Some Results on k-vertex Duplication Self Switching

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Abstract

By a graph $G_1 = (V, E)$, we specify a simple finite graph. Let a graph be G_1 having $\sigma \neq \phi$ as a subset of V. The graph ${G_1}^{\sigma}$ is generated from G_1 by deleting all edges connecting σ to $V - \sigma$ and all non-present edges between two subsets σ and $V - \sigma$ are added as new edges. In case of $G_1 \cong {G_1}^{\sigma}, \sigma$ is stated to be a self switching of G_1. A self-switching σ of G_1 with $|\sigma| = k$ is also referred to as k-vertex self switching. The collection of all k-vertex self switchings of the graph is represented by $SS_k(G_1)$ and its cardinality by $ss_k(G_1)$. Duplication of a vertex v of graph G_1 results in a new graph ${G_1}'$ where a vertex v' is added and connected to the same neighbourhood as v. This paper presents essential properties for σ to be a k-vertex duplication self switching for a graph G_1 and utilizing these properties, we determine the cardinality $dss_k(G_1)$ for path P_p and complete graph K_m.

Keywords: Switching, self switching, duplication,duplication self vertex switching, path, complete graph.

Subject Classification Number: 05C60.

1 Introduction

For an undirected finite graph $G_1 = (V, E)$, the degree of vertex v in G_1 is symbolized by $\deg_{G_1}(v)$ is described as the count of incident edges on v. Seidel defined switching and provided overview of two graphs in [8] that is termed as Seidel switching. And for a simple graph $G_1 = (V, E)$ that is finite undirected with subset σ of V which is non-empty, the switching of G_1 by σ denoted by $G_1^{\sigma} = (V, E')$ is constructed from G_1 by deleting all edges connecting σ to $V - \sigma$ and inserting every non-present edges between σ and $V - \sigma$ as new edges. While σ consists of a single vertex v, the switching is specifically refered to as vertex switching denoted by G_1^v. In [6], vertex switching was initiated by Lint and Seidel. If $G_1 \cong G_1^\sigma$, then it is called self vertex switching. C. Jayasekaran introduced self vertex switching [10]. Further results on self vertex switching can be found in [3, 9]. Switching classes are discussed by A. Ehrenfeuct, J. Hage and T. Harju [1]. For details of k-vertex self switching, we refer [4]. The duplication self vertex switching was conceptualized by C. Jayasekaran and V. Prabavathy [5]. A vertex v in a graph G_1 is considered as duplication self vertex switching of G_1 if the vertex v is duplicated and the resultant graph contains a self vertex switching on v and $dss_1(G_1)$ denotes the number of such duplication self vertex switching. We rely on [2,11] for fundamental definitions.

2 Preliminaries

Definition 2.1. [9] *Let $G_1(V,E)$ be a undirected finite graph contains $\sigma \subseteq V$. The switching of G_1 by σ is explained as the graph $G_1^\sigma(V,E')$ that is generated from G_1 by deleting the existing edges connecting σ to $V-\sigma$ and inserting the non-edges between σ and $V-\sigma$ as new edges. Whenever σ contains a single vertex v, the resulting switching $G_1^{\{v\}}$ is termed as vertex switching and is symbolized by G_1^v.*

Definition 2.2. [10] *$\sigma \subseteq V(G_1)$ is considered as a self switching of G_1 if $G_1 \cong G_1^\sigma$. $SS_k(G_1)$ denotes the set of all self switchings of G_1 with cardinality k and $ss_k(G_1)$ denotes the cardinality of $SS_k(G_1)$.*
Self switching is termed as self vertex switching when $k=1$.

Definition 2.3. [5] *Duplication of a vertex v of a graph G_1 generates a graph $G_1{}'$ by inserting a vertex v' so that Neighbourhood of v' is same as the Neighbourhood of v. $D(vG_1)$ denotes the graph generated after duplication of v. For example, the graph $G_1 = P_3$ and the duplication of each vertex of P_3 are given in the figures 2.1 to 2.4.*

