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CERTAIN SUBCLASS OF HARMONIC MULTIVALENT FUNCTIONS DEFINED BY DERIVATIVE OPERATOR

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ABSTRACT. In the present paper, we investigate new properties of a new subclass of multivalent harmonic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, under certain conditions involving a new generalized differential operator. Furthermore, a representation theorem, an integral property and convolution conditions for the subclass denoted by $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ are also obtained. Finally, we will give an application of neighborhood.

Keywords: differential operator, harmonic function, extreme points, convolution, neighborhood.
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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See also Clunie and Sheil-Small [5] for more details.)

Denote by $S_{\mathcal{H}}(p, n), (p, n \in \mathbb{N} = \{1, 2, \ldots\})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{1.1}$$

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Let $\tilde{S}_{\mathcal{H}}(p,n,m),(p,n\in\mathbb{N},m\in\mathbb{N}_0\cup\{0\})$ denote the family of functions $f_m=h+\bar{g}_m$ that are harmonic in D with the normalization

(1.2)
$$h(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k,\quad g_m(z)=(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|z^k,\quad |b_{p+n-1}|<1.$$

Definition 1.1. [4] Let $H(U)$ denote the class of analytic functions in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging to $H(U)$ of the form

$$h(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k.$$

For $m\in\mathbb{N}_0$, $\lambda\geq 0$, $\delta\in\mathbb{N}_0$, $l\geq 0$ we define the generalized differential operator $I^m_{\lambda,\delta}(p,l)$ on $\mathcal{A}(p)$ by the following infinite series

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k, \tag{1.3}$$

where

$$C(\delta,k)=\begin{pmatrix} k+\delta-1\\ \delta\end{pmatrix}=\frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}. \tag{1.4}$$

Remark 1.2. When $\lambda=1$, $p=1$, $l=0$, $\delta=0$ we get Sălăgean differential operator [13]; $p=1$, $m=0$ gives Ruscheweyh operator [12]; $p=1$, $l=0$, $\delta=0$ implies Al-Oboudi differential operator of order m (see [1]); $\lambda=1$, $p=1$, $l=0$ operator (1.3) reduces to Al-Shaqsi and Darus differential operator [2] and when $p=1$, $l=0$ we reobtain the operator introduced by Darus and Ibrahim in [6].

Definition 1.3. [4] Let $f\in S_{\mathcal{H}}(p,n)$, $p\in\mathbb{N}$. Using the operator (1.3) for $f=h+\bar{g}$ given by (1.1) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z)=I^m_{\lambda,\delta}(p,l)h(z)+(-1)^m\overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{1.5}$$

where

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k \tag{1.6}$$

and

$$I^m_{\lambda,\delta}(p,l)g(z)=\sum_{k=p+n-1}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)b_kz^k. \tag{1.7}$$

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Remark 1.4. When $\lambda = 1$, $l = 0$, $\delta = 0$ the operator (1.5) reduces to the operator introduced earlier in [8] by Jahangiri et al.

Definition 1.5. [4] A function $f \in S_{\mathcal{H}}(p,n)$ is said to be in the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if

$$\frac{1}{p+l}\mathrm{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^{m}_{\lambda,\delta}(p,l)f(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{1.8}$$

where $I^m_{\lambda,\delta}f$ is defined by (1.5), for $m \in \mathbb{N}_0$.

Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \equiv AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_{\mathcal{H}}(p,n,m). \tag{1.9}$$

Remark 1.6. The class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}(p,n)$. For example, letting $n = 1$ we get $AL_{\mathcal{H}}(1,1,0,\alpha,1,0) \equiv HK(\alpha)$ in [7], for $n = 1$, $AL_{\mathcal{H}}(1,m-1,0,\alpha,1,0) \equiv S_H(t,u,\alpha)$ in [14], $AL_{\mathcal{H}}(p,n+p,0,\alpha,1,0) \equiv SH_p(n,\alpha)$ in [11] and $n = 1$, $AL_{\mathcal{H}}(1,m,\delta,\alpha,1,0) \equiv M_{\mathcal{H}}(m,\delta,\alpha)$ in [3].

Theorem 1.7. [4] *Let $f_m = h + \bar{g_m}$ be given by (1.2). Then $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{1.10}$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \leq 1,$$

where $\lambda n \geq \alpha(p+l)$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda \geq 0$ and

$$d_{p,k}(m,\lambda,l) = [p+\lambda(k-p)+l]^m. \tag{1.11}$$

Remark 1.8. The harmonic function

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}x_k z^k + \tag{1.12}$$

$$+\sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\overline{y_k z^k},$$

where $\sum_{k=p+n}^{\infty}|x_k| + \sum_{k=p+n-1}^{\infty}|y_k| = 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda n \geq \alpha(p+l)$, $\lambda \geq 0$ and $d_{p,k}(m,\lambda,l)$ is given in (1.11), show that the coefficient bound expressed by (1.10) is sharp.

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2. Convex combination and extreme points

In this section, we show that the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination of its members.

