k-vertex Self Switching of Graphs

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Abstract

For a finite undirected graph $G(V, E)$ and a non-empty set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V, E')$ which is obtained from G by removing all edges between σ and its complement $V - \sigma$ and adding as edges all non-edges between σ and $V - \sigma$. If $G \cong G^{\sigma}$, then σ is called as self switching of G and if $|\sigma| = k$, then it is called as k-vertex self switching. The set of all k-vertex self switchings of G is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. In this paper, we give a sufficient condition for σ to be a k-vertex self switching and we find $ss_k(G)$ of path and star graphs.

***Keywords* :** Switching, self switching, k-vertex self switching, $SS_k(G)$, $ss_k(G)$

***AMS Subject Classification* :** *05C38, 05C60.*

1 Introduction

Switching, also known as Seidel switching or $|\sigma|$-vertex switching, has been explained by Seidel [2, 6]. For a finite undirected graph $G(V, E)$ with $|V| = p$ and a non-empty set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V, E')$ which is obtained from G by removing all edges between σ and its complement $V - \sigma$ and adding as edges all non-edges between σ and $V - \sigma$. In 1998, Hertz emphasized that if we take into account the restricted form of switching, we can get more results. For example, the cardinality of σ is 1 or 2. When $\sigma = \{v\} \subset V$, the corresponding switching $G^{\{v\}}$ is called as vertex switching and is denoted by G^v. Jayasekaran was the first person to propose the idea of self switching in 2007. If $G \cong G^{\sigma}$, then σ is said to be a self switching of G. It is sometimes referred to as $|\sigma|$-vertex self switching. If $k = 1$, then the self switching σ is termed as self vertex switching [4, 5] and if $k = 2$, then the corresponding self switching is termed as 2-vertex self switching [1]. If $|\sigma| = k$, then it is called as k-vertex self switching. The set of all k-vertex self switchings of G is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$.

In this paper, we prove that if $\sigma = \{v_1, v_2, \dots, v_k\} \subset V$ is a k-vertex self switching of G, then $deg(v_1) + deg(v_2) + \dots + deg(v_k) = \frac{k(p-k)}{2} + 2$(The number of edges between the vertices of σ in G) for $k \geq 2$; an even order graph has no odd k-vertex self switching; $V - \sigma$ is a $(p-k)$-vertex self switching of G; if σ is a k-vertex self switching of G, then σ is also a k-vertex self switching of G^{σ} and $(G^{\sigma})^{\sigma} = G$ and the number of k-vertex self switchings for path and star graphs has been found.

2 Preliminaries

The following results are required in the subsequent section.

Definition 2.1. *The set of positive integers, each is the degree of a vertex in a graph G, is denoted by $DS(G)$. That is, $DS(G) = \{n : n = deg(u), u \in V(G)\}$.*

Theorem 2.2. *[7] Let $G(V,E)$ be a graph and let $\sigma \subset V$ be a self switching of G. Then the number of edges between the vertices of σ and $V - \sigma$ in G is $k(p-k)/2$ where $k = |\sigma|$.*

Lemma 2.3. *[3] Let $G(V,E)$ be a graph and $\sigma \subseteq V$ be a switching of G. Then*

i. $G[\sigma] = G^{\sigma}[\sigma]$,

ii. $G[V - \sigma] = G^{\sigma}[V - \sigma]$ *and*

iii. $G^{\sigma} = G^{V-\sigma}$.

Remark 2.4. *[7] $ss_1(P_2) = ss_1(K_{1,1}) = 0$; $ss_1(P_3) = ss_1(K_{1,2}) = 2$; $ss_1(P_4) = 0$; $ss_1(P_5) = 1$ and $ss_1(P_6) = 0$.*

Theorem 2.5. *[1] For $p \geq 2$, $ss_2(P_p) = \begin{cases} 1 & if\ p = 2, 6 \\ 2 & if\ p = 3, 4, 5 \\ 0 & otherwise \end{cases}$*

Definition 2.6. *A Branch at v in G is a maximal connected subgraph B of G such that the intersection of B with the vertex v is v and $B - v$ is connected and maximal.*

Notation 2.7. *[5] Let G be a connected graph and v be a cut vertex of G. Let B_1 , $B_2, \ldots, B_r$ be the branches with $n_1, n_2, \ldots, n_r$ copies, respectively at v in G. The graph G is denoted by $G(v; n_1B_1, n_2B_2, \ldots, n_rB_r)$.*