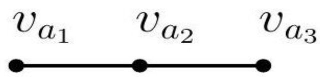


Fig. 2.1. $G_1 = P_3$

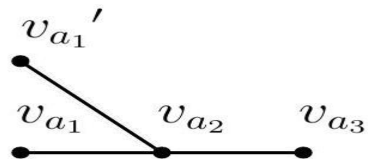


Fig. 2.2. $D\left(v_{a_1}G_1\right)$

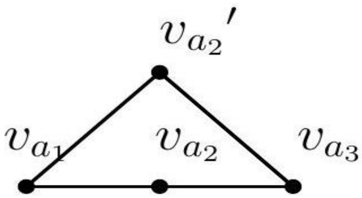


Fig. 2.3. $D\left(v_{a_2}G_1\right)$

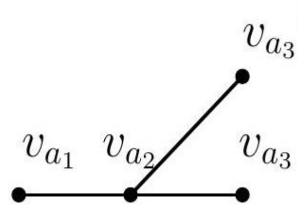


Fig. 2.4. $D\left(v_{a_3}G_1\right)$

Definition 2.4. [5] *A vertex v is termed as duplication self vertex switching of a graph G_1 if the vertex v is duplicated and the graph generated after duplication contains a self vertex switching on v. $dSS_1(G_1)$ denotes the set of all duplication self vertex switching and $dss_1(G_1)$ denotes the cardinality of $dSS_1(G_1)$.*

Theorem 2.5. [10] *Let $G_1(V, E)$ be a graph and let $\sigma \subset V$ be a self switching of G_1. Then the number of edges between σ and $V - \sigma$ in G_1 is $\frac{k(p-k)}{2}$ where $k = |\sigma|$.*

Theorem 2.6. [7] $dss(P_p) = \begin{cases} 2 & if\, p = 2,4 \\ 0 & otherwise \end{cases}$.

Theorem 2.7. [7] $dss(C_n) = \begin{cases} 4 & if\, n = 4 \\ 0 & otherwise \end{cases}$.

3 Main Results

Definition 3.1. *A k-vertex duplication of a graph G_1 generates a new graph $G_1{}'$ by inserting new k vertices $u_1{}', u_2{}', \dots, u_k{}'$ as the duplication of any k vertices $u_1, u_2, \dots, u_k$ of G_1 such that $N(u_i) = N(u_i{}')$, where $i = 1,2,3, \dots, k$. The graph obtained by duplicating the k vertices $u_1, u_2, \dots, u_k$ is denoted by $D\left((u_1, u_2, \dots, u_k)G_1\right)$. If $\sigma = \{u_1, u_2, u_3, \dots, u_k\} \subseteq V(G_1)$, then the duplication of G_1 by σ is denoted by $D(\sigma G_1)$.*

Definition 3.2. *Let $\sigma \subseteq V(G_1)$ be such that $|\sigma| = k$. Then σ is termed as k-vertex duplication self switching of graph G_1 if $D(\sigma G_1) \cong D(\sigma G_1)^\sigma$ where $D(\sigma G_1)$ is the duplication graph of G_1 by σ and $D(\sigma G_1)^\sigma$ is the switching graph of $D(\sigma G_1)$ by σ.*

The set of all k-vertex duplication self switchings of G_1 is denoted by $dSS_k(G_1)$ and the $dss_k(G_1)$ denotes the cardinality of $dSS_k(G_1)$.

Example 3.3. *Refer the graph $G_1 = P_4$ given in the figure 3.1. Let $\sigma = \{u_\gamma, v_\gamma\} \subseteq V(G_1)$. The 2-vertex duplication $D(\sigma G_1)$ of the graph G_1 shown in the figure 3.2 and the graph $D(\sigma G_1)^\sigma$ is shown in the figure 3.3 imply that $D(\sigma G_1) \cong D(\sigma G_1)^\sigma$. Henceforth, σ is a duplication self switching of G_1 on 2 vertices.*

Fig. 3.1. G_1

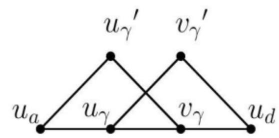


Fig. 3.2. $D(\sigma G_1)$

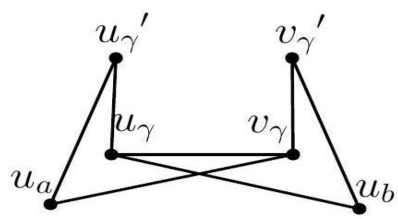


Fig. 3.3. $D(\sigma G_1)^\sigma$

Theorem 3.4. *Let G_1 be a (p, q) graph and let $\sigma \subseteq V(G_1)$ where $|\sigma| = k$. Then $D(\sigma G_1)$ is a $\left(p + k, q + \sum_{u \in \sigma} deg_{G_1}(u)\right)$ graph.*

