Advanced Optimization Techniques for Solution of Integer Quadratic Programming Problems via State Variables Reduction

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Abstract: This paper addresses critical challenges in nonlinear programming and integer quadratic programming problems (IQPPs) by presenting innovative solution methodologies. It introduces advanced decision-variable reduction techniques that optimize solutions by minimizing the number of state variables. The study establishes necessary and sufficient conditions for IQPPs and proposes strategies to identify and eliminate dominated terms in problem formulations. The variable reduction approach is further refined through an in-depth analysis of problem data and upper bounds, allowing certain variables to be fixed at zero. Comprehensive computational analysis demonstrates the efficiency of these methods across diverse IQPP scenarios. Furthermore, the paper delves into separable IQPPs, providing a streamlined framework to enhance understanding and facilitate intuitive problem-solving. MATLAB-based simulations and graphical representations validate the practical applicability and robustness of the proposed techniques.

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Keywords: Nonlinear programming problems, Quadratic programming, Simplex methods, Separable programming algorithms.

1. Introduction

Integer quadratic programming problems (IQPPs) represent a class of optimization challenges defined by a quadratic objective function and decision variables restricted to integer values. These problems are critical in fields such as engineering, economics, and operations research, where discrete decision-making is required. Recent advancements in IQPP research have focused on addressing computational complexities through innovative solution methodologies, including exact algorithms, heuristics, and metaheuristics. Mixed-integer quadratic programming (MIQP), which involves both integer and continuous variables, has also gained attention, with effective approaches such as decomposition techniques, branch-and-bound algorithms, and reformulation strategies emerging as key solutions. Research in this area prioritizes the development of efficient, scalable methods that balance solution quality with computational feasibility, tackling real-world optimization and computational intelligence challenges ([1], [2]).
Optimization problems, in general, are mathematical models designed to identify optimal solutions from a set of feasible alternatives, typically by minimizing or maximizing an objective function under specific constraints. Such problems find widespread applications in disciplines like engineering, economics, operations research, and machine learning. Many real-world scenarios—such as capacity planning, material

cutting, and logistics network design—can be formulated as IQPPs, which are often high-dimensional and computationally intensive. Addressing these challenges necessitates innovative methods for solving large-scale IQPPs efficiently ([3], [4], [5]).

Variable reduction techniques have emerged as a promising approach for tackling complex, large-scale optimization problems. A notable contribution to this area was made by Babayev and Mardanov in 1994, who introduced a novel method comparing pairs of columns in the constraint matrix for integer programming problems. This method significantly reduces the number of integer variables in various problems, including Knapsack Problems (KP), Multi-dimensional KPs, and general integer programming problems. The authors provided specific conditions under which a variable could be fixed at zero in the optimal solution and validated their approach through empirical studies on di-verse datasets. Lawrence J. Watters also advanced the field by reducing Integer Polynomial Programming Problems to Zero-One Linear Programming Problems, contributing further to variable reduction techniques ([6], [7]).

Applying variable reduction methods before implementing the Hashian algorithm has been shown to decrease prob-lem complexity and mitigate overflow risks associated with direct application. However, many existing techniques pri-marily focus on linear or binary optimization problems ([8], [9]). In 2007, Hua proposed a variable reduction approach tailored to convex integer quadratic programming problems (IQPPs), leveraging continuous relaxation values and fea-sible solutions to identify reducible variables. Zhu and Broughan expanded this work by establishing necessary and sufficient conditions for identifying reducible variables in general integer linear programming matrices, advancing the understanding of reduction strategies ([10], [11]). Further contributions include Sun and Gu's work on nonlinear integer programming for postal design problems and Billionnet and Soutif's exact Lagrangian decomposition technique for 0–1 quadratic knapsack problem ([12], [13]). Wang have established the path relinking for unconstrained binary quadratic programming in his article in 2012 and Kumar et al have discussed the advanced solution technique of quadratic programming problems with neural network modeling in 2024 ([14], [15]).