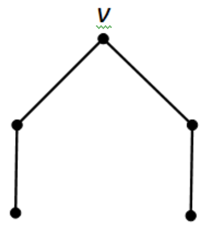

Fig 1.1 $G(v;2P_3)$

Theorem 2.8 . *[5] Let G be a tree of odd order $p = 2n + 1$, $n \in N$. Then G has a self vertex switching v if and only if $G = G(v; nP_3)$.*

Corollary 2.9. *[7] If v is a self vertex switching of a graph G of order p, then $deg_G(v) = (p - 1)/2$.*

Theorem 2.10. *[7] For $p \geq 7$, $ss_1(P_p) = 0$.*

Theorem 2.11. *[1] For $m, n \in N$, $ss_2(K_{m,n}) \geq mn$. In particular,*

$$ss_2(K_{m,n}) = \begin{cases} mn & if\ n \neq m + 2 \\ mn + \binom{n}{2} & if\ n = m + 2 \end{cases}$$

Theorem 2.12. *[7] For $n \geq 3$, $ss_1(K_{1,n}) = 0$.*

3 *k*-vertex Self Switching of Graphs

Theorem 3.1. *Let G be a graph and let $\sigma = \{v_1, v_2, \ldots, v_k\} \subset V$ be a k-vertex self switching of G. Then for $k \geq 2$, $deg(v_1) + deg(v_2) + \ldots + deg(v_k) = \frac{k(p-k)}{2} + 2$(The number of edges between the vertices of σ in G).*

Proof. Let σ be a k-vertex self switching of G. Then $G \cong G^{\sigma}$. By Theorem 2.2, the number of edges between the vertices of σ and $V - \sigma$ is $k(p - k)/2$. Also, each edge in $G[\sigma]$ contributes 2 to the sum of degrees. Hence, $deg(v_1) + deg(v_2) + \ldots + deg(v_k)$ =the number of edges between σ

and $V - \sigma + 2$(the number of edges in $G[\sigma]) = k(p-k)/2 + 2$(the number of edges between the vertices in σ).

Remark 3.2. *The converse of the above theorem need not be true. For example, consider the graph G given in figure 3.1. Let $\sigma = \{v_1, v_4, v_5\}$.*

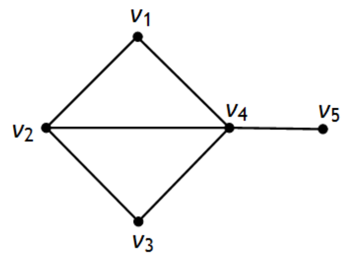

Fig 3.1 G

Here, $p = 5$, $q = 6$. Now, $deg(v_1) + deg(v_4) + deg(v_5) = 2 + 4 + 1 = 7 = k(p-k)/2 +$ 2(The number of edges between the vertices of σ). The graph G^σ is given in figure 3.2. Clearly, σ is not a 3-vertx self switching of G as $G \ncong G^\sigma$.

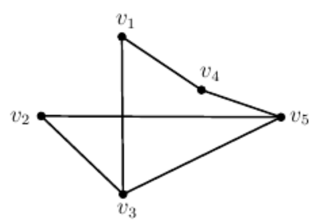

Fig 3.2 G^σ

Corollary 3.3. *For even order graph, there is no odd k-vertex self switching.*

Proof. Let G be a graph with even order p. Let $\sigma = \{v_1, v_2, \dots, v_k\}$ be a k-vertex self switching of G where k is odd. Then $k(p-k)/2$ is not an integer and thereby $deg(v_1) + deg(v_2) + \dots + deg(v_k)$ is not an integer, which is a contradiction to the sum of degrees is an integer. Hence, G has no odd k-vertex self switching.

Theorem 3.4. *Let G be a connected graph and let $\sigma \subset V$ be a k-vertex self switching of G. Then there is no vertex in $V - \sigma$ which is adjacent only to all the k vertices in σ.*

Proof. Let $\sigma = \{v_1, v_2, \dots, v_k\}$ be a k-vertex self switching of a connected graph G. Then $G \cong G^\sigma$. Let w be a vertex in $V - \sigma$ which is adjacent only to $v_1, v_2, \dots, v_k$ in G. Then $deg_G(w) = k$ and in G^σ, w is an isolated vertex which is a contradiction to G^σ is connected. Hence, no vertex in $V - \sigma$ is adjacent only to all the k vertices in σ.