Proof. Let G_1 be a (p, q) graph. Let $\sigma = \{u_{a_1}, u_{a_2}, \dots, u_{a_k}\} \subseteq V(G_1)$. By Definition 3.1, $D(\sigma G_1)$ is the duplication graph attained by inserting k duplication vertices $v_{a_1}', v_{a_2}', \dots, v_{a_k}'$ to the graph G_1. Hence, $\left|V\left(D(\sigma G_1)\right)\right| = p + k$. $\left|E\left(D(\sigma G_1)\right)\right| = |E(G_1)| +$ the count of edges added after duplication of the k vertices. By Definition 3.1, $N\left(v_{a_i}\right) = N\left(v_{a_i}'\right)$ in $D(\sigma G_1)$ where $i = 1$ to k. Thus, the number of edges added after duplication of the k vertices $= \sum_{v_a \in \sigma} \deg_{G_1}(v_a)$. Therefore, $\left|E\left(D(\sigma G_1)\right)\right| = q + \sum_{u_a \in \sigma} \deg_{G_1}(u_a)$. Hence the desired result.

Result 3.5. *Let $u \in \sigma \subseteq V(G_1)$ with $|\sigma| = k$. Then $deg_{G_1}(u) = deg_{D(\sigma G_1)}(u')$.*

Proof. By Definition 3.1, $N(u) = N(u')$ in $D(\sigma G_1)$. This means that the vertices connected to u in G_1 are connected to u' in $D(\sigma G_1)$. That is, $\deg_{G_1}(u) = \deg_{D(\sigma G_1)}(u')$.

Theorem 3.6. *If a graph G_1 is with n components, then $D(\sigma G_1)$ also has n components.*

Proof. Let G_1 be a graph with n components and $\sigma = \{v_{a_1}, v_{a_2}, \dots, v_{a_k}\} \subseteq V(G_1)$. Assume $v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'$ is the duplication vertices of $v_{a_1}, v_{a_2}, \dots, v_{a_k}$, respectively. As G_1 has n components, v_{a_i} 's are either in one component or in different components. Suppose v_{a_i} and v_{a_j} are in two distinct components of G_1. Let v_{a_i} be in the component C_1 and v_{a_j} be in the component C_2 different from C_1. The duplication $v_{a_i}{}'$ of v_{a_i} must be adjacent to the vertices of $N(v_{a_i})$ and so $v_{a_i}{}'$ lies in a component which contains C_1. Similarly, the duplication $v_{a_j}{}'$ of v_{a_j} must be adjacent to the vertices of $N\left(v_{a_j}\right)$ and so the duplication vertex $v_{a_j}{}'$ lies in the component which contains C_2. As a result, v_{a_i} and its duplication vertex $v_{a_i}{}'$ are in the same component and so the duplication graph $D(\sigma G_1)$ of G_1 by σ remains disconnected with n components.

Theorem 3.7. *For any disconnected graph $G_1, dss_k(G_1) = 0$.*

Proof. Since G_1 is disconnected, G_1 has at least two components, say $C_1, C_2, \dots C_n, n \geq 2$. Consider a k-vertex duplication switching, $D(\sigma G_1)$ where $\sigma = \{v_{a_1}, v_{a_2}, \dots, v_{a_k}\}$. Assume $v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'$ is the duplication vertices of $v_{a_1}, v_{a_2}, \dots, v_{a_k}$, respectively. By Theorem 3.6, $D(\sigma G_1)$ has n components. In $D(\sigma G_1), v_{a_i}$ is non-adjacent to $v_{a_i}{}'$ and all the vertices of other components. Thus in $D(\sigma G_1)$, v_{a_1} is not adjacent to minimum one vertex in every component. This means that $D(\sigma G_1)^\sigma$ is connected and so $D(\sigma G_1) \not\cong D(\sigma G_1)^\sigma$ which contradicts our assumption, $\sigma = \{v_{a_1}, v_{a_2}, \dots, v_{a_k}\}$ is a duplication self switching on k vertices of G_1. As a result, $dss_k(G_1) = 0$.

