ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. TThe purpose of the present paper is to investigate some argument properties for certain analytic functions in the open unit disk. The main results presented in here generalize some previous those concerning starlike function of reciprocal of order beta and strongly starlike functions.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}). \tag{1.1}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $C(\alpha)$ of convex functions of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1) \tag{1.2}$$

and is said to be in the class $\mathbb{S}^*(\alpha)$ of starlike functions of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1). \tag{1.3}$$

We note that $\mathcal{C}(0) = \mathcal{C}$ and $\mathbb{S}^*(0) = \mathbb{S}^*$, where $\mathcal{C}$ and $\mathbb{S}^*$ are, respectively, the well-known classes of convex and starlike functions.

The classical result of Marx [5] and Strahhäcker [8] asserts that a convex function is starlike of order $1/2$, that is,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{1.4}$$

If $f(z) \in \mathbb{S}^*$ satisfies the condition

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}), \tag{1.5}$$

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then $f(z)$ is said to be starlike of reciprocal of order β (see Nunokawa et al. [4]).

In [7] Sakaguchi proved that: If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

In [6] Pommerenke generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$ and

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \le \frac{\pi}{2}\alpha \quad (0 < \alpha \le 1; z \in \mathbb{U}), \tag{1.7}$$

then

$$\left|\arg \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right| \le \frac{\pi}{2}\alpha \quad (|z_1| < 1, |z_2| < 1). \tag{1.8}$$

Recently, Nunokawa et al. [4] generalized Pommerenke's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$, then $g(z)$ is starlike of reciprocal of order β and

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \le \frac{\pi}{2}\alpha + \tan^{-1} \frac{\alpha\beta}{1+\alpha} \quad (z \in \mathbb{U}; 0 < \alpha \le 1; 0 \le \beta < 1), \tag{1.9}$$

then

$$\left|\arg \frac{f(z)}{g(z)}\right| \le \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{1.10}$$

Also Kanas et al. [1] generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\right\} > 0 \ (z \in \mathbb{U}; 0 \le \alpha \le 1) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \alpha \ (z \in \mathbb{U}), \tag{1.11}$$

where the powers in (1.11) are meant as the principal values.

Also Kanas et al. [1] defined the class $\mathcal{H}(\alpha)$ as follows.

$$\mathcal{H}(\alpha) = \left\{f(z) \in \mathcal{A}, g(z) \in \mathbb{S}^* : \operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > 0 \ (0 \le \alpha \le 1)\right\}. \tag{1.12}$$

In the present paper, we extend some results obtained by Kanas et al. [1], Liu [2], Nunokawa et al. [4], Pommerenke [6] and Sakaguchi [7] by using Nunowawa's lemma [3].

2. Main results

To derive our results, we need the following lemma due to Nunokawa [3].

Lemma 2.1. *[3] Let a function $p(z)$ with $p(0)=1$ and $p(z)\neq 0$ be analytic in $\mathbb{U}$. If there exists a point $z_0\in\mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z|<|z_0|\,,\ \alpha>0),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha \ \textit{and}\ |\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

where

$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) \geq 1 \ \textit{when}\ \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \leq -1 \ \textit{when}\ \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia (a>0).$$

Theorem 2.2. *Let $f(z)\in\mathcal{A}$, $g(z)\in\mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β. Suppose that*

$$\left|\arg\left[(1-\lambda)\frac{f(z)}{g(z)}+\lambda\frac{f'(z)}{g'(z)}-\gamma\right]\right| < \frac{\pi}{2}\rho \quad (0\leq\lambda\leq 1;\ 0\leq\gamma<1;\ z\in\mathbb{U}), \tag{2.1}$$

where

$$\rho = \alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\beta\lambda}{1+\alpha\lambda}\right)(0<\alpha\leq 1;\ 0\leq\gamma<1). \tag{2.2}$$

Then we have

$$\left|\left(\arg\frac{f(z)}{g(z)}-\gamma\right)\right| < \frac{\pi}{2}\alpha \quad (z\in\mathbb{U}). \tag{2.3}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)}-\gamma\right). \tag{2.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $p(z)\neq 0$. It follows from (2.4) that

$$\frac{f'(z)}{g'(z)} = \gamma+(1-\gamma)p(z)\left[1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.5}$$

Also, from (2.4) and (2.5), we have

$$(1-\lambda)\frac{f(z)}{g(z)}+\lambda\frac{f'(z)}{g'(z)}-\gamma = (1-\gamma)p(z)\left[1+\lambda\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.6}$$

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If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1).$$

Then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,$$

where

$$k \geq \frac{1}{2}(a + a^{-1}) \geq 1 \text{ when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}(a + a^{-1}) \leq -1 \text{ when} \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where $(p(z_0))^{1/\alpha} = \pm ia (a > 0)$. Since $g(z) \in \mathcal{C}$, from Marx-Strohhäcker's theorem [5,8], we have

$$\text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 1/2 \quad (z \in \mathbb{U}),$$

so that $g(z) \in \mathbb{S}^*(1/2)$. Putting $\frac{zg'(z)}{g(z)} = u + iv$, where $u > 1/2$. Then

$$\left|\frac{g(z)}{zg'(z)} - 1\right|^2 = \left|\frac{1 - u - iv}{u + iv}\right|^2 = \frac{1 - 2u + u^2 + v^2}{u^2 + v^2} < 1.$$

