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ORTHOGONALLY EULER-LAGRANGE TYPE CUBIC FUNCTIONAL EQUATIONS IN ORTHOGONALITY NORMED SPACES

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Abstract. In this paper, we investigate the orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax+by)+f(ax-by)-ab^2[f(x+y)+f(x-y)]-2a(a^2-b^2)f(x)\\ &+c[f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)]=0,\ \ x\perp y \end{aligned}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real number c with $a^2 \neq b^2$ and $a \neq \pm 1$ and prove the generalized Hyers-Ulam stability for it by using the fixed point method,

1. Introduction

Assume that X is a real inner product space and $f : X \longrightarrow \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x+y) = f(x)+f(y)$, $< x, y >= 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The orthogonally Cauchy functional equation

$$f(x+y) = f(x)+f(y),\ \ x\perp y$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [5]. Rätz [16] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [17] investigated the problem in a rather more general framework.

Definition 1.1. [17] Let X be a real vector space with $\dim X \geq 2$ and $\perp$ a binary relation on X with the following properties:

(O1) totality for zero: $x\perp 0$ and $0\perp x$ for all $x \in X$;

(O2) independence: if $x, y \in X - \{0\}$, $x\perp y$, then x, y are linearly independent;

(O3) homogeneity: if $x, y \in X$, $x\perp y$, then $\alpha x\perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and a non-negative real number k, then there exists an $y \in P$ such that $x\perp y$ and $x + y\perp kx - y$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

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Remark 1.2. (i) The trivial orthogonality on a vector space X defined by (O1) and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, < \cdot, \cdot >)$ given by $x \perp y$ if and only if $< x, y >= 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + ky\| \geq \|x\|$ for all $k \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Then clearly examples (i) and (ii) are symmetric but example (iii) is not. However, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

In 1940, S. M. Ulam proposed the following stability problem (cf. [19]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference. The result of Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam stability of functional equations. Ger and Sikorska [4] investigated the orthogonal stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y \tag{1.1}$$

and Vajzović [20] investigated the orthogonally additive-quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y \tag{1.2}$$

when X is a Hilbert space, Y is a scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, many mathematicians have investigated the orthogonal stability of functional equations ([3], [9], [10], [11], [12], [13], and [18]).

In 2001, Rassias [15] introduced the following cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0 \tag{1.3}$$

and every solution of the cubic functional equation is called *a cubic mapping* and Jun, Kim, and Chang [8] introduced the Euler-Lagrange cubic functional equation.

In this paper, we consider the following orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax + by) + f(ax - by) - ab^2[f(x + y) + f(x - y)] - 2a(a^2 - b^2)f(x) \\ &+ c[f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y)] = 0, \quad x \perp y. \end{aligned} \tag{1.4}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real numbers c with $a^2 \neq b^2$ and $a \neq \pm 1$ and prove the generalized Hyers-Ulam stability for it. Every solution of (1.4) is called *an orthogonally Euler-Lagrange type cubic mapping*.

Throughtout this paper, $(X, \perp)$ is an orthogonality normed space with the norm $\| \cdot \|_X$ and $(Y, \| \cdot \|)$ is a Banach space.

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2. Solutions of (1.4)

In this section, we investigate solutiuons of (1.4). We will show that a mapping f satisfying (1.4) is an orthogonally cubic mapping.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. If f satisfies (1.4) and $c \neq 0$, then f is an orthogonally cubic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $y = 0$ in (1.4), we have

$$f(ax) = a^3 f(x) \tag{2.1}$$

for all $x \in X$ and setting $x = 0$ and $y = x$ in (1.4), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(x) + (ab^2 + c)f(-x) - cf(2x) \tag{2.2}$$

for all $x \in X$. Replacing x by $-x$ in (2.2), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(-x) + (ab^2 + c)f(x) - cf(-2x) \tag{2.3}$$

for all $x \in X$. Since $c \neq 0$, by (2.2) and (2.3), we have

$$f(2x) - f(-2x) = 8[f(x) - f(-x)] \tag{2.4}$$

for all $x \in X$. Relpacing y by ay in (1.4), by (2.2), we have

$$\begin{aligned} &a^3[f(x + by) + f(x - by)] - (ab^2 + 3c)f(x + ay) - (ab^2 + c)f(x - ay) \\ &+ cf(x + 2ay) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(ay) = 0 \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $x \perp y$ and letting $y = \frac{y}{b}$ in (2.5), we have