For $i=1,2,3,...$, let the functions $f_{m_i}(z)$ be

$$f_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}|a_{k,i}|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_{k,i}|\bar{z}^k. \tag{2.1}$$

Theorem 2.1. *The class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination.*

Proof. For $i=1,2,3,...$, let $f_{m_i}(z)\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, where the functions $f_{m_i}(z)$ are defined by (2.1). Then by (1.10) we have

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+ \tag{2.2}$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\le 1.$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)z^k+(-1)^m\sum_{k=p+n-1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)\bar{z}^k.$$

Then by (2.2) one obtains

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)=$$
$$\sum_{i=1}^{\infty}t_i\cdot\left\{\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+\right.$$
$$\left.+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\right\}\le\sum_{i=1}^{\infty}t_i=1,$$

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and therefore $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$. □

Further, we will determine a representation theorem for functions in $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ from which we also establish the extreme points of closed convex hulls of $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ denoted by $clco\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$.

Theorem 2.2. *Let $f_m(z)$ given by (1.2). Then $f_m(z) \in \widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ if and only if*

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z), \tag{2.3}$$

where $h_p(z) = z^p$

$$h_k(z) = z^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha) + \lambda(k-p)]d_{p,k}(m, \lambda, l)C(\delta, k)} z^k, \tag{2.4}$$

$$k = p+n, p+n+1, ...,$$

and

$$g_{m_k}(z) = z^p + (-1)^m \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha) + \lambda(k-p)]d_{p,k}(m, \lambda, l)C(\delta, k)} \bar{z}^k, \tag{2.5}$$

$$k = p+n-1, p+n, ...,$$

with $X_k \geq 0, Y_k \geq 0, X_p = 1 - \sum_{k=p+n}^{\infty} X_k - \sum_{k=p+n-1}^{\infty} Y_k$.

In particular, the extreme points of $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. For the functions f_m of the form (2.3), we have

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z) =$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha) + \lambda(k-p)]d_{p,k}(m, \lambda, l)C(\delta, k)} X_k z^k +$$

$$+(-1)^m \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha) + \lambda(k-p)]d_{p,k}(m, \lambda, l)C(\delta, k)} Y_k \bar{z}^k.$$

Consequently,

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha) + \lambda(k-p)]d_{p,k}(m, \lambda, l)C(\delta, k)}{(p+l)^{m+1}(1-\alpha)} a_k +$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}b_k=$$

$$=\sum_{k=p+n}^{\infty}X_k+\sum_{k=p+n-1}^{\infty}Y_k=1-X_p\leq 1,$$

where

$$a_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}X_k$$

$$b_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}Y_k$$

and therefore $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Conversely, suppose that $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.
Setting

$$X_k=\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|, \tag{2.6}$$

$$k=p+n,p+n+1,...,$$

$$Y_k=\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|$$

$$k=p+n-1,p+n,...,$$

and $X_p=1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k$. We note by Theorem 1.7 that $0\leq Y_k\leq 1, 0\leq X_k\leq 1,\ and\ X_p\geq 0$.

We obtain the required representation since f_m can be written as

$$f_m(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)X_k}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}z^k+$$

$$+(-1)^m\sum_{k=p+n-1}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)Y_k}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}(z^p-h_k(z))X_k+\sum_{k=p+n-1}^{\infty}(g_{m_k}(z)-z^p)Y_k=$$

$$=\sum_{k=p+n}^{\infty}h_k(z)X_k+\sum_{k=p+n-1}^{\infty}g_{m_k}(z)Y_k+z^p\left(1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k\right)=$$

$$= X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z),$$

as required. □

3. Integral property and convolution conditions

In this section we will examine the closure properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ under the generalized Bernardi-Libera-Livingston integral operator and also convolution properties of the same class.

Now, for $f = h + \bar{g}$ given by (1.1) we define the modified generalized Bernardi-Libera-Livingston integral operator of f as

$$\mathcal{L}_c(f(z)) = \mathcal{L}_c(h(z)) + \overline{\mathcal{L}_c(g(z))}, \quad c > -p, \tag{3.1}$$

where

$$\mathcal{L}_c(h(z)) = \frac{c+p}{z^c}\int_0^z t^{c-1}h(t)dt$$

and

$$\mathcal{L}_c(g(z)) = \frac{c+p}{z^c}\int_0^z t^{c-1}g(t)dt.$$

Putting $g = 0$ in (3.1), we get the definition of the generalized Bernardi-Libera-Livingston integral operator on analytic functions, (see [9], [10]).