Theorem 3.8. *Let G_1 be a graph and $\sigma \subseteq V$ be a k-vertex duplication self switching of G_1. Then the count of edges linking σ and $V\big(D(\sigma G_1)\big) - \sigma$ in $D(\sigma G_1)$ is $\frac{kp}{2}$ where $k = |\sigma|$.*

Proof. Assume σ is a duplication self switching of G_1 on k vertices. Accordingly, $D(\sigma G_1) \cong D(\sigma G_1)^{\sigma}$. By Theorem 3.4, $D(\sigma G_1)^{\sigma}$ is a graph with $p + k$ vertices. Hence by Theorem 2.5, the count of edges linking of σ and $V\big(D(\sigma G_1)\big) - \sigma$ in $D(\sigma G_1)$ is $\frac{k(p+k-k)}{2} = \frac{kp}{2}$.

Theorem 3.9. *Let G_1 be a graph with σ as a k-vertex duplication self switching of G_1 and σ' as the set of duplication vertices of σ. Then the count of edges linking σ and σ' in $D(\sigma G_1)$ is 2 (count of edges linking the points of σ in G_1).*

Proof. Assume $\sigma = \{v_{a_1}, v_{a_2}, \dots, v_{a_k}\} \subseteq V(G_1)$ is a k-vertex duplication self switching of G_1 and $\sigma' = \{v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'\}$ where $v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'$ are the duplication vertices of $v_{a_1}, v_{a_2}, \dots, v_{a_k}$, respectively. By the definition of duplication, each edge between the vertices of σ in G_1 contributes 2 edges between the vertices of σ and σ' in $D(\sigma G_1)$. Hence, the count of edges linking σ and σ' in $D(\sigma G_1)$ is 2 (count of edges linking the points of σ in G_1).

Theorem 3.10. *Let G_1 be a graph and σ be a k-vertex duplication self switching of G_1. Then the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1)$ is $\frac{kp}{2} - 2$ (count of edges linking the points of σ in G_1).*

Proof. Assume $\sigma = \{v_{a_1}, v_{a_2}, \dots, v_{a_k}\} \subseteq V(G_1)$ is a duplication self switching of G_1 on k vertices and $\sigma' = \{v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'\}$ in which $v_{a_1}{}', v_{a_2}{}', \dots, v_{a_k}{}'$ are the duplication vertices of $v_{a_1}, v_{a_2}, \dots, v_{a_k}$, respectively. Obviously, $V\big(D(\sigma G_1)\big) = V(G_1) \cup \sigma' = (V(G_1) - \sigma) \cup \sigma \cup \sigma'$ implies that $V\big(D(\sigma G_1)\big) - \sigma = \sigma' \cup (V(G_1) - \sigma)$. Hence, the count of edges linking σ and $V\big(D(\sigma G_1)\big) - \sigma$ in $D(\sigma G_1) =$ the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1) +$ the count of edges linking σ and σ' in $D(\sigma G_1)$. That is, the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1) =$ the count of edges linking σ and

$V\big(D(\sigma G_1)\big) - \sigma$ in $D(\sigma G_1)$ the count of edges linking σ and σ' in $D(\sigma G_1)$. By Theorem 3.8, the count of edges linking σ and $V\big(D(\sigma G_1)\big) - \sigma$ in $D(\sigma G_1) = \frac{kp}{2}$. By Theorem 3.9, the count of edges linking σ and σ' in $D(\sigma G_1) = 2$ (the count of edges linking the points of σ in G_1). Hence, the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1)$ is $\frac{kp}{2} - 2$ (count of edges linking the points of σ in G_1).

Observation 3.11. *Refer graph G_1 illustrated in figure 3.4. The graph ${G_1}^{\sigma}$ is illustrated in figure 3.5. Undoubtedly, ${G_1}^{\sigma}$ is the union of two induced subgraphs of ${G_1}^{\sigma}$ namely ${G_1}^{\sigma}[\sigma]$ and ${G_1}^{\sigma}[V - \sigma]$ together with the edges joining σ and $V - \sigma$ in ${G_1}^{\sigma}$.*