Theorem 3.5. *Let G be a graph with p vertices. Let $\sigma \subset V$ be a k-vertex self switching of G. Then $V - \sigma$ is a $(p-k)$-vertex self switching of G and hence $G^\sigma = G^{V-\sigma}$ and $ss_k(G) =$*

$ss_{p-k}(G)$.

Proof. Since σ is a k-vertex self switching of G, $G \cong G^{\sigma}$. By lemma 2.3, $G^{\sigma} = G^{V-\sigma}$ and thereby $G \cong G^{V-\sigma}$. Hence, $V - \sigma$ is a $(p-k)$-vertex self switching of G and it is true for each σ. Hence, $G^{\sigma} = G^{V-\sigma}$ and $ss_k(G) = ss_{p-k}(G)$.

Result 3.6. *If σ is a k-vertex self switching of G, then σ is also a k-vertex self switching of G^{σ} and $(G^{\sigma})^{\sigma} = G$.*

Note 3.7. *Since for any graph G, $ss_k(G) = 1$ when $k = p$, we find the value of $ss_k(G)$ for $1 \leq k < p$.*

Theorem 3.8. *For a path P_p, $ss_k(P_p) =$*

$$\begin{cases} 1 & if\ p = 5\ and\ k = 1,4\ or\ p = 6\ and\ k = 2,4\ or\ p = 7\ and\ k = 3,4 \\ 2 & if\ p = 3\ and\ k = 1,2\ or\ p = 4\ and\ k = 2\ or\ p = 5\ and\ k = 2,3 \\ 0 & otherwise \end{cases}$$

Proof. Let P_p be a path with p vertices. Since $1 \leq k < p$, we have $p > 1$. Now, we consider the following eight cases.

Case 1. $p = 2$

By Corollary 3.3, $ss_1(P_2) = 0$.

Case 2. $p = 3$

By Remark 2.4, $ss_1(P_3) = 2$. Also, by Theorem 2.5, $ss_2(P_3) = 2$. Hence, $ss_1(P_3) = ss_2(P_3) = 2$.

Case 3. $p = 4$

By Corollary 3.3, $ss_1(P_4) = ss_3(P_4) = 0$. Also, by Theorem 2.5, $ss_2(P_4) = 2$.

Case 4. $p = 5$

Then $G = P_5 = G(v; 2P_3)$. By Theorem 2.8, P_5 has the self vertex switching v and thereby $ss_1(P_5) = 1$. The self vertex switching of P_5 is given in figure 3.3. By Theorem 3.5, $ss_4(P_5) = 1$. Also, By Theorem 2.5, $ss_2(P_5) = 2$, which implies that $ss_3(P_5) = 2$.

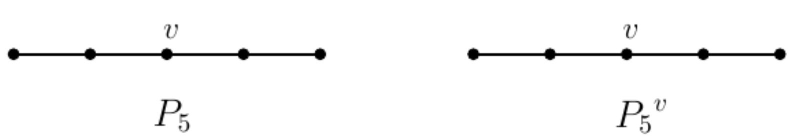

Fig 3.3

Case 5. $p = 6$

By Corollary 3.3, P_6 has no odd k-vertex self switching. Therefore, $ss_1(P_6) = ss_3(P_6) = ss_5(P_6) = 0$. Also, by Theorem 2.5, $ss_2(P_6) = 1$, which implies that $ss_4(P_6) = 1$.

Case 6. $p = 7$

By Theorem 2.10, $ss_1(P_7) = 0$, which implies that $ss_6(P_7) = 0$. Also, by Theorem 2.5, $ss_2(P_7) = 0$, which implies that $ss_5(P_7) = 0$.

Let $\sigma = \{u, v, w\}$ be a 3-vertex self switching of P_7. By Theorem 3.1, $deg(u) + deg(v) + deg(w) = \frac{k(p-k)}{2} + 2$(number of edges between u, v and w) $= 6 + 2$(number of edges between u, v and w in P_7). But, in P_7, $4 \leq deg(u) + deg(v) + deg(w) \leq 6$ and thereby the number of edges between u, v and w in P_7 should be 0 and $deg(u) + deg(v) + deg(w) = 6$. The only possibility is that $\sigma = \{v_2, v_4, v_6\}$. Clearly, ${P_7}^{\sigma} \cong P_7$ and so σ is a 3-vertex self switching of P_7. Then $ss_3(P_7) = 1 = ss_4(P_7)$.

