Improved Approaches for Common Fixed Points in Bi-S-Metric Spaces

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Abstract. We have formulated new definitions and concepts for Bi-S-metric spaces, specifically highlighting our discovery of the common fixed-points for two mappings that concentrate on Bi-S-metric spaces and originate from the same set. Applications, including Fredholm equations, substantiate our conclusions. This research develops and improves upon prior hypotheses regarding Bi-S-metric spaces.

Keywords: S-metric spaces; Common fixed-points; Existence solutions and Fredholm equations.

1. Introduction

It is commonly acknowledged that a key concept in fixed point theory is the introduction of new iterative algorithms and contractive conditions that expand previously published mappings and algorithms (for references, see [1, 2, 3, 4, 5]).

In 1979, Mishra [6] proved that the fixed-point theorem of Maia could be expanded to apply to bi-metric spaces. In their paper in 1984, J. Appell and M. P. Pera build on the work of Anselon and Ansorge on the approximate solution of nonlinear equations involving incompressible operators [7]. The investigation of two-metric spaces was initiated in the year 1992 by a group of academics, the most notable of them was Dhage [8], who was the pioneer in the concept of D-spaces. Various fixed-points were demonstrated by Mustafa and Sims [9] in 2006, and the concept of G-metric spaces was introduced as generalizing of classical metric spaces.

Following the concepts outlined in [10], which relate to a generalization of D^*-metric space. In their publication [11], Sedghi et al. demonstrated a range of characteristics pertaining to S-metric spaces. In 2012, numerous fixed-point results were established for self-maps defined on S-metric spaces. In 2013, Gupta [12] introduced the concept of cyclic contraction in S-metric spaces and established numerous fixed-point results. Theorems presented here act as suitable generalizations of the discoveries made by Sedghi et al. [11].

For multivalued mappings on complete S-metric spaces, Sedghi et al. proved a common fixed points in 2014 [13]. In 2018, Suzuki extended the idea of fixed points in metric spaces to the context of metric S-spaces, building on the work done in [14].

In 2022, Pourgholam and colleagues construct several well-known fixed-point results for single-valued and multi-valued mappings via employing a generalization of coincidence points in S-metric spaces [15].

Using the C-class function, Saluja demonstrated in 2022 that there are shared fixed points on S-metric spaces for two sets of weakly compatible mappings [16].

In 2024, Hashoosh and Dasher unveiled groundbreaking discoveries in the realm of fixed-points specifically tailored in S-metric spaces. The results were achieved through the use of the tri-simulation function, as outlined in [5].

This study aims to uncover new findings regarding common fixed points in double S-metric spaces, referred to as Bi-S-metric spaces. Specific instances of the Fredholm integral equation will be demonstrated.

2. Preliminaries

To make sure we cover everything, we will review the topic's basic description and outcomes in the part that follows.

Definition 2.1[11]. Consider that Ξ is considered a non-empty set and $S: \Xi^3 \to [0, \infty)$ be a function meeting the given requirements, for each $\sigma, \varrho, \omega, a \in \Xi$.

1. $S(\sigma, \varrho, \omega) \geq 0$;
2. $S(\sigma, \varrho, \omega) = 0$ if and only if $\sigma = \varrho = \omega$;
3. $S(\sigma, \varrho, \omega) \leq S(\sigma, \sigma, a) + S(\varrho, \varrho, a) + S(\omega, \omega, a)$.

Subsequently, S is designated as an S-metric space on Ξ, and the pair (Ξ, S) is referred to as an S-metric space.

Definition 2.2 [11]. If (Ξ, S) denote an S-metric space, what follows is the definition of a sequence $\{\sigma_n\}$ of Ξ.

1. $\{\sigma_n\}$ in Ξ converges to σ if, and just if $S(\sigma_n, \sigma_n, \sigma) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ in which $S(\sigma_n, \sigma_n, \sigma) < \varepsilon$ for each $n \geq n_0$. This is indicated by $\lim_{n\to\infty} \sigma_n = \sigma$;

2. $\{\sigma_n\}$ in Ξ is defined as a Cauchy if for all $\varepsilon > 0$,there is $n_0 \in \mathbb{N}$ whereby $S(\sigma_n, \sigma_n \sigma_m) < \varepsilon$ for each $n, m \geq n_0$;

3. If all Cauchy sequence converges, we consider a space is complete.

Example 2.3 ([5]). Assuming $\|.\|$ is a norm on $\mathbb{R}^n$, the S-metric on $\mathbb{R}^n$ is defined as $S(x, y, \lambda) = \|y + \lambda - 2x\| + \|y - \lambda\|$.

:**Lemma 2.4** [11] In the space of S-metrics (Ξ, S), the following features manifest

1. For all $x, y \in \Xi$, we have $S(x, x, y) = S(y, y, x)$.

2. If $x_n \to x$ and $y_n \to y$ such as $n \to \infty$ then $S(x_n, x_n, y_n) \to S(x, x, y)$ where $n \to \infty$.

Lemma 2.5 [17] Consider (Ξ, d) as a complete b-metric space, if $\{x_n\}$ denote a sequence within Ξ, and $r \in [0; 1)$ such that the next inequality holds,

$$\delta(x_{n+1}, x_{n+2}) \leq r\, \delta(x_n, x_{n+1}) \leq \text{ for any } n \in \mathbb{N}.$$

Then, $\{\eth_n\}$ constitutes a Cauchy sequence in Ξ.

3. MAIN RESULT

Within the context of S-metric spaces, we present the idea of Bi-S metric spaces in this chapter. We examine whether common fixed-point theorems exist in these recently constructed spaces and whether they are unique. Our findings open the door for more research in mathematical analysis by improving and generalizing many well-established results from the body of existing literature.

Definition.3.1. (Ξ, S_1, S_2) is referred to as a Bi-S-metric space if (Ξ, S_1) and (Ξ, S_2) are two S-metric spaces.

Below is an example that clarifies the definition of Bi-S-metric space

Example 3.2. Assume that $(\mathbb{R}, S_1)$ and$(\mathbb{R}, S_2)$ be S_i −metric spaces, for all i= $\overline{1,2}$ and $S_1, S_2: R^3 \to [0,\infty)$ is described via:

$S_1(ð, y, ƛ) = |y + ƛ - 2ð| + |y - ƛ|$ and $S_2 = |ð + ƛ - 2y|$ for all $ð, y, ƛ \in \mathbb{R}$. Then $(\mathbb{R}, S_1, S_2)$ is a Bi-S-metric space.

The next Lemma is an improvement of the Lemma 2.5 that contributes to solving our main results.

Lemma 3.3 Consider $\{ð_n\}$ be a sequence in Ξ and that space (Ξ, S) is a complete S-metric, and assuming $\{ð_n\}$ is a sequence in Ξ that fulfills

$$S(ð_{n+1}, ð_{n+2}, ð_{n+3}) \le rS(ð_n, ð_{n+1}, ð_{n+2}) \quad \forall n \in \mathbb{N}, \text{and } r \in [0,1)$$

then, a sequence $\{ð_n\}$ is Cauchy in Ξ.

Proof. For any n, from the inequality provided:

$$S(ð_{n+1}, ð_{n+2}, ð_{n+3}) \le rS(ð_n, ð_{n+1}, ð_{n+2}) \quad \forall n \in \mathbb{N}.$$

Then,

$$S(ð_{n+2}, ð_{n+3}, ð_{n+4}) \le r^2 S(ð_n, ð_{n+1}, ð_{n+2}