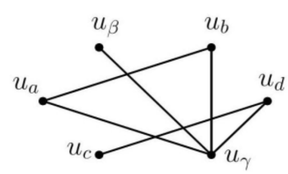


Fig 3.4. G_1

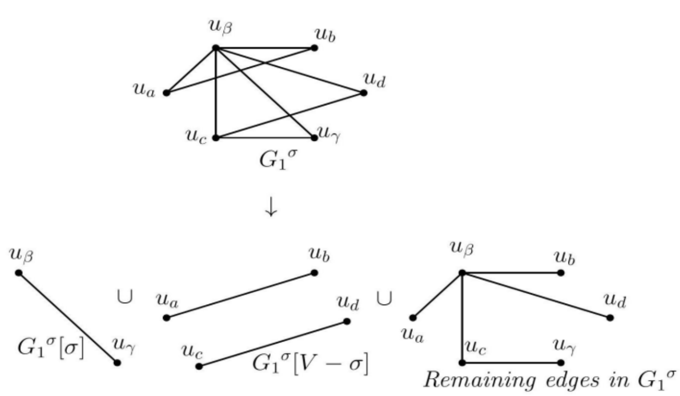


Fig 3.5

Theorem 3.12. *Let G_1 be a graph and $\sigma \subseteq V(G_1)$ where $|\sigma| = k$. If σ is a k-vertex duplication self switching of the graph G_1, then $\sum_{u\in\sigma} deg_{G_1}(u) = \frac{kp}{2}$.*

Proof. Let $\sigma = \{v_{a_1}, v_{a_2}, \dots., v_{a_k}\} \subseteq V(G_1)$ be a k-vertex duplication self switching of the graph G_1. Accordingly, $D(\sigma G_1) \cong D(\sigma G_1)^\sigma$ and thereby $|E(D(\sigma G_1))| = |E(D(\sigma G_1)^\sigma)|$. By Theorem 3.4, $| E(D(\sigma G_1) | = q + \sum_{u\in\sigma} \deg_{G_1}(u)$.

Let $\sigma' = \{v_{a_1}', v_{a_2}', \dots, v_{a_k}'\}$ in which $v_{a_1}', v_{a_2}', \dots, v_{a_k}'$ are the duplication vertices of $v_{a_1}, v_{a_2}, \dots., v_{a_k}$, respectively.

Now, $|E(D(\sigma G_1)^\sigma)|$ = the count of edges linking the points of σ in $D(\sigma G_1)^\sigma$ + the count of edges linking the points of $V(G_1) - \sigma$ in $D(\sigma G_1)^\sigma +$ the count of edges linking the points of σ' in $D(\sigma G_1)^\sigma +$ the count of edges linking σ' and $V(G_1) - \sigma$ in $D(\sigma G_1)^\sigma +$ the count of edges linking σ and σ' in $D(\sigma G_1)^\sigma$ + the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1)^\sigma$.

Obviously, the count of edges linking the points of σ in $G_1, D(\sigma G_1)$ and $D(\sigma G_1)^\sigma$ are equal. Also, the count of edges linking the points of $V(G_1) - \sigma$ in $G_1, D(\sigma G_1)$ and $D(\sigma G_1)^\sigma$ are equal and the count of edges linking the points of σ' in $D(\sigma G_1)$ and $D(\sigma G_1)^\sigma$ is 0 .

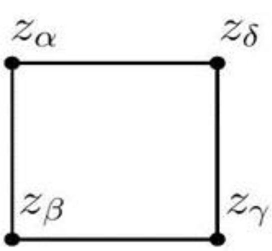
Since $N(u_i) = N(u_i')$ where $1 \leq i \leq k$, the count of edges linking σ' and $V(G_1) - \sigma$ in $D(\sigma G_1)^\sigma =$ the count of edges linking σ' and $V(G_1) - \sigma$ in $D(\sigma G_1) =$ the count of edges linking σ and $V(G_1) - \sigma$ in G_1.

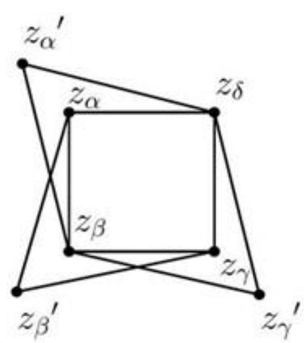
The count of edges linking σ and σ' in $D(\sigma G_1)^\sigma$ is obviously equal to the count of non-edges between the points of σ and σ' in $D(\sigma G_1) =$ all possible edges linking σ and σ' in $D(\sigma G_1)$-the count of edges linking σ and σ' in $D(\sigma G_1) = k^2 - 2$ (count of edges linking the points of σ in G_1)(by Theorem 3.9).

It is clear that the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1)^\sigma =$ the count of non-edges between the points of σ and $V(G_1) - \sigma$ in $D(\sigma G_1) =$ all possible edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1)$-the count of edges linking σ and $V(G_1) - \sigma$ in $D(\sigma G_1) = k(p - k) - \left[\frac{kp}{2} - 2\right.$ (count of edges linking the points of σ in G_1)] (by Theorem 3.10).