Building upon these foundational advancements, this paper introduces a novel variable reduction technique for general integer quadratic programming problems (GIQPPs). The proposed method enables certain variables to be fixed at zero while maintaining the optimality of the solution. Specific conditions for identifying removable decision variables in quadratic integer programming problems are presented, with criteria established based on problem data analysis and variable upper bounds. The effectiveness of these conditions is validated through extensive computational experiments using MATLAB and is illustrated with graphical representations of quadratic programming problems. This innovative approach contributes to the ongoing development of efficient, scalable optimization solutions for GIQPPs.

The paper is organized as follows: Section 1 is an introduction to the topic, setting the context for the study. Section 2 explores the fundamental concepts of nonlinear and quadratic programming problems. Section 3 presents the derivation of necessary and sufficient conditions for identifying dominated terms and offers a detailed explanation of the separable technique for integer quadratic programming problems. Section 4 assesses the effectiveness of the proposed technique through computational experiments on randomly generated GIQPPs and separable IQPPs. Finally, Section 5 concludes the paper, summarizing key findings and implications.

2 Mathematical Formulation of General Quadratic Programming Problems

2.1 An Overview of Nonlinear Programming:

The general nonlinear programming problem describes as:

Optimize (max or min) $Z = f(x) = f(x_j)$;

subject to constraints, $g(x) = g_i(x_i) \leq b_i, \quad i = 1,2,\ldots,m;$

$= b_i \quad i = 1,2,\ldots,m;$ (1)

$\geq b_i,$

$i = 1,2,\ldots,m;$

and $x_i = (x_1, x_2,\ldots,x_n); \forall_j \geq 0, j = 1,2,\ldots,n;$

where $f(x) = f(x_i)$ is real valued nonlinear objective function of n decision variables and $g(x) = g_i(x_i)$ is real-valued functions of n decision variables.

2.2 Necessary Kuhn-Tucker Conditions of Nonlinear Programming Problems (NLPs)

Maximize $Z = f(x)$,

subject to the constraints: $g_i(x) = 0, \; i = 1,2,\ldots,m;$

and $x = x_i, \geq 0$ for all i. (2)

In a nutshell, this is written as:.

1. $\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} g_i = 0, j = 1,2,\ldots,n$
2. $\sum_{i=1}^{m} \lambda_i g_i = 0, \quad i = 1,2,\ldots,m;$
3. $\sum_{i=1}^{m} \lambda_i g_i \leq 0, \quad i = 1,2,\ldots,m;$
4. $\lambda_i \geq 0$

Note: If $\lambda \leq 0$, these conditions also apply to minimization of nonlinear programming (NLP) problems. The non negativity conditions $x = (x_1, x_2 \ldots\ldots x_n) \geq 0$ is taken for all these conditions 1-4. This represents the feasibilities conditions.

The Kuhn-Tucker conditions for maximization NLP problem is rewritten as:

Maximize Z = f(x),

subject to the constraints $g_i \leq 0,$ (3)

and $-x \leq 0, \; i = 1,2,\ldots,m;$

where $x = (x_1, x_2 \ldots\ldots x_n)$

In this NLP problem, taking the m+n inequalities into equations. Take the m+n slack variables $s_i^2 \geq 0$ ($i = 1,2,\ldots,m,m+1,\ldots,m+n$) as:

$g_i(x) + s_i^2 = 0, \qquad i = 1,2,\ldots,m;$

$-x_j + s_{m+j,}^2 \quad j = 1,2,\ldots,n$

The Kuhn-Tucker necessary conditions for the maximum of $f(x)$ is obtained as:

1. $\frac{\partial}{\partial x_j} f(x) = \lambda \frac{\partial}{\partial x_j} g_i(x),\ \sum_{i=1}^{m} \lambda_i\, g_i = \lambda_{m+j}, j\ = 1,2,\ldots,m$
2. $\lambda_i g_i(x) = 0,\ \ i = 1,2,\ldots\ldots,m$
3. $\lambda_{m+j} x_j = 0$
4. $g_i(x) \le 0,\ \lambda_i, \lambda_{m+j}, x_j \ge 0,\ for\ all\ i$ and j.

The Kuhn-Tucker necessary conditions is taken into sufficient conditions when f(x) is concave and $g_i(x)$ is convex with respect to x. For minimization NLPPs, f(x) is taken as convex, while gi(x) is taken as concave in relation to x.

