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Some Properties of the q-Exponential Functions

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Abstract. This paper aims to investigate some striking properties of the q-exponential functions more profoundly. To achieve this, at first, the Gauss q-binomial formula is generalized and based on the formula, important properties of the q-exponential functions are established.

Keywords. q-Exponential function, q-Binomial formula.
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1 Introduction

The q-analogue of any real number t is defined as $[t]_q = \frac{1-q^t}{1-q}$ and the q-factorial, denoted by $[n]_q!$, is defined [1, 2] as

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q \times [n-1]_q \times \cdots \times [1]_q & \text{if } n = 1, 2, \ldots. \end{cases} \tag{1}$$

The q-analogue of $(a+x)^n$, denoted by $(a+x)_q^n$, is defined [3] as

$$(a+x)_q^n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(a+q^m x) & n = 1, 2, \ldots. \end{cases} \tag{2}$$

It is also defined for any complex number α as

$$(a+x)_q^\alpha = \frac{(a+x)_q^\infty}{(a+q^\alpha x)_q^\infty}, \tag{3}$$

where $(a+x)_q^\infty := \lim_{n\to\infty} \prod_{m=0}^{n}(a+q^m x)$, and the principal value of q^α is considered, $0 < q < 1$. Yet, the q-Maclaurin series expansion of $(a+x)_q^n$ is

$$(a+x)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} x^k q^{\binom{k}{2}} \tag{4}$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ are called q-binomial coefficients. Expression (4) is called Gauss q-binomial formula (see [3], p. 15). In the q-binomial coefficients, if $|q| < 1$ and n tends to infinity (see [3], p. 30) we obtain $\lim_{n\to\infty} \binom{n}{k}_q = \frac{1}{(1-q)_q^k}$. More details about the identities involving q-binomial coefficients can be found in reference [4].
One can also recall definitions of the q-functions [2, 5, 6] as follows:

$$e_q^x = \frac{1}{(1-(1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n, \qquad |x| < 1, \tag{5}$$

$$E_q^x = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n q^{\binom{n}{2}}, \qquad x \in \mathbb{C}. \tag{6}$$

It can be seen that $e_q^x E_q^{-x} = 1$ and $e_{q^{-1}}^x = E_q^x$. The product of the two functions are investigated in a more detailed way in [6, 7, 8]. The contribution of the corresponding references can be summarized in the following theorem:

Theorem 1. *For all $x, y \in \mathbb{C}$ the following equation holds*

$$e_q^x\, E_q^y = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} (x+y)_q^n = e_q^{(x+y)_q} \tag{7}$$

where $(x+y)_q^n$ is defined in (4).

In the light of aforementioned preliminaries, this paper aims at studying about the q-exponential functions more closely. At first, the Gauss q-binomial formula is generalized and based on the formula, some properties of the q-exponential functions are established.

2 q-Exponential Functions

First, let us generalize the q-binomial formula given in (4). The generalization of the q-binomial can then be carried out as follows.

Theorem 2. *For any $x, y, z \in \mathbb{C}$ and positive integer n, the following identity holds:*

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k}. \tag{8}$$

Proof. The induction is used to prove the theorem. Equation (8) is valid for $n = 1$. Assuming that (8) holds for any n and we show that it holds for $n+1$. Then

$$\begin{aligned}
(x+y)_q^{n+1} &= (x+y)_q^n \left(q^k(z+q^{n-k}y) + (x-q^k z)\right) \\
&= \sum_{k=0}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{k+1} (z+y)_q^{n-k} \\
&= (z+y)_q^{n+1} + (x-z)_q^{n+1} + \sum_{k=1}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k} \\
&\qquad + \sum_{k=1}^{n} \binom{n}{k-1}_q (x-z)_q^k (z+y)_q^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k}_q (x-z)_q^k (z+y)_q^{n+1-k}.
\end{aligned}$$

Thus, the proof is complete. □

It is realized that the identity in Theorem 2 can be re-written as

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{n-k} (z+y)_q^k. \tag{9}$$

Its proof can be readily derived form the proof of Theorem 2.

Theorem 2 and its re-expression (9) allow one to conclude the striking identities given as follows:

- For $y = 0$ and $z = 1$, the q-Taylor expansion of x^n about $x = 1$, (see [3], p. 23) becomes

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)_q^k.$$

- For $x = 1$, $y = -ab$ and $z = a$, the following identity (see [2], p. 25) is obtained

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} (1-a)_q^k (1-b)_q^{n-k}.$$

- For $y = -x$, the identity

$$\sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z-x)_q^{n-k} = 0.$$

  is found.