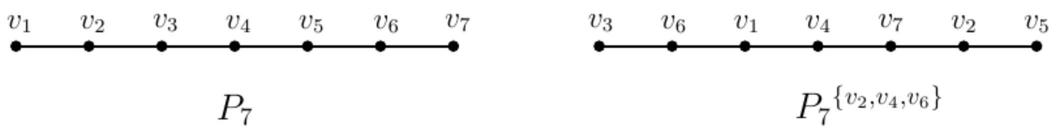

Fig 3.4

Case 7. $p = 8$

We need to find the value of ss_k for $1 \leq k \leq 4$ since $ss_k(P_8) = ss_{p-k}(P_8)$.

By Corollary 3.3, P_8 has no odd k-vertex self switching. Therefore, $ss_1(P_8) = ss_3(P_8) = 0$. Also, by Theorem 2.5, $ss_2(P_8) = 0$.

Let $k = 4$. Suppose that $\sigma = \{u, v, w, x\}$ is a 4-vertex self switching of P_8. By Theorem 3.1, $deg(u) + deg(v) + deg(w) + deg(x) = \frac{k(p-k)}{2} + 2$(number of edges between u, v, w and x) $= 8 + 2$(number of edges between u, v, w and x in P_8). But, in P_8, $6 \leq deg(u) + deg(v) + deg(w) + deg(x) \leq 8$ and thereby the number of edges between u, v, w and x in P_8 should be 0 and $deg(u) + deg(v) + deg(w) + deg(x) = 8$. But, in P_8, there is no such $\{u, v, w, x\}$ exists and thereby $ss_4(P_8) = 0$.

Hence, $ss_k(P_8) = 0$ for $1 \leq k \leq 4$, which implies that $ss_k(P_8) = 0$ for all values of k.

Case 8. $p > 8$

By Theorem 2.10, we have $ss_1(P_p) = 0$. Also, by Theorem 2.5, $ss_2(P_p) = 0$. So let us calculate the value of $ss_k(P_p)$ for $3 \leq k \leq \frac{p}{2}$ when p is even and $3 \leq k \leq \frac{p-1}{2}$ when p is odd.

Suppose $\sigma = \{v_1, v_2, \ldots, v_k\}$ is a k-vertex self switching of P_p. Then $\sum_{i=1}^{k} deg_G(v_i) \leq 2k$. But, since $p > 8$, $3 \leq k \leq \frac{p}{2}$ when p is even and $3 \leq k \leq \frac{p-1}{2}$ when p is odd, we have $p - k > 4$, which implies that $\frac{2k(p-k)}{4} > 2k$. That is, $2k < \frac{k(p-k)}{2}$. Therefore,

$\sum_{i=1}^{k} deg_G(v_i) < \frac{k(p-k)}{2}$ for $p > 8$, $3 \leq k \leq \frac{p}{2}$ when p is even and $3 \leq k \leq \frac{p-1}{2}$ when p is odd. Then by Theorem 3.1, σ cannot be a k-vertex self switching for $p > 8$, $3 \leq k \leq \frac{p}{2}$ when p is even and $3 \leq k \leq \frac{p-1}{2}$ when p is odd, which implies that σ cannot be a k-vertex self switching of P_p $(p > 8)$ for all k.

The theorem follows from the cases 1 to 8.

Remark 3.9. *Complete graph K_n has no k-vertex self switching.*

Proof. Suppose that $\sigma = \{v_1, v_2, \dots, v_k\}$ is a k-vertex switching of K_n. Then ${K_n}^{\sigma} = K_k \cup K_{n-k}$, which is a disconnected graph and so $K_n \ncong {K_n}^{\sigma}$, which is a contradiction to σ is a k-vertex self switching of K_n. This completes the proof.