Lagrangian function is rewritten as:

$$L(x,s,\lambda) = f(g_i) - \sum_{i=1}^{m} \lambda_i\, s_i^2$$

Where $\lambda = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_m)T$ is the vector of Lagrange multiplier. The necessary conditions for an extreme point to be local optimum (max or min) can be obtained by solving the following equations:

$$\frac{\partial}{\partial x_j} L = \lambda \frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \frac{\partial}{\partial x_j} g_i(x)\, \lambda = 0, \qquad j\ = 1,2,\ldots,n$$

$$\frac{\partial}{\partial x_j} L = -[g_i(x) + s_i^2] \quad i = 1,2,\ldots\ldots,n$$

$$\frac{\partial}{\partial x_j} L = -2x_i\lambda_i, \quad i = 1,2,\ldots\ldots,m$$

Thus, the Kuhn-Tucker necessary conditions to be satisfied at a local optimum (max or min) point is stated as follows:

$$\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \frac{\partial}{\partial x_j} g_i(x)\, \lambda = 0, \qquad j\ = 1,2,\ldots,n$$

$$\lambda_i g_i(x) = 0$$

$$g_i(x) \le 0,$$

$$\lambda_i \ge 0, \qquad i = 1,2,\ldots\ldots,m$$

Remark If the provided nonlinear programming (NLP) problem is a minimization problem or if the constraints are of the form $g_i(x) \ge 0$, then $\lambda_i \le 0$. Conversely, if the NLP problem is a maximization problem with constraints of the form $g_i(x) \le 0$, then $\lambda_i \ge 0$.

Kuhn-Tucker Sufficient Conditions:

Theorem 2.1 The Kuhn-Tucker necessary conditions for the problem.

Maximize $Z = f(x)$,

subject to the constraints: $g_i(x) \le 0$, $i = 1,2,\ldots,m;\ x \ge 0,$

are also sufficient conditions if $f(x)$ is concave and all $g_i(x)$ are convex functions of x.

2.3 Quadratic Programming Problems (QPP)

The mathematical modeling of quadratic programming (QP) problems is expressed as:

Optimize (Maximize or Minimize) $Z = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk} x_j x_k$,

subject to the constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i; \quad i = 1,2,\dots\dots,m \qquad (4)$$

$$x_j \ge 0\,,\; j = 1,2,\dots,n$$

In matrix notation, the quadratic programming problem is reformulated as:

Optimize (Maximize or Minimize) $Z = c^T x + \frac{1}{2} x^T D x$,

subject to the constraints: $Ax \le b$, (5)

and $x \ge 0$,

where

$x = (x_1, x_2 \dots\dots x_n)^T$ is the vector of decision variables,

$c = (c_1, c_2, \dots, c_n)^T$ is the vector of linear coefficients,

$b = (b_1, b_2, \dots, b_m)^T$ is the vector of constraint bounds,

$D = [d_{jk}]$ is an $n \times n$ symmetric matrix, where., $d_{jk} = d_{kj}$,

$A = [a_{ij}]$ is an $m \times n$ matrix.

3 General Integer Quadratic Programming Problems (GIQPPs)

In this section, let us take the general quadratic programming problem (GIQPP):

$GQPP_1$ $\min f(x) = x^T Q_{x+} c^T x$

subject to constraint: $A_1 x \le b_1$;

$x \in Z^n$; *(6)*

$x \ge 0$.

Where, $Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$

Q is a symmetric matrix; $A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$a_{mj} = (a_{1j}, a_{2j}, \dots, a_{mj})^T > 0$; $A_2 = \begin{pmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{m1} & \hat{a}_{m2} & \cdots & \hat{a}_{mn} \end{pmatrix}$

$\hat{a}_{mj} = (\hat{a}_{1j}, \hat{a}_{2j}, \dots\dots, \hat{a}_{mj})^T > 0$

$$c = (c_1, c_2, \dots, c_n)^T \in R^n;$$

$$b_1 = (b_1, b_2, \dots, b_{m1})^T \in R^{m1};$$

$$b_2 = (\hat{b}_1, \hat{b}_2, \ldots\ldots, \hat{b}_{m2})^T \in R^{m2};$$

$N = \{1,2, \ldots ,n\}; M_1 = \{1,2, \ldots ,m_1\}; M_2 = \{1,2, \ldots ,m_2\}$. Let S be the feasible range of (GIQPP). If we remove some variables from (GIQPP), generally, the optimal solution and optimal value will change. However, if the optimal value of the problem which has been removed a variable, is equal to that of the original problem (GQPP), then we should only consider the new problem with lower dimensions.