Thus, (2) indicates that $|E(D(\sigma G_1)^{\sigma})|$ = the count of edges linking the points of σ in G_1 + the count of edges linking the points of $V(G_1) - \sigma$ in $G_1 + 0$ + the count of edges linking σ and $V(G_1) - \sigma$ in $G_1 + k^2$-2(count of edges linking the points of σ in G_1) $+ k(p-k) - \left\{\frac{kp}{2} - 2\right.$ (the count of edges linking the points of σ in G_1)} = the count of edges linking the points of σ in G_1 + the count of edges linking the points of $V(G_1) - \sigma$ in G_1 + the count of edges linking σ and $V(G_1) - \sigma$ in $G_1 + k^2 - 2$ (count of edges linking the points of σ in G_1) $+ kp - k^2 - \frac{kp}{2} + 2$ (the count of edges linking the points of σ in G_1) = the count of edges in G_1(by Observation 3.11) $+kp - \frac{kp}{2} = q + \frac{kp}{2}$. Since $\left|E\left(D(\sigma G_1)\right)\right| = |E(D(\sigma G_1)^{\sigma})|$, from (1) and (3) we get, $q + \sum_{u\in\sigma} \deg_{G1}(u) = q + \frac{kp}{2}$ which implies that $\sum_{u\in\sigma} \deg_{G_1}(u) = \frac{kp}{2}$. Hence the desired result.

Remark 3.13. *The above theorem does not hold for its converse. For example, refer the graph* $G_1 = C_4$ *with 4 vertices given in the figure 3.6 and let* $\sigma = \{z_\alpha, z_\beta, z_\gamma\}$. *Then* $deg_{G_1}(z_\alpha) + deg_{G_1}(z_\beta) + deg_{G_1}(z_\gamma) = 6 = \frac{3\times4}{2} = \frac{kp}{2}$. *The graphs* $D(\sigma G_1)$ *and* $D(\sigma G_1)^{\sigma}$ *are given in figure 3.7 and figure 3.8 respectively shows that* $D(\sigma G_1) \subsetneqq D(\sigma G_1)^{\sigma}$. *Thus, the converse of the above theorem does not hold.*





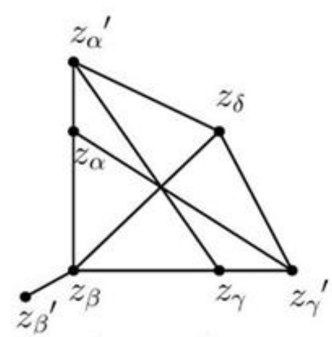


Fig. 3.6. $G_1 = C_4$ Fig. 3.7. $D\left((z_\alpha, z_\beta, z_\gamma)G_1\right)$ Fig. 3.8. $D\left((z_\alpha, z_\beta, z_\gamma)G_1\right)^{\sigma}$

Corollary 3.14. *A graph of odd order has no odd order duplication self switching.*

Proof. Let a graph be G_1 with order p. Suppose $\sigma \subseteq V(G_1)$ is a duplication self switching of G_1 where $|\sigma| = k$ is odd. By Theorem 3.12, $\sum_{u\in\sigma} deg_{G_1}(u) = \frac{kp}{2}$. Since $\frac{kp}{2}$ is an integer and p is odd, k must be even which contradicts k is odd. Hence the desired result.

Theorem 3.15. *For* $p > 1$,

$$dss_k(P_p) = \begin{cases} 1 & if\, k = 2\ and\ p = 2\ or\ 4 \\ 2 & if\, k = 1\ and\ p = 2\ or\ 4;\, k = 2\ and\ p = 3. \\ 0 & otherwise \end{cases}.$$

Proof. Let $v_{a_1}v_{a_2} \dots . v_{a_p}$ be the path P_p with two end vertices with degree 1 and the rest with degree 2. Let $\sigma \subseteq V$ and $|\sigma| = k$. Then clearly, $2k - 2 \leq \sum_{u_a\in\sigma} \deg_{G_1}(u_a) \leq 2k$.

To prove the required results, we look at the following two cases, $p \leq 4$ and $p > 4$

Case 1. $p \leq 4$

Obviously, $\frac{kp}{2} \leq \frac{4k}{2} = 2k$ and $\sum_{u_a\in\sigma} \deg_{G_1}(u_a) \leq 2k$. This implies that $\sum_{u_a\in\sigma} \deg_{G_1}(u_a)$ may be equal to $\frac{kp}{2}$ in certain situations. Now consider the following three subcases: $p = 2$,3, and 4.