$$\begin{aligned} &a^3[f(x + y) + f(x - y)] - (ab^2 + 3c)f(x + py) - (ab^2 + c)f(x - py) \\ &+ cf(x + 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(py) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in X$ with $x \perp y$, where $p = \frac{a}{b}$. Letting $y = -y$ in (2.6), we have

$$\begin{aligned} &a^3[f(x - y) + f(x + y)] - (ab^2 + 3c)f(x - py) - (ab^2 + c)f(x + py) \\ &+ cf(x - 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(-py) = 0 \end{aligned} \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. By (2.6) and (2.7), we have

$$\begin{aligned} &c[f(x + 2py) - f(x - 2py)] - 2c[f(x + py) - f(x - py)] \\ &- 6c[f(py) - f(-py)] = 0 \end{aligned} \tag{2.8}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = \frac{1}{p}y$ in (2.8), we have

$$[f(x + 2y) - f(x - 2y)] - 2[f(x + y) - f(x - y)] - 6[f(y) - f(-y)] = 0 \tag{2.9}$$

for all $x, y \in X$ with $x \perp y$.

Let $f_o(x) = \frac{f(x) - f(-x)}{2}$. Then f_o satisfies (2.9). Letting $x = 0$ in (2.9), we have

$$f_o(2y) = 8f_o(y) \tag{2.10}$$

for all $y \in X$. Letting $x = 2x$ in (2.9), by (2.10), we have

$$4[f_o(x + y) - f_o(x - y)] = f_o(2x + y) - f_o(2x - y) + 6f_o(y) \tag{2.11}$$

for all $x, y \in X$ with $x \perp y$. Interchanging x and y in (2.11), we have

$$4[f_o(x + y) + f_o(x - y)] = f_o(x + 2y) + f_o(x - 2y) + 6f_o(x) \tag{2.12}$$

for all $x, y \in X$ with $x \perp y$. By (2.9) and (2.12), we have

$$f_o(x + 2y) - 3f_o(x + y) + 3f_o(x) - f_o(x - y) - 6f_o(y) = 0$$

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for all $x, y \in X$ with $x \perp y$ and hence f_0 is an orthogonally cubic mapping.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_e satisfies (2.9) and so we have

$$f_e(x+2y) - f_e(x-2y) - 2[f_e(x+y) - f_e(x-y)] = 0 \tag{2.13}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = x$ in (2.13), we have

$$f_e(3x) = 2f_e(2x) + f_e(x)$$

for all $x \in X$ and letting $y = 2x$ in (2.13), we have

$$f_e(4x) = 2f_e(3x) - 2f_e(x)$$

for all $x \in X$. Hence we have $f_e(4x) = 4f_e(2x)$ for all $x \in X$ and so

$$f_e(2x) = 4f_e(x), \;\; f_e(3x) = 9f_e(x), \;\; f_e(4x) = 16f_e(x)$$

for all $x \in X$. By induction on n, we have

$$f_e(nx) = n^2 f_e(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence

$$f_e(rx) = r^2 f_e(x)$$

for all $x \in X$ and all rational number r. By (2.1), since a is a non-zero rational number with $a \neq 1$, $f(x) = 0$ for all $x \in X$. Hence $f = f_o + f_e = f_o$ is an orthogonally cubic mapping. □

3. The Generalized Hyers-Ulam stability for (1.4)

In this section, we prove the generalized Hyers-Ulam stability for the orthogonally cubic functional equation (1.4) by using the fixed point method.

In 1996, Isac and Rassias [7] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 3.1. [1], [2] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$\begin{aligned} Df(x,y) = f(ax+by) + f(ax-by) - ab^2[f(x+y) + f(x-y)] - 2a(a^2-b^2)f(x) \\ + c[f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y)] \end{aligned}$$

for all $x, y \in X$.