Let (GIQPPk) be the new problem after removing the term k of (GIQPP):

(GIQPPk) min $f_k(y) = y^T Q_k y + d^T y$,

subject to constraints: $A^r{}_1 y \le b_1$,

$A^r{}_2 y = b_2$, (7)

$y \in Z^{n-1}$,

$y \ge 0$,

where

$Q_r = \{q^r{}_{ij}\}_{(n-1)\times(n-1)}$ and $d = (c_1, \ldots ,c_{r-1},c_{r+1}, \ldots ,c_n)^T \in R^{n-1}$;

$A_1k = (a_1 \cdots a_{k-1}\ a_{k+1} \cdots a_n)$;

$A_2k = (\hat{a}_1 \ldots\ldots \hat{a}_{k-1}\ \hat{a}_{k+1} \ldots\ldots, \hat{a}_n)$.

Let S_k be the feasible range of (GIQPPk).

Definition 3.1. Let x^* be the optimal solution of (GIQPP) and $f(x^*)$ be the corresponding optimal value. y^* is the optimal solution of (GIQPPk) and $f_k(y^*)$ is the corresponding optimal value. If $f_k(y^*) = f(x^*)$, then we say term k can be removed. The corresponding integer variable x_k is called a dominated decision variable.

Theorem 3.1. Let $x \in R^n$ be a feasible integer solution of (GIQPP). Suppose $k \in N$ and for all $j \in N\backslash\{k\}$, there exist nonnegative integers l_j satisfying

$$\sum_{j\in n\backslash\{k\}} a_{kj} l_j \le a_{kr} \; for\; k \in M_1; \text{ and } \sum_{j\in n\backslash\{k\}} \hat{a}_{kj} l_j = \hat{a}_{kr} \text{ for } k \in M_2.$$

If for all $j \in N \backslash \{k\}$, we set $y_j = x_j + l_j x_r$, then $y \in R^{n-1}$ is a feasible integer solution of (GQPPk). Additionally, $x_0 = (y_1, \ldots ,y_{r-1},0,y_{r+1}, \ldots ,y_n)^T$ is also a feasible integer solution of (GIQPP).

Theorem 3.2. If there exists a nonnegative integer vector $l \in S_k$ such that

$$l^T Q^k l \le q_{kk};\; 2a^T Q^+{}_k l + d^T l \le 2a^T q_k^- \le c_k;$$

then x_k is a dominated variable in (GIQPP). Here $a = (a_1, \ldots ,a_{r-1},a_{r+1}, \ldots ,a_n)^T$, $Q^+{}_k = \{q^+{}_{ij}\}$, $Q^{-k} = \{q^-{}_{ij}\}$.

Notes. If a_j is the other upper bound of x_j, then the result of Theorem 3.2 also holds.

Denote $u = max_{i,\, j\in N\backslash\{k\}}\{q^+{}_{ij}\}$, $v = min_{j\in N\backslash\{k\}}\{q^-{}_{rj}\}$, $b = max_{j\in N\backslash\{k\}}\{a_j\}$, and $e = (1,1, \ldots ,1)^T{}_{(n-1)\times 1}$. With these symbols, we obtain the following result.

Corollary 3.1. If there exists a nonnegative integer vector $l \in S_k$ such that

$$l^T Q_k l \leq q_{kk};\ 2b(n-1)(ue^T l - v) + d^T l \leq c_k;$$

then x_k is a dominated variable in GIQPP.

Corollary 3.2. Assume for all $i, j \in N \setminus \{k\}$, $q_{ij} < 0$ and $q_{rj} > 0$ in GIQPP. If there exists a nonnegative integer vector $l \in S_k$ such that $l^T Q_k l \leq q_{kk};\ d^T l \leq c_k;$

then x_k is a dominated variable in GIQPP.

