, {b}, X}. Since {a} is minimal $\widehat{D_\alpha}$ closed set but not minimal closed set and {a, c} is minimal closed set but not minimal $\widehat{D_\alpha}$-closed set.

Definition 4.4. A proper nonempty $\widehat{D_\alpha}$-open subset U of X is said to be a maximal $\widehat{D_\alpha}$-open set if any $\widehat{D_\alpha}$-open set containing U is either X or U.

Remark 4.5. Maximal open set and maximal $\widehat{D_\alpha}$-open set are independent. It is shown by the following example.

Example 4.6. Let X and τ be defined as in the Example 4.2.3. Then {a, b} is maximal $\widehat{D_\alpha}$-open set but not maximal open set and {b} is maximal open set but not maximal $\widehat{D_\alpha}$-open set.

Theorem 4.7. A proper nonempty subset U of X is maximal $\widehat{D_\alpha}$-open set iff X − U is a minimal $\widehat{D_\alpha}$-closed set.

Proof. Let U be a maximal proper $\widehat{D_\alpha}$-open set.

Suppose X − U is not a minimal $\widehat{D_\alpha}$-closed set.

Then there exists a $\widehat{D_\alpha}$-closed set F ≠ X − U such that φ ≠ F ⊂ X − U.

That is U ⊂ X − F and X − F is a $\widehat{D_\alpha}$-open set which is a contradiction for U is a maximal $\widehat{D_\alpha}$-open set.

Conversely, let X − U be a minimal $\widehat{D_\alpha}$-closed set.

Suppose U is not a maximal $\widehat{D_\alpha}$-open set.

Then there exists a $\widehat{D_\alpha}$-open set E ≠ U such that U ⊂ E ≠ X.

That is φ ≠ X − E ⊂ X − U and X − E is a $\widehat{D_\alpha}$-closed set which is a contradiction for X − U is a minimal $\widehat{D_\alpha}$-closed set.

Theorem 4.8.

i) Let F be a maximal $\widehat{D_\alpha}$-open set and W be a $\widehat{D_\alpha}$-open set. Then F ∪ W = X or W ∪ F.

ii) Let F and S be maximal $\widehat{D_\alpha}$-open sets. Then F ∪ S = X or F = S.

Proof.

i) Let F be a maximal $\widehat{D_\alpha}$-open set and W be a $\widehat{D_\alpha}$-open set. If F ∪ W = X, then there is nothing to prove. Suppose F ∪W 6 = X. Then F ⊆ F ∪W. Therefore F ∪ W = F as F is a maximal $\widehat{D_\alpha}$-open set in X. Hence F ∪ W = W ∪ F.

ii) Let F and S be maximal $\widehat{D_\alpha}$-open sets. If F ∪ S ≠ X, then we have F ⊆ S and S ⊆ F by (i). Therefore F = S.

5.Minimal $\widehat{D_\alpha}$-continuous functions and maximal $\widehat{D_\alpha}$-continuous functions

Definition 5.1. Let X and Y be topological spaces. A function f : X → Y is called

1. minimal $\widehat{D_\alpha}$-continuous (briefly, min-$\widehat{D_\alpha}$-continuous) if f^{-1} (A) is $\widehat{D_\alpha}$-open set in X for every minimal open set A in Y .

2. maximal $\widehat{D_\alpha}$-continuous (briefly, max-$\widehat{D_\alpha}$-continuous) if f^{-1} (A) is $\widehat{D_\alpha}$-open set in X for every maximal open set A in Y .

3. minimal $\widehat{D_\alpha}$-irresolute (briefly, min-$\widehat{D_\alpha}$-irresolute) if f^{-1} (A) is minimal $\widehat{D_\alpha}$-open set in X for every minimal $\widehat{D_\alpha}$-open set A in Y .

4. maximal $\widehat{D_\alpha}$-irresolute (briefly, max-$\widehat{D_\alpha}$-irresolute) if f^{-1} (A) is maximal $\widehat{D_\alpha}$-open set in X for every maximal $\widehat{D_\alpha}$-open set A in Y .

5. minimal-maximal $\widehat{D_\alpha}$-continuous (briefly, min-max-$\widehat{D_\alpha}$-continuous) if f^{-1} (A) is maximal $\widehat{D_\alpha}$-open set in X for every minimal open set A in Y .