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Theorem 3.2. *Assume that* $\phi : X^2 \longrightarrow [0, \infty)$ *is a function such that*

$$\phi(x, y) \le \frac{L}{|a|^3}\phi(ax, ay) \tag{3.1}$$

for all $x, y \in X$ *and some real number* L *with* $0 < L < 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\|Df(x, y)\| \le \phi(x, y) \tag{3.2}$$

for all $x, y \in X$ *with* $x \perp y$. *Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \le \frac{L}{2|a|^3(1 - L)}\phi(x, 0) \tag{3.3}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g, h) = \inf\{c \in [0, \infty) \mid \|g(x) - h(x)\| \le c\ \phi(x, 0), \forall x \in X\}.$$

Then (S, d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = a^3 g(\frac{x}{a})$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g, h) \le c$ for some $c \in [0, \infty)$. Then by (3.1), we have

$$\|Tg(x) - Th(x)\| = |a|^3\Big\|g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big)\Big\| \le cL\phi(x, 0)$$

for all $x \in X$. Hence we have $d(Tg, Th) \le Ld(g, h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax) - 2a^3 f(x)\| \le \phi(x, 0)$$

for all $x \in X$ and hence

$$\Big\|f(x) - a^3 f\Big(\frac{x}{a}\Big)\Big\| \le \frac{L}{2|a|^3}\phi(x, 0)$$

for all $x \in X$ and hence $d(f, Tf) \le \frac{L}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n f, F) \to 0$ as $n \to \infty$ and

$$\|F(x) - f(x)\| \le \frac{L}{2|a|^3(1 - L)}\phi(x, 0)$$

for all $x \in X$. Replacing x, y by $\frac{x}{a^n}$, $\frac{y}{a^n}$ in (3.2), respectively, and multiplying (3.2) by $|a|^{3n}$, by (O3), we have

$$\Big\|a^{3n} Df\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big\| \le L^n \phi(x, y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

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Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}\|G(x)-F(x)\| &= \|T^nG(x)-T^nF(x)\|\\ &\le \|T^nG(x)-T^nf(x)\|+\|T^nF(x)-T^nf(x)\|\\ &\le \frac{L^{n+1}}{|a|^3(1-L)}\phi(x,0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. □

Related with Theorem 3.2, we can also have the following theorem. And the proof is similar to that of Theorem 3.2.

Theorem 3.3. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a function such that*

$$\phi(ax, ay) \le |a|^3 L\phi(x, y) \tag{3.4}$$

for all $x, y \in X$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that satisfying (3.2). Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x)-f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g,h) = \inf\{c \in [0,\infty) \mid \|g(x)-h(x)\| \le c\ \phi(x,0), \forall x \in X\}.$$

Then (S, d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = \frac{1}{a^3}g(ax)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g, h) \le c$ for some $c \in [0,\infty)$. Then by (3.4), we have

$$\|Tg(x)-Th(x)\| = \frac{1}{|a|^3}\|g(ax)-h(ax)\| \le cL\phi(x,0)$$

for all $x \in X$. Hence we have $d(Tg, Th) \le Ld(g, h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax)-2a^3f(x)\| \le \phi(x,0)$$

for all $x \in X$ and hence

$$\left\|f(x)-\frac{1}{a^3}f(ax)\right\| \le \frac{1}{2|a|^3}\phi(x,0)$$

for all $x \in X$ and hence $d(f, Tf) \le \frac{1}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^nf, F) \to 0$ as $n \to \infty$ and

$$\|F(x)-f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0)$$

for all $x \in X$. Replacing x, y by a^nx, a^ny in (3.2), respectively, and multiplying (3.2) by $|a|^{-3n}$, by (O3), we have

$$\left\|a^{-3n}Df(a^nx, a^ny)\right\| \le L^n\phi(x,y)$$

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for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}
\|G(x) - F(x)\| &= \|T^n G(x) - T^n F(x)\| \\
&\le \|T^n G(x) - T^n f(x)\| + \|T^n F(x) - T^n f(x)\| \\
&\le \frac{L^n}{|a|^3(1-L)}\phi(x, 0)
\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. □

As an example of $\phi(x, y)$ in Theorem 3.2 and Theorem 3.3, we can take $\phi(x, y) = \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})$ for some positive real numbers ϵ and p. Then we can formulate the following corollary :

Corollary 3.4. *Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y)\| \le \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p}) \tag{3.6}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \ne \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

By Theorem 2.1, if $c = -\frac{1}{3}ab^2$, then we have the following orthogonally Euler-Lagrange type cubic functional equation :

$$\begin{aligned}
&f(ax + by) + f(ax - by) - \frac{2}{3}ab^2 f(x - y) - \frac{1}{3}ab^2 f(x + 2y) \\
&- a(2a^2 - b^2)f(x) + 2ab^2 f(y) = 0
\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. By Corollary 3.6, we have the following exmaple.