Theorem 3.3. In $GIQPP_1$, if x_k is a dominated decision variable, then ∃ a nonnegative integer vector $l \in S_k$ such that

$$l^T Q_k l + d^T l \leq q_{kk} + c_k;$$

where $S_k = \{y : A_1 k y \leq b_1, y \in Z^{n-1}, y \geq 0\}$.

Theorem 3.4. In $GIQPP_1$, if for $k, s \in N;\ s \neq r$, there exists a nonnegative integer ls such that

$$\forall k \in M_1,\ a_k l_s \leq a_{kr},\ \ q_{ss} l^2_s \leq q_{kk}$$

$$2 \sum_{i \in n \setminus \{k\}} a_i q_{is}^+ + c_s l_s \leq c_k + 2 \sum_{j \in n \setminus \{k\}} a_j q_{rj}^-,$$ then x_k is a dominated decision variable.

Corollary 3.4. If x_k is a dominated decision variable in GIQPP1, then there exists a feasible solution x_0 of GIQPP1, such that $f(x_0) \leq q_{kk} + c_k$. Furthermore, the optimal value of GIQPP1, $f(x^*)$, satisfies the inequality $f(x^*) \leq q_{kk} + c_k$.

Corollary 3.5. In GIQPP1, for $k \in N$, if $q_{kk} \geq 0$ and $c_k + 2 \sum_{j \in n \setminus \{k\}} a_j q_{rj}^- \geq 0$, then x_k is a dominated decision variable.

Corollary 3.6. In GIQPP1, if there exist $k \in N$ and $s \in N \setminus \{k\}$ satisfying $q_k \geq 0$, $q_{ss} \leq 0$, and

$$2 \sum_{j \in n \setminus \{k\}} a_i q_{is}^+ + c_s - \min_{k \in M1} \left(\left| \frac{a_{kr}}{a_{ks}} \right|, \left| \frac{q_{kr}}{q_{ks}} \right| \right) \geq c_k + 2 \sum_{j \in n \setminus \{k\}} a_j q_{kj}^-,$$

then x_k is a dominated decision variable.

Corollary 3.7. In GIQPP1, if $\exists\, k \in N$ and $s \in N \setminus \{k\}$ satisfying $q_{kk} \geq 0$, $q_{ss} \geq 0$, and

$$(2 \sum_{j \in n \setminus \{k\}} a_i q_{is}^+ + c_s) \min_{k \in M1} \left(\left| \frac{a_{kr}}{a_{ks}} \right|, \left| \frac{q_{kr}}{q_{ks}} \right|^{\frac{1}{2}} \right) \geq c_k + 2 \sum_{j \in n \setminus \{k\}} a_j q_{kj}^-;\ \frac{q_{kk}}{q_{ss}} \geq \frac{1}{2},$$

then x_k is a dominated decision variable.

Corollary 3.8. In GIQPP1, suppose $\exists\, k \in N$ satisfying $q_{kk} \geq 0$ and for all $i, j \in N \setminus \{k\}$ satisfying

$q_{ij} < 0$. Assume $s \in N \setminus \{k\}$, if

$$c_s - \min_{k \in M1} \left(\left| \frac{a_{kr}}{a_{ks}} \right|, \left| \frac{q_{kr}}{q_{ks}} \right| \right) \geq c_k + 2 \sum_{j \in n \setminus \{k\}} a_j q_{rj}^-,$$

holds, then x_k is a dominated variable.

Corollary 3.9. In GIQPP1, suppose $\exists\, k \in N$ and $s \in N \setminus \{k\}$ satisfying $q_{kk} < 0$ and $q_{ss} < 0$. If there exists an integer ls satisfying

$$\left(\frac{q_{kk}}{q_{ss}}\right)^{\frac{1}{2}} \le l_s \le \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|,\left|\frac{q_{kr}}{q_{ks}}\right|\right), \text{ and } \; 2\sum_{j\in n\setminus\{k\}} a_i q_{is}^{+} + c_s - l_s + 2\sum_{j\in n\setminus\{k\}} a_j q_{rj}^{-},$$

then x_k is a dominated decision variable.