Subcase 1.a. $p = 2$

By Theorem 2.6, $dss_1(P_2) = 2$. When $k = 2, \frac{kp}{2} = 2 = \sum_{i=1}^{k} \deg_{G_1}(v_{a_i})$. By Theorem 3.12, σ may be a 2-vertex duplication self switching. Refer the graph P_2 given in figure

3.9. Clearly, $\sigma = \{v_{a_1}, v_{a_2}\}$. Let $v_{a_1}{}'$ and $v_{a_2}{}'$ be the corresponding duplications of v_{a_1} and v_{a_2}. The graphs $D(\sigma G_1)$ and $D(\sigma G_1)^{\sigma}$ are given in figure 3.9. Clearly, $D(\sigma G_1) \cong D(\sigma G_1)^{\sigma}$. Henceforth, $\sigma = \{v_{a_1}, v_{a_2}\}$ is a duplication self switching of P_2 on 2 vertices and so $dss_2(P_2) = 1$.

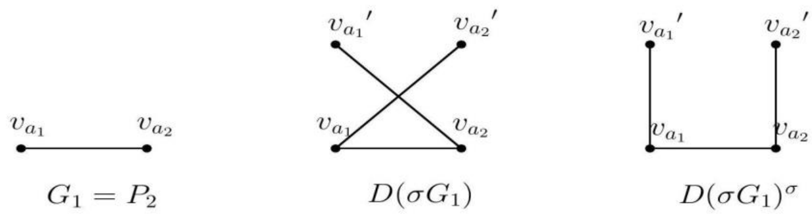


Fig. 3.9

Subcase 1.b. $p = 3$

By Corollary 3.14, $dss_1(P_3) = dss_3(P_3) = 0$. When $k = 2, \frac{kp}{2} = 3$. Refer the graph P_3 given in figure 3.10. Now, σ can be either $\{v_{a_1}, v_{a_3}\}$ for which $\deg_{P_3}(v_{a_1}) + \deg_{P_3}(v_{a_3}) = 2$ or $\{v_{a_1}, v_{a_2}\}$ for which $\deg_{P_3}(v_{a_1}) + \deg_{P_3}(v_{a_2}) = 3$. By Theorem 3.12, $\sigma = \{v_{a_1}, v_{a_2}\}$ might be a duplication self switching of P_3 on 2 vertices. Take $v_{a_1}{}'$ and $v_{a_2}{}'$ as the duplications of v_{a_1} and v_{a_2}, respectively.

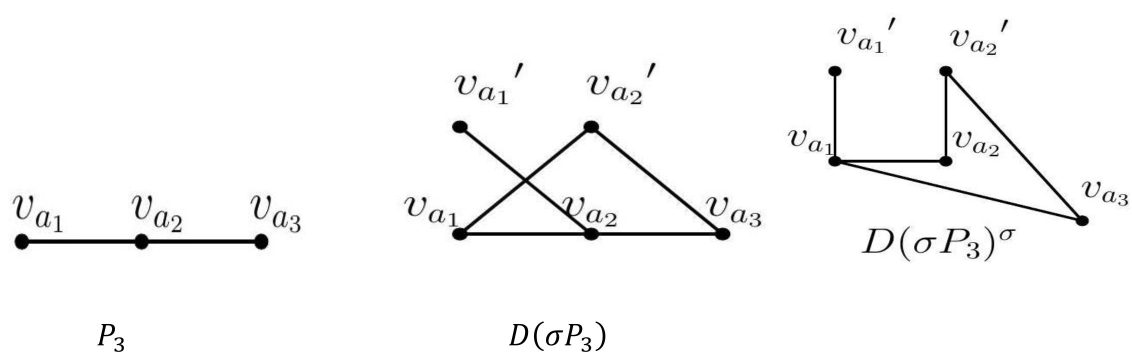
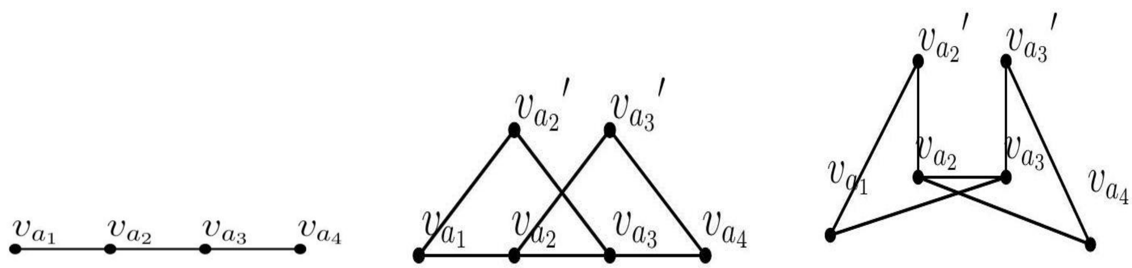