Theorem 3.1. *Let $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Then $\mathcal{L}_c(f)$ belongs to the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.*

Proof. From the representation of $\mathcal{L}_c(f)$, it follows that

$$\mathcal{L}_c(f(z)) = \frac{c+p}{z^c}\int_0^z t^{c-1}(h(t) + \bar{g}_m(t))dt =$$

$$= \frac{c+p}{z^c}\left[\int_0^z t^{c-1}\left(t^p - \sum_{k=p+n}^{\infty}|a_k|t^k\right)dt + (-1)^m \overline{\int_0^z t^{c-1}\left(\sum_{k=p+n-1}^{\infty}|b_k|t^k\right)dt}\right] =$$

$$= z^p - \sum_{k=p+n}^{\infty}|A_k|z^k + (-1)^m \sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k,$$

where

$$A_k = \frac{c+p}{c+k}a_k, \ \ B_k = \frac{c+p}{c+k}b_k.$$

Further, one obtains

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} \cdot \frac{c+p}{c+k}|a_k| +$$

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$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\frac{c+p}{c+k}|b_k|\leq$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\leq 1.$$

Since $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, by Theorem 1.7 we have $\mathcal{L}_c(f) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

For the harmonic functions

$$f_1(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k,\ |b_{p+n-1}|<1, \tag{3.2}$$

and

$$f_2(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k,\ |B_{p+n-1}|<1, \tag{3.3}$$

we define the convolution of f_1 and f_2 as

$$(f_1*f_2)(z)=f_1(z)*f_2(z)=z^p-\sum_{k=p+n}^{\infty}|a_kA_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_kB_k|\bar{z}^k.$$

In the following theorem, we examine the convolution properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 3.2. *For $0\leq\beta\leq\alpha<1$ let $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. Then $f_1*f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$.*

Proof. Let $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. Obviously, the coefficients of f_1 and f_2 must satisfy similar conditions to the inequality (1.10). Therefore, for the coefficients of f_1*f_2 we can write

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_kA_k|+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_kB_k|\leq$$
$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_k|+$$

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$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_k|\leq$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)+}|b_k|\leq 1,$$

because $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. In view of Theorem 1.7, it follows that $f_1*f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. □

4. An application of neighborhood

Let us define a generalized (n,η)-neighborhood of a function f given in (1.2) to be the set

$$N_{n,\eta}(f)=\Bigg\{F_m(z)\in\tilde{S}_{\mathcal{H}}(p,n,m):$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k-A_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k-B_k|\leq\eta\Bigg\}$$

where $F_m(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k$.

Theorem 4.1. *Let $f_m=h+\bar{g}_m$ be given by (1.2). If the functions f_m satisfy the conditions*

$$\sum_{k=p+n}^{\infty}k\cdot\Bigg[\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$
$$+\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\Bigg]\leq 1-U_{p,\delta}^{\alpha}(m,\lambda,l)\tag{4.1}$$

and

$$\eta\leq\frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U_{p,\delta}^{\alpha}(m,\lambda,l)\right),\tag{4.2}$$

$\lambda n\geq\alpha(p+l)$, where

$$U_{p,\delta}^{\alpha}(m,\lambda,l)=\frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,p+n-1}(m,\lambda,l)C(\delta,p+n-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|$$

then $N_{n,\eta}(f)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

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Proof. Let f_m satisfy (4.1) and $F_m \in N_{n,\eta}(f)$. We have

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|A_k|+$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|B_k| \leq$$

$$\leq \eta + \sum_{k=p+n}^{\infty} \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\right.$$

$$\left.\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\eta + \frac{1}{p+n-\alpha} \sum_{k=p+n}^{\infty} k \cdot \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\right.$$

$$\left.\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\leq \eta + \frac{1}{p+n-\alpha}(1-U^{\alpha}_{p,\delta}(m,\lambda,l)) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq 1.$$

Hence, for $\eta \leq \frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U^{\alpha}_{p,\delta}(m,\lambda,l)\right)$ we deduce that $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., **27**(2004), 1429-1436.

[2] K. Al-Shaqsi, M. Darus, *An operator defined by convolution involving polylogarithms functions*, Journal of Math. and Statistics, **4**(1)(2008), 46-50.

[3] K. Al-Shaqsi, M. Darus, *On Harmonic Functions Defined by Derivative Operator*, Journal of Inequalities and Applications, vol. 2008, Article ID 263413, doi: 10.1155/2008/263413.

[4] A. Cătaş, R. Şendruţiu *On harmonic multivalent functions defined by a new derivative operator*, Journal of Computational Analysis and Applications, Volume: 28, Issue: 5, Pages: 775-780, 2020.

[5] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn, Ser. A I. Math. **9**(1984), 3-25.

[6] M. Darus, R. W. Ibrahim, *On new classes of univalent harmonic functions defined by generalized differential operator*, Acta Universitatis Apulensis, **18**(2009), 61-69.

[7] J. M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklowdowska Sect. A, **52**(1998), 57-66.

[8] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya *Sălăgean type harmonic univalent functions* South. J. Pure Appl. Math., **2**(2002), 77-82.

[9] R. J. Libera, *Some classes of regular univalent functions*, Proc. Am. Math. Soc. 63(1965), 755-758.

[10] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Am. Math. Soc. 17(1966), 352-357.

[11] Om P. Ahuja and J. M. Jahangiri, *Multivalent harmonic starlike functions with missing coefficients*, Math. Sci. Res. J., **7**(9)(2003), 347-352.

[12] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[13] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, Heidelberg and New York, 1013(1983), 362-372.

[14] Sibel Yalcin, *A new class of Sălăgean-type harmonic univalent functions* Appl. Math. Letters, **18**(2005), 191-198.

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