Example 3.5. Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that

$$\begin{aligned}
&\|f(ax + by) + f(ax - by) - \frac{2}{3}ab^2 f(x - y) - \frac{1}{3}ab^2 f(x + 2y) \\
&- a(2a^2 - b^2)f(x) + 2ab^2 f(y)\| \le \epsilon(\|x\|_X^p \|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})
\end{aligned}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \ne \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

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It should be remarked that if a functional inequality can be deformed into the type of (3.2), then a solution of the original functional equation is cubic. In the following theorems, we give a simple example.

Theorem 3.6. *Let* $\phi : X^2 \longrightarrow [0, \infty)$ *be a function such that*

$$\phi(x, y) \le \frac{1}{8} L\phi(2x, 2y) \tag{3.7}$$

for all $x, y \in X$*, some real number* L *with* $0 < L < 1$ *and* $f : X \longrightarrow Y$ *a mapping such that* $f(0) = 0$ *and*

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \phi(x, y) \tag{3.8}$$

for all $x, y \in X$ *with* $x \perp y$*. Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \le \frac{L}{16(1 - L)}[3\phi(x, 0) + 8\phi(0, x)]$$

for all $x \in X$*.*

Proof. Letting $x = 0$ in (3.8), we have

$$\|f(y) + f(-y)\| \le \phi(0, y) \tag{3.9}$$

for all $y \in X$ and letting $y = 0$ in (3.8), we have

$$\|f(2x) - 8f(x)\| \le \frac{1}{2}\phi(x, 0) \tag{3.10}$$

for all $y \in X$. Letting $y = 2y$ in (3.8), by (3.10), we have

$$\begin{aligned} &\|8f(x + y) + 8f(x - y) - 2f(x + 2y) - 2f(x - 2y) - 12f(x)\| \\ \le\ &\frac{1}{2}\phi(x + y, 0) + \frac{1}{2}\phi(x - y, 0) + \phi(x, 2y) \end{aligned} \tag{3.11}$$

for all $x, y \in X$ with $x \perp y$. Interchang x and y in (3.8), by (3.9), we get

$$\begin{aligned} &\|f(x + 2y) - f(x - 2y) - 2f(x + y) + 2f(x - y) - 12f(y)\| \\ \le\ &\phi(y, x) + \phi(0, x - 2y) + 2\phi(0, x - y) \end{aligned} \tag{3.12}$$

for all $x, y \in X$ with $x \perp y$. Putting $a = 2$, $b = 1$, and $c = -4$ in $Df(x, y)$, by (3.8), (3.11), and (3.12), we have

$$\|Df(x, y)\| \le \psi(x, y)$$

for all $x, y \in X$, where

$$\begin{aligned} \psi(x, y) = \phi(x, y) + 2\phi(y, x) + \frac{1}{2}\phi(x + y, 0) + \frac{1}{2}\phi(x - y, 0) + \phi(x, 2y) \\ + 2\phi(0, x - 2y) + 4\phi(0, x - y) \end{aligned}$$

Since ψ satisfies (3.1), by Theorem 3.2, we get the result. □

Similar to Theorem 3.6, we have the following theorem :

Theorem 3.7. *Let* $\phi : X^2 \longrightarrow [0, \infty)$ *be a function such that*

$$\phi(2x, 2y) \le 8L\phi(2x, 2y) \tag{3.13}$$

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for all $x, y \in X$*, some real number* L *with* $0 < L < 1$ *and* $f : X \longrightarrow Y$ *a mapping satisfying* $f(0) = 0$ *(3.8). Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \leq \frac{1}{16(1-L)}[3\phi(x,0) + 8\phi(0,x)]$$

for all $x \in X$*.*

By Theorem 3.6 and Theorem 3.7, we have the following corollary :

Corollary 3.8. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \leq \|x\|^p\|y\|^p+\|x\|^{2p}+\|y\|^{2p}.$$

for all $x, y \in X$ *and a fixed positive real number* p *with* $p \neq \frac{3}{2}$*. Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \leq \frac{11}{2|8-2^{2p}|}\|x\|^{2p}$$

for all $x \in X$*.*

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