Fig. 3.10

Figure 3.10 indicates that $D(\sigma P_3) \cong D(\sigma P_3)^\sigma$. Henceforth, $\sigma = \{v_{a_1}, v_{a_2}\}$ is a duplication self switching of P_3 on 2 vertices. Similarly, $\sigma = \{v_{a_2}, v_{a_3}\}$ is also a duplication self switching of P_3 on 2 vertices. As there are two such possible pairs, $dss_2(P_3) = 2$.

Subcase 1.c. $p = 4$

By Theorem 2.6, $dss_1(P_4) = 2$. When $k = 2, \frac{kp}{2} = 4$. Let $\sigma = \{v_{a_i}, v_{a_j}\} \subseteq V(P_4)$. Then $\deg_{P_4}(v_{a_i}) + \deg_{P_4}(v_{a_j}) = 2$ or 3 or 4. By Theorem 3.12, σ might be a 2 -vertex duplication self switching only when $\sum_{u_a \in \sigma} \deg_{P_4}(u_a) = 4$. Refer the graph P_4 given in figure 3.11. The only possibility for σ is $\{v_{a_2}, v_{a_3}\}$ for which $\sum_{u_a \in \sigma} \deg_{P_4}(u_a) = 4$. Let v_{a_2}' and v_{a_3}' be the duplications of v_{a_2} and v_{a_3}, respectively. The graphs $D(\sigma G_1)$ and $D(\sigma G_1)^\sigma$ are given in figure 3.11.



P_4 $D(\sigma P_4)$ $D(\sigma P_4)^\sigma$

Fig. 3.11

From figure 3.11, we see that $D(\sigma P_4) \cong D(\sigma P_4)^\sigma$. Henceforth, $\sigma = \{v_{a_2}, v_{a_3}\}$ is a duplication self switching of P_4 on k vertices. As there is only one such pair, $dss_2(P_4) = 1$.

For $k = 3, \frac{kp}{2} = 6$ and there does not exist a set σ where $|\sigma| = k = 3$ for which $\sum_{u_a \in \sigma} deg_{P_4}(u_a) = 6$ and hence $dss_3(P_4) = 0$. Also, when $k = 4, \frac{kp}{2} = 8$ and there does not exist a set σ where $|\sigma| = 4$ for which $\sum_{u_a \in \sigma} \deg_{P_4}(u_a) = 8$. Hence, $dss_4(P_4) = 0$

Case 2. $p > 4$

Let $\sigma \subseteq V(G)$ be such that $|\sigma| = k$. Now, $\sum_{u_a \in \sigma} \deg_G(u_a) \leq 2k < \frac{kp}{2}$. By Theorem 3.12, σ can't be a k-vertex duplication self switching of P_p.

Based on the foregoing discussions, we conclude that

$$dss_k(P_p) = \begin{cases} 1 & \text{if k = 2 and p = 2 or 4} \\ 2 & \text{if k = 1 and p = 2 or 4; k = 2 and p = 3.} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.16. *For* $m \geq 3$, $dss_k(K_m) = 0$.

Proof. Assume K_m is a complete graph having m vertices. Let $\sigma \subseteq V(K_m)$ be such that $|\sigma| = k$. Then $\sum_{u \in \sigma} \deg_{K_m}(u) = k(m-1) \neq \frac{km}{2}$. By Theorem 3.12, $dss_k(K_m) = 0$.

Applications

k-vertex duplication self-switching in graph theory is mainly used in areas such as graph isomorphism testing, detecting graph automorphisms, and analyzing the structural properties of graphs.

Conclusion

In this paper, we found the conditions for σ to be a k-vertex duplication self switching for a graph G_1 and using this, we determined the cardinality $dss_k(G_1)$ for path P_p and complete graph K_m.

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