6. maximal-minimal $\widehat{D_\alpha}$-continuous (briefly, max-min-$\widehat{D_\alpha}$-continuous) if f^{-1} (A) is minimal $\widehat{D_\alpha}$-open set in X for every maximal open set A in Y .

Theorem 5.2. Every continuous function is minimal $\widehat{D_\alpha}$-continuous function but not conversely.

Proof. Let f : X → Y be a continuous function.

To prove that f is minimal $\widehat{D_\alpha}$-continuous.

Let U be any minimal open set in Y .

Since every minimal open set is an open set and every open set is $\widehat{D_\alpha}$-open set, U is a $\widehat{D_\alpha}$-open set in Y . Since f is continuous, f^{-1}(U) is a $\widehat{D_\alpha}$-open set in Y .

Hence f is a minimal $\widehat{D_\alpha}$-continuous.

Example 5.3. Let X = Y = {a, b, c} be with τ = {φ, {a, b}, X} and μ = {φ, {a}, {b}, {a, b}, Y }. Define a map f : X → Y by an f(a) = b, f(b) = a and f(c) = c. Then f is a minimal $\widehat{D_\alpha}$-continuous function but it is not a continuous function, since for the open set {b} in Y , $f^{-1}(\{b\}) = \{a\}$ which is not an open set in X.

Theorem 5.4. Let X and Y be topological spaces. A function f : X → Y is minimal $\widehat{D_\alpha}$-continuous if and only if the inverse image of each maximal closed set in Y is a $\widehat{D_\alpha}$-closed set in X.

Proof. suppose f : X → Y is minimal $\widehat{D_\alpha}$-continuous. Let F be a maximal closed set in Y . Then F^c is a minimal open set in Y . Therefore $f^{-1}(F^c)$ is $\widehat{D_\alpha}$-open set in X. Since $(f^{-1}(F))^c = f^{-1}(F^c)$ and so $f^{-1}(F)$ is $\widehat{D_\alpha}$-closed set in X.

Conversely, let U be a minimal open set in Y . Then U^c is maximal closed set in Y . By hypothesis, $f^{-1}(U^c)$ is a $\widehat{D_\alpha}$-closed set in X. Since $(f^{-1}(U))^c = f^{-1}(U^c)$, $f^{-1}(U)$ is $\widehat{D_\alpha}$-open in X. Therefore f is minimal $\widehat{D_\alpha}$-continuous.

Theorem 5.5. If f : X → Y is $\widehat{D_\alpha}$-irresolute function and g : Y → Z is minimal $\widehat{D_\alpha}$-continuous function, then g ∘ f : X → Z is a minimal $\widehat{D_\alpha}$-continuous.

Proof. Let U be any minimal open set in Z.

Since g is minimal $\widehat{D_\alpha}$-continuous, $g^{-1}(U)$ is a $\widehat{D_\alpha}$-open set in Y .

Again since f is $\widehat{D_\alpha}$-irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a $\widehat{D_\alpha}$-open set in X.

Hence g ∘f is a minimal $\widehat{D_\alpha}$-continuous.

Theorem 5.6. Let X and Y be the topological spaces. A function f : X → Y is maximal $\widehat{D_\alpha}$-continuous if and only if the inverse image of each minimal closed set in Y is a $\widehat{D_\alpha}$-closed set in X.

Proof. Obviously true by Theorem 5.4.

Theorem 5.7. If f : X → Y is $\widehat{D_\alpha}$-irresolute function and g : Y → Z is maximal $\widehat{D_\alpha}$-continuous functions, then g ∘ f : X → Z is a maximal $\widehat{D_\alpha}$-continuous.

Proof. Obviously true by Theorem 5.5.

Theorem 5.8. Let X and Y be the topological spaces. A function f : X → Y is minimal $\widehat{D_\alpha}$-irresolute if and only if the inverse image of each maximal $\widehat{D_\alpha}$-closed set in Y is a maximal $\widehat{D_\alpha}$-closed set in X.

Proof. Obviously true by Theorem 5.4.

Theorem 5.9. If f : X → Y and g : Y → Z are minimal $\widehat{D_\alpha}$-irresolute functions, then g ∘ f : X → Z is a minimal $\widehat{D_\alpha}$-irresolute function.

Proof. Let U be any minimal $\widehat{D_\alpha}$-open set in Z. Since g is minimal $\widehat{D_\alpha}$-irresolute, g^{-1} (U) is a minimal $\widehat{D_\alpha}$-open set in Y . Again since f is minimal $\widehat{D_\alpha}$-irresolute, f^{-1} (g^{-1} (U)) = $(g \circ f)^{-1}$ (U) is minimal $\widehat{D_\alpha}$-open set in X. Therefore g ∘ f is minimal $\widehat{D_\alpha}$-irresolute.