Therefore,

$$\left|\frac{g(z)}{zg'(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.7}$$

which implies that

$$\left|\text{Im}\left\{\frac{g(z)}{zg'(z)}\right\}\right| < 1 \quad (z \in \mathbb{U}), \tag{2.8}$$

and from the assumption of the theorem, we have

$$\text{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \beta \quad (0 \leq \beta < 1; \ z \in \mathbb{U}). \tag{2.9}$$

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For the case $|\arg p(z_0)| = \frac{\pi}{2}\alpha$, from (2.5), (2.6) and (2.8), we have

$$\begin{aligned} & \arg\left\{(1-\lambda)\frac{f(z_0)}{g(z_0)} + \lambda\frac{f'(z_0)}{g'(z_0)} - \gamma\right\} \\ = \ & \arg p(z_0) + \arg\left\{1 + \lambda\frac{z_0 p'(z_0)}{p(z_0)}\left(\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\ = \ & \frac{\pi}{2}\alpha + \arg\left\{1 + i\alpha k\lambda\left(\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)} + i\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\} \\ = \ & \frac{\pi}{2}\alpha + \arg\left\{1 - \alpha k\lambda\left(\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right) + ik\alpha\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}\right\} \\ = \ & \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}}{1 + \alpha k\lambda\left|\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right|}\right\} \\ \geq \ & \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\beta}{1 + \alpha k\lambda}\right\} \geq \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha\lambda\beta}{1 + \alpha\lambda}\right\}. \end{aligned}$$

This contradicts the assumption of the theorem, then

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

For the case $|\arg p(z_0)| = -\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.2. ∎

Remark. Putting $\lambda = 1$ in Theorem 1, we get the result obtained by Liu [2, Theorem 2.1]. Also, from Theorem 1, we have the results obtained by Kanas [1], Nunokawa [4] and Sakaguchi [7].

Theorem 2.3. *Let $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β $(0 \leq \beta < 1)$. Suppose that*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}\right| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}), \tag{2.10}$$

where

$$\rho = (\mu + \gamma)\alpha + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right) \quad (z \in \mathbb{U}), \tag{2.11}$$

μ and γ are fixed positive real numbers with $0 < \mu + \gamma \leq 1$ and $0 < \alpha \leq 1$. Then

$$\left|\arg\frac{f(z)}{g(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.12}$$

Proof. Let us define the function $p(z)$ by (2.4). It follows from (2.4) and (2.5) that

$$\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}=(p(z))^{\mu+\gamma}\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)^{\gamma}$$

and

$$\begin{aligned}\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma} &= \mu\arg\frac{f(z)}{g(z)}+\gamma\arg\frac{f'(z)}{g'(z)}\\ &= (\mu+\gamma)\arg p(z)+\gamma\arg\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right).\end{aligned}\tag{2.13}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\arg p(z)|<\frac{\pi}{2}\alpha\ \ (|z|<|z_0|)\ \ and\ \ |\arg p(z_0)|=\frac{\pi}{2}\alpha\ \ (0<\alpha\le 1).$$

Then, using Lemma 1, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\beta.$$

For the case $\arg p(z)=\frac{\pi}{2}\alpha$, from (2.7), (2.8) and (2.13), we have

$$\begin{aligned}\arg\left(\frac{f(z_0)}{g(z_0)}\right)^{\mu}\left(\frac{f'(z_0)}{g'(z_0)}\right)^{\gamma} &= (\mu+\gamma)\arg p(z_0)+\gamma\arg\left(1+\frac{z_0p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0g'(z_0)}\right)\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1+i\alpha k\left(\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}+i\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1-\alpha k\left(\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)+i\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}}{1+\alpha k\left|\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right|}\right\}\\ &\ge (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta k}{1+\alpha k}\right\}\\ &\ge (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta}{1+\alpha}\right\}.\end{aligned}$$

This contradicts the assumption of the theorem, then we have

$$|\arg p(z)|<\frac{\pi}{2}\alpha\ \ (z\in\mathbb{U}).$$

For the case $|\arg p(z_0)|=-\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.3.

Putting $\mu = 1 - \gamma$ $(\gamma > 0)$ in Theorem 2, we obtain the following corollary.

Corollary 1. Let $f(z) \in \mathcal{A}, g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β $(0 < \beta \leq 1)$. Suppose that

$$\left|\arg \left(\frac{f(z)}{g(z)}\right)^{1-\gamma} \left(\frac{f'(z)}{g'(z)}\right)^{\gamma}\right| < \frac{\pi}{2}\rho \quad (\gamma > 0;\ z \in \mathbb{U}),$$

$$\rho = \alpha + \frac{2\gamma}{\pi} \tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right) \quad (0 < \alpha \leq 1).$$

Then

$$\left|\arg \frac{f(z)}{g(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Remark. Putting $\gamma = 1$ in Corollary 1, we have the result obtained by Nunokawa et al. [[4], Theorem 2.3].

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