- For the case of $z = 0$ in (9), the q-binomial formula in (4) is reached.

- For $x = 1$, $y = -ab$ and $z = b$ in (9); the identity (see [2], p. 25)

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q b^k (1-a)_q^k (1-b)_q^{n-k}$$

  is stated.

Theorem 3. *For $x, y, z \in \mathbb{C}$, the following equations hold*

$$\frac{(x+y)_q^{\infty}}{(z+y)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k} = e_q^{\frac{(x-z)_q}{(1-q)z}}, \tag{10}$$

and

$$\frac{(x+y)_q^{\infty}}{(x-z)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z+y)_q^k}{(1-q)^k} \frac{1}{x^k} = e_q^{\frac{(z+y)_q}{(1-q)x}}. \tag{11}$$

Proof. As $n \to \infty$ in equation (8), it is arrived at

$$\begin{aligned}
(x+y)_q^{\infty} &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k \frac{(z+y)_q^n}{(z+yq^{n-k})_q^k} \\
&= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{1}{(1-q)^k} (x-z)_q^k \frac{(z+y)_q^{\infty}}{z^k}.
\end{aligned}$$

Dividing both sides of the last equation by $(z+y)_q^{\infty}$ gives

$$\frac{(x+y)_q^{\infty}}{(z+y)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k}.$$

By using Theorem 1, the right hand side of the previous equation can be re-written as $e_q^{\frac{(x-z)_q}{(1-q)z}}$ which completes the proof of equation (10). In a similar manner, the latter can be proven. □

Example 1. *If we take $x = 1$ and $y = -az$ in equation (11), we will get (see [2], p. 8)*

$$\frac{(1-az)_q^{\infty}}{(1-z)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z-az)_q^k}{(1-q)^k} = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} z^k = {}_1\phi_0(a; -; q, z).$$

The function on the right hand side of the above equation is called *basic hypergeometric series* and more details about it can be found in [2].
Now we concentrate about the q-exponential functions. At first, product of the q-exponential functions is given in the next theorem and then some properties of the q-exponential functions are derived.

Remark 1. *For $|x| < 1$ and $|q| < 1$, the following identity holds*

$$\frac{(1-y)_q^{\infty}}{(1-x)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{(x-y)_q^k}{(1-q)_q^k}. \tag{12}$$

Theorem 4. *For $x, y, z \in \mathbb{C}$, the following identity holds*

$$e_q^{(x+y)_q} = e_q^{(x-z)_q}\, e_q^{(z+y)_q}. \tag{13}$$

Proof. The identity (7) is taken to expand the q-exponential functions on the right hand side of (13), and thus

$$\begin{aligned}
e_q^{(x-z)_q}\, e_q^{(z+y)_q} &= \Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!}(x-z)_q^n\Big)\Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!}(z+y)_q^n\Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_q!}(x+y)_q^n = e_q^{(x+y)_q}.
\end{aligned}$$

□

Corollary 1. *For $x, z, \in \mathbb{C}$, the following identity holds*

$$e_q^{-(x+z)_q} = \frac{1}{e_q^{(z+x)_q}}.$$

Proof. By taking $y := -x$ and $z := -z$ in Theorem 4, the requirement can be easily carried out. □

Theorem 5. *For $x \in \mathbb{C}$ and $m, n \in \mathbb{Z}$, the following identity*

$$e_q^{(m-n)_q x} = \begin{cases} \prod_{j=n}^{m-1} e_q^{((j+1)-j)_q x} & if \ m > n \\ \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x} & if \ m < n \end{cases}$$

holds.