Corollary 3.10. In GIQPP1, suppose $\exists k\in N$ satisfying $q_{kk} < 0$ and for all $i,j \in N\setminus \{k\}$ satisfying

$q_{ij} < 0$. Assume $s \in N\setminus\{k\}$, if there exists an integer ls satisfying

$$\left(\frac{q_{kk}}{q_{ss}}\right)^{\frac{1}{2}} \le l_s \le \min_{k\in M1}\left(\left|\frac{a_{kr}}{a_{ks}}\right|,\left|\frac{q_{kr}}{q_{ks}}\right|\right), \text{and } \; \frac{c_s}{l_s} \le c_k + 2\sum_{j\in n\setminus\{k\}} a_j q_{kj}^{-},$$

then x_k is a dominated decision variable.

4 Simulation Results Analysis

Example 4.1. Consider the following problem:

$$\min f(x)=f(x_1,x_2,x_3,x_4)=(-10x_1^2+20x_2^2+9x_3^2+8x_4^2+4x_1x_2-15x_1x_3+x_1x_4+3x_2x_3+9x_2x_4+2x_3x_4+x_1-4x_2+2x_3+5x_4)$$

Subject to constraints: $15x_1 - 40x_2 + 24x_3 + 7x_4 \le 50$

$$19x_1 + 23x_2 + 16x_3 + 7x_4 \le 40$$

$$x = (x_1, x_2, x_3, x_4) \ge 0,\ x \in X^4$$

In the matrix form, the above can be written as,

$$\min f(x) = \frac{1}{2}x^T\begin{pmatrix}-20 & 4 & -15 & 1\\ 4 & 40 & 3 & 9\\ -15 & 3 & 18 & 2\\ 1 & 9 & 2 & 16\end{pmatrix}x + \begin{pmatrix}1\\-4\\2\\5\end{pmatrix}^T x$$

Subject to constraints: $Ax \le b$

where,

$$A = \begin{pmatrix}15 & -40 & 24 & 7\\ 19 & 23 & 16 & 7\end{pmatrix};\ b = \begin{pmatrix}50\\40\end{pmatrix};\ x = (x_1,x_2,x_3,x_4)^T$$

$$x_1 \ge 0;\ x_2 \ge 0;\ x_3 \ge 0;\ x_4 \ge 0;\ x \in X^4.$$

We find that $q_{44} + c_4 = 21$ is larger. Thus, we should let $x_4 = 0$ in the optimal solution from Theorem 3.1. The optimal solution is $x = (1,0,1,0)^T$. Using the above Corollaries, For k = 4, $q_{44} = 6 > 0$ and $q_{4j} = 0$ for $j = 1,2,3$; $c_4 = 5 > 0$. x_4 is a dominated decision variable. We remove x_4 first. For $k = 2$, $s = 1$; $c_1 \le \frac{a_{22}}{a_{21}} - c_2.x_2$ is a dominated decision variable. Remove x_2. The new problem can be written as:

$$\min f(z) = \frac{1}{2}z^T\begin{pmatrix}-20 & -15\\ -15 & 18\end{pmatrix}z + \begin{pmatrix}1\\2\end{pmatrix}^T z$$

Subject to constraints: $15z_1 + 24z_2 \le 50;$

$$19z_1 + 16z_2 \le 40;$$

$$z = [z_1,z_2];z_1 \ge 0;z_2 \ge 0;z \ge 0;$$

$z \in Z^2$.

The optimal solution is

$$Z^* = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \textit{ and } g(z^*) = -4.$$

Thus, the optimal solution for the original problem is

$$x^* = \frac{1}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \textit{ and } f(x^*) = -4$$

Example 4.2. Consider the following problem:

$f(x) = f(x_1,x_2,x_3,x_4, x_5,x_6) = (10x_1^2+12.5x_2^2+10x_3^2-8x_4^2-3x_5^2+12.5x_6^2+12x_1x_2$ $+18x_1x_3 +15x_1x_4 +4x_1x_5 +2x_1x_6 +24x_2x_3 +19x_2x_4 +3x_2x_5 +14x_2x_6 -6x_3x_4 +10x_3x_5 +4x_3x_6$ $+3x_4x_5 +8x_4x_6 +7x_5x_6 -4x_1 +8x_2 +6x_3 +8x_4 +8x_5 +25x_6)$