Theorem 5.10. If f : X → Y and g : Y → Z are maximal $\widehat{D_\alpha}$-irresolute functions, then g ∘ f : X → Z is a maximal $\widehat{D_\alpha}$-irresolute function.

Proof. Obviously true by Theorem 5.9.

Theorem 5.11. Every min-max $\widehat{D_\alpha}$-continuous function is minimal $\widehat{D_\alpha}$-continuous function but not conversely.

Proof. Let f : X → Y be a min-max $\widehat{D_\alpha}$-continuous function.

Let U be any minimal open set in Y.

Since f is min-max $\widehat{D_\alpha}$-continuous, f^{-1} (U) is a maximal $\widehat{D_\alpha}$-open set in X.

Since every maximal $\widehat{D_\alpha}$-open set is a $\widehat{D_\alpha}$-open set, f^{-1} (U) is a $\widehat{D_\alpha}$-open set in X.

Hence f is a minimal $\widehat{D_\alpha}$-continuous.

Example 5.12. Let (X, τ) and (Y, σ) be defined as in example 3.5.7. Let X = Y = {a, b, c} be with τ = {φ, {a}, {b}, {a, b}, X} and σ = {φ, {a}, {a, b}, Y }. Define a map f : X → Y by f(a) = a, f(b) = c and f(c) = b. Then f is a minimal $\widehat{D_\alpha}$-continuous function but it is not a min-max $\widehat{D_\alpha}$-continuous, since for the minimal open set {a} in Y , f^{-1} ({a}) = {a} which is not a maximal $\widehat{D_\alpha}$-open set in X.

Theorem 5.13. Every max-min $\widehat{D_\alpha}$-continuous function is maximal $\widehat{D_\alpha}$-continuous function but not conversely.

Proof. Obviously true by theorem 5.11.

Example 5.14. Let (X, τ) and (Y, σ) be defined as in Example 3.6.3. Let X = Y = {a, b, c}, τ = {φ, {a, b}, X} and σ = {φ, {a}, {b}, {a, b}, Y }. Define a map f : X → Y by f(a) = b, f(b) = c and f(c) = a. Then f is a maximal $\widehat{D_\alpha}$ continuous function but it is not a max-min $\widehat{D_\alpha}$-continuous, since for the maximal open set {a, b} in Y , f^{-1} ({a, b}) = {a, c} which is not a minimal $\widehat{D_\alpha}$-open set in X. **Theorem 5.15.** If f : X → Y is maximal $\widehat{D_\alpha}$-irresolute and g : Y → Z is min-max $\widehat{D_\alpha}$-continuous functions, then g ∘ f : X → Z is a min-max $\widehat{D_\alpha}$-continuous function.

Proof. Let U be any minimal open set in Z. Since g is min-max $\widehat{D_\alpha}$-continuous, g^{-1} (U) is a maximal $\widehat{D_\alpha}$-open set in Y . Again since f is maximal $\widehat{D_\alpha}$irresolute, f^{-1} (g^{-1} (U)) = $(g \circ f)^{-1}$ (U) is a maximal $\widehat{D_\alpha}$-open set in X. Hence g ∘ f is a min-max $\widehat{D_\alpha}$-continuous.

Theorem 5.16. If f : X → Y is maximal $\widehat{D_\alpha}$-irresolute and g : Y → Z is min-max $\widehat{D_\alpha}$-continuous functions, then g ∘ f : X → Z is a minimal $\widehat{D_\alpha}$-continuous.

Proof. Let U be any minimal $\widehat{D_\alpha}$-open set in Z. Since g is min-max $\widehat{D_\alpha}$-continuous, g^{-1} (U) is a maximal $\widehat{D_\alpha}$-open set in Y . Again since f is maximal $\widehat{D_\alpha}$-irresolute f^{-1} (g^{-1} (U)) = $(g \circ f)^{-1}$ (U) is maximal $\widehat{D_\alpha}$-open. Since every maximal $\widehat{D_\alpha}$-open set in $\widehat{D_\alpha}$-open, $(g \circ f)^{-1}$ (U) is $\widehat{D_\alpha}$-open set in X. Hence g ∘ f is a minimal $\widehat{D_\alpha}$-continuous.

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