Proof. First, consider the case of $m > n$. The theorem is proven by induction. For the basis step, $m = n + 1$, the theorem is valid. Take the case $m = k$, $k > n$. Then it needs to be proven that it holds for the case $m = k + 1$. By using identity (13) and the induction, it can be reached

$$e_q^{((k+1)-n)_q x} = e_q^{((k+1)-k)_q x} e_q^{(k-n)_q x} = e_q^{((k+1)-k)_q x} \prod_{j=n}^{k-1} e_q^{((j+1)-j)_q x} = \prod_{j=n}^{k} e_q^{((j+1)-j)_q x}$$

which completes the proof of the first part.
For the case of $m < n$, Corollary 1 is used. Then the result of the first part is applied to get

$$e_q^{(m-n)_q x} = \frac{1}{e_q^{(n-m)_q (x)}} = \frac{1}{\prod_{j=m}^{n-1} e_q^{((j+1)-j)_q x}} = \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x}$$

which completes the proof. □

Corollary 2. *For $x \in \mathbb{C}$, and positive integers m and n, the following identities hold:*

$$e_q^{mx} = \prod_{j=0}^{m-1} e_q^{((j+1)-j)_q x}, \tag{14}$$

$$E_q^{-nx} = \prod_{j=0}^{n-1} e_q^{(j-(j+1))_q x} \tag{15}$$

Proof. Consideration of (7) with $n = 0$ and m any positive integer in Theorem 5 leads to the complete proof of the first identity. Replacing m and n values between each other in the first identity gives the proof of the second one. □

Now then, the n-*th* q-derivative of the q-exponential functions is found in the next theorem.

Theorem 6. *For $\alpha, \beta, x \in \mathbb{C}$ and positive integer n,*

$$D_q^n e_q^{(\alpha+\beta)_q x} = (\alpha + \beta)_q^n \ e_q^{(\alpha+q^n\beta)_q x}. \tag{16}$$

Proof. We use the induction to prove the theorem. For the case of $n = 1$, we need to get the q-derivative of $e_q^{(\alpha+\beta)_q x}$. So we use equation (7) and then take the q-derivative to obtain

$$D_q e_q^{(\alpha+\beta)_q x} = D_q \Big(\sum_{k=0}^{\infty} \frac{1}{[k]_q!} (\alpha+\beta)_q^k x^k \Big) = (\alpha+\beta) \sum_{k=0}^{\infty} \frac{1}{[k]_q!} (\alpha + q\beta)_q^k x^k = (\alpha+\beta) \ e_q^{(\alpha+q\beta)_q x}.$$

Assuming that (16) holds for a given k and to prove that it holds for $k + 1$, we need to obtain the q-derivative of $D_q^k e_q^{(\alpha+\beta)_q x}$. Hence

$$D_q^{k+1} e_q^{(\alpha+\beta)_q x} = D_q \Big(D_q^k e_q^{(\alpha+\beta)_q x} \Big) = (\alpha+\beta)_q^k \ D_q \Big(e_q^{(\alpha+q^k\beta)_q x} \Big) = (\alpha+\beta)_q^{k+1} \ e_q^{(\alpha+q^{k+1}\beta)_q x}.$$

Thus the proof is complete. □

Theorem 7. *For $|x| < 1$, $|q| < 1$ and any arbitrary α, the following identity holds*

$$e_q^{(1-q^\alpha)_q x} = \frac{1}{(1-(1-q)x)_q^\alpha} \tag{17}$$

Proof. To prove the theorem, we use equations (3), (5), (6) and (7). Then we have

$$e_q^{(1-q^\alpha)_q x} = e_q^x \; E_q^{-q^\alpha x} = \frac{1}{(1-(1-q)x)_q^\infty} \; (1-(1-q)q^\alpha x)_q^\infty = \frac{1}{(1-(1-q)x)_q^\alpha}$$

which completes the proof. □

Remark 2. *Equation (17) can be rewritten as $e_q^{(q^\alpha-1)_q x} = (1-(1-q)x)_q^\alpha$.*

In the next example, we show that the q-binomial theorem (see: [1] P. 247 or [9] P. 488) can be proven shortly by using Theorem 1.

Example 2. *For $|x| < 1$ and $|q| < 1$,*

$$\sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{[k]_q!} \big(\frac{x}{1-q}\big)^k = e_q^{\frac{(1-a)_q x}{(1-q)}} = e_q^{(\frac{x}{1-q})} E_q^{(\frac{-ax}{1-q})} = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}.$$

Note that to reach this result; (7) in the second and third equations, and (5) and (6) in the last equation have been considered.

3 Conclusions and Recommendation

Some striking properties of the q-exponential functions have been analyzed in detail. In doing so, the Gauss q-binomial identity has generalized and based on it, remarkable properties of the q-exponential have been established. For further studies, similar discussion can be carried out for q-trigonometric functions.

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