Subject to constraints: $3x_1 +4x_2 +6x_3 +3x_4 +2x_5 +1x_6 \leq 200$

$4x_1 +8x_2 +4x_3 +7x_4 +2x_5 +3x_6 \leq 40$

$x = (x_1,x_2,x_3,x_4) \geq 0,\ x \in X^4$

In the matrix form, the above can be written as,

$$\min f(x) = \frac{1}{2}x^T\begin{pmatrix}20 & 12 & 18 & 15 & 4 & 2\\12 & 25 & 24 & 19 & 3 & 14\\18 & 24 & 20 & -6 & 10 & 4\\15 & 19 & -6 & -16 & 3 & 8\\4 & 3 & 10 & 3 & -6 & 7\\2 & 14 & 4 & 8 & 7 & 25\end{pmatrix}x + \begin{pmatrix}-4\\8\\6\\8\\8\\25\end{pmatrix}^T x$$

Subject to constraints: $Ax \leq b$,

where,

$$A = \begin{pmatrix}3 & 4 & 6 & 3 & 2 & 1\\4 & 8 & 4 & 7 & 2 & 3\end{pmatrix};\ b = \begin{pmatrix}10\\40\end{pmatrix};\ x = (x_1,x_2,x_3,x_4,x_5,x_6)^T.$$

$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_4 \geq 0;\ x_5 \geq 0;\ x_6 \geq 0\ \ x \in X^6.$

Applying the above corollaries, for $k = 2$, $q_{22} = 25 > 0$ and $q_{4j} = 4$ for $j = 1,3,4,5, 6$; $c_2 = 8 > 0$. x_4 is a dominated decision variable. For the same reason, there exists $k = 6$ satisfying above corollaries, so x_6 is a dominated decision variable. Remove x_2 first and x_6.

Then the new problem can be written as:

$$\min g(x) = \frac{1}{2}z^T\begin{pmatrix}20 & 18 & 15 & 4\\18 & 20 & -6 & 10\\15 & -6 & -16 & 3\\4 & 10 & 3 & -3\end{pmatrix}z + \begin{pmatrix}-4\\6\\8\\8\end{pmatrix}^T z$$,

Subject to constraints: $3z_1 +6z_2 +3z_3 +2z_4 \leq 10;$

$4z_1 +4z_2 +7z_3 +7z_4 \leq 40;$

$z \geq 0;$

$z = [z_1, z_2, z_3, z_4]; z_1 \geq 0; z_2 \geq 0; z_3 \geq 0; z_4 \geq 0; z \geq 0;$

$z \in Z^4.$

The optimal solution is $z^* = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *and* $g(z^*) = -12$

Thus, the optimal solution for the originals problems is $x^* = \frac{1}{2}\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *and* $f(x^*) = -12$.

We used MATLAB to compute two examples with continuous variables, yielding identical optimal solutions and values for both cases. This confirms the applicability of the proposed technique to general integer quadratic programming problems (GIQPPs). The results show that uncorrelated data produces results like strongly correlated data in numerical analysis. The decision variable reduction technique requires minimal problem data and can be broadly applied to GIQPPs with linear inequality constraints. For examples GIQPP1 and GIQPP3, we analyzed randomly generated test problems using uncorrelated and correlated data, where q_{jj} and c_j were uniformly distributed in $[-10, 10]$, and a_{kj} in $[0, 100]$.

Experimental conclusions highlight the average remaining variables and dominated rate. For separable integer quadratic programming problems, integer variables identified as dominated can be fixed at zero before applying solution methods. Additionally, the problem size and the number of constraints significantly impact on the dominated rate (percentage) and remaining variables. Throughout this article, the following notations are used: $q_{ij} = \max\{q_{ij}, 0\}$ and $q_{ij} = \min\{q_{ij}, 0\}$. Figure 1 is plotted for QPPs 4.1.

Figure 2 is plotted for QPPs 4.2.