Theorem 3.10. *Let G be a star graph $K_{1,n}$ $(n \geq 2)$. Then*

(i) For $n = 3$, $ss_k(G) = ss_{p-k}(G) = \begin{cases} 2n & if\ k = 2 \\ 0 & otherwise \end{cases}$

(ii) For $n \neq 3$, $ss_k(G) = ss_{p-k}(G) = \begin{cases} n & if\ k = 2 \\ 0 & otherwise \end{cases}$

Proof. Let $G = K_{1,n}$ be a star graph for $n \geq 2$ with $p = n + 1$ vertices. Let $V = V_1 \cup V_2$ where $V_1 = \{v\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$ be the bipartition of V. Clearly, $deg(v) = n$ and $deg(v_i) = 1$, $1 \leq i \leq n$ and so $DS(G) = \{1, n\}$.

Case 1. $n = 3$

Then $G = K_{1,3}$. If $k = 2$, then by Theorem 2.11, $ss_2(K_{1,3}) = 1 \times 3 + \binom{3}{2} = 6 = 2 \times 3$.

If $k \neq 2$, then by Theorem 2.12, $ss_1(K_{1,3}) = 0$. Also, by Theorem 3.5, $ss_3(K_{1,3}) = 0$. Therefore, $ss_k(K_{1,3}) = ss_{p-k}(K_{1,3}) = 0$ if $k \neq 2$.

Hence, we have, for $n = 3$,

$$ss_k(K_{1,n}) = ss_{p-k}(K_{1,n}) = \begin{cases} 2n & if\ k = 2 \\ 0 & otherwise \end{cases}$$

Case 2. $n \neq 3$

Subcase 2.1. $n = 2$

Then $G = P_3$. By Theorem 3.8, $ss_1(G) = ss_2(G) = 2$. Hence, for $k = 1, 2$, $ss_k(G) = ss_{p-k}(G) = n$.

Subcase 2.2. $n \geq 4$

Let $k = 2$. Now, star graph is a complete bipartite graph $K_{m,n}$ with $m = 1$. Then for $n \geq 4$, we have $n \neq m + 2 = 1 + 2$. By Theorem 2.11, $ss_2(G) = 1 \times n = n$. Also, by Theorem 3.5, $ss_2(G) = ss_{p-2}(G) = n$ for $n \geq 4$. Therefore, $ss_k(G) = ss_{p-k}(G) = n$ if

$k = 2$.

Let $k \neq 2$. If $k = 1$, then by Theorem 2.12, $ss_1(K_{1,n}) = 0$. Also, by Theorem 3.5, we have $ss_n(K_{1,n}) = 0$. Therefore, the result is true for $k = 1$ and $k = n$ and hence let us take σ to be a k-vertex self switching of G when $2 < k < n$.

If the vertex v of V_1 is in σ, then in G^σ, v is adjacent to the remaining $k - 1$ vertices of σ and thereby $deg_{G^\sigma}(v) = k - 1$. If $k - 1 = n$, then $k > n$, whivh is a contradiction. If $k - 1 = 1$, then $k = 2$, which is a contradiction. Hence, $deg_{G^\sigma}(v)$ is equal to neither n nor 1 and thereby $DS(G^\sigma) \neq DS(G)$. Then G^σ is not isomorphic to G and thereby σ cannot be a k-vertex self switching of G. Thus, all the vertices of σ are in V_2.

Then in G^σ, v has degree $n - k$, each vertex in σ has degree $n - k$ and each vertex in $V - \sigma$ other than v has degree $k + 1$ and thereby $G^\sigma = K_{1,n}^\sigma = K_{n-k,k+1}$. Now, $K_{n-k,k+1} = K_{1,n}$ if and only if $n - k = n$ or 1 and $k + 1 = n$ or 1 if and only if $n - k = 1$ and $k + 1 = n$ since $n - k \neq n$ and $k + 1 \neq 1$. That is, σ is a k-vertex self switching if and only if $n = k + 1$ if and only if $p - k = 2$ since $p = n + 1$. By Theorem 3.5, σ is a k-vertex self switching if and only if $k = 2$. Therefore, $ss_k(G) = ss_{p-k}(G) = 0$ for $k \neq 2$.

From both the cases, we have, for $n = 2$ and $n \geq 4$,

$$ss_k(G) = ss_{p-k}(G) = \begin{cases} n & if\ k = 2 \\ 0 & otherwise \end{cases}$$

Conclusion

In this paper, we have given sufficient condition for σ to be a k-vertex self switching. Also, we have found k-vertex self switching of path and star graphs.

Application

k-vertex self switching can be used to optimize network structures such as communication networks or transportation networks by generating isomorphic graphs and comparing their properties.

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