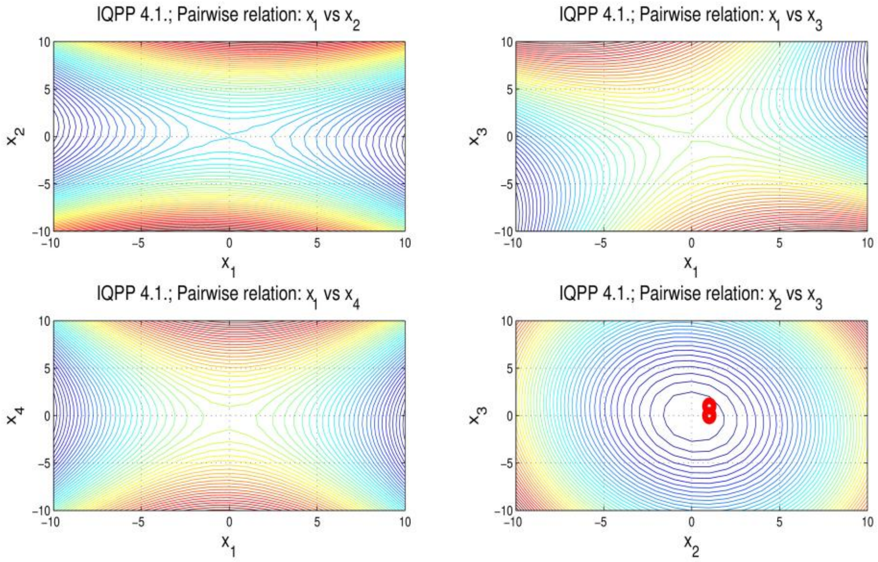

Fig. 1: Phase portrait of Quadratic Programming Problem 4.1 with different axes

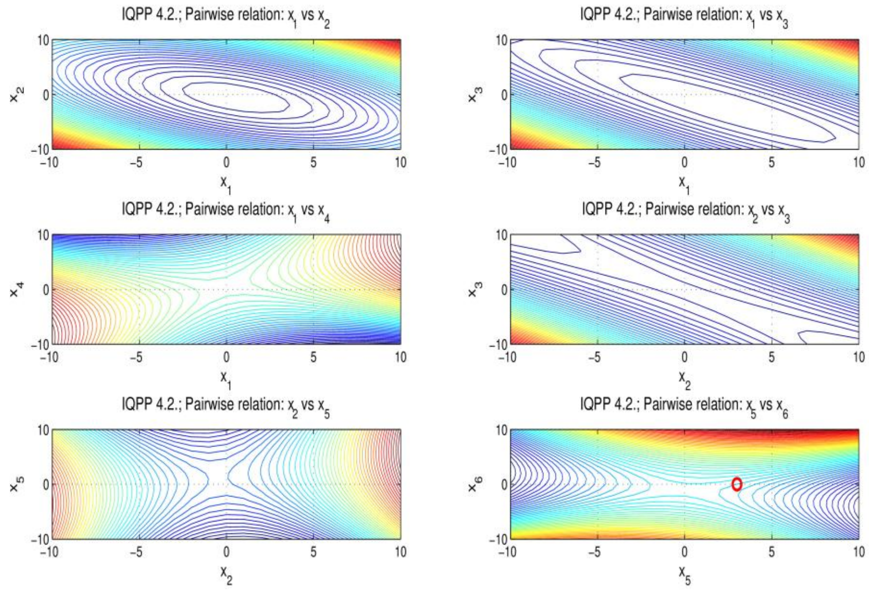

Fig. 2: Phase portrait of Quadratic Programming Problem 4.2 with different axes.

5 Conclusions

This paper explores nonlinear programming and general integer quadratic programming problems (GIQPPs), emphasizing advanced optimization techniques. It introduces a novel variable reduction method for GIQPPs, which reduces problem dimensionality by fixing specific decision variables at zero while preserving optimality. Additionally, it establishes necessary and sufficient conditions for identifying and removing dominated terms within GIQPPs. Extensive computational experiments conducted in MATLAB confirm the proposed approach's effectiveness, showcasing enhanced solution efficiency and accuracy. The findings offer valuable insights into optimizing GIQPPs and present a robust framework for tackling large-scale, real-world nonlinear programming challenges.

6 Declarations and Statements

6.1 Conflict of Interest
This is not applicable.
6.2 Funding
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6.3 Author Contributions
All authors were actively involved in the conceptualization and development of this research article. Each author has reviewed and approved the final version of the manuscript.
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6.5 Data Availability
The authors affirm that all data supporting the findings of this study are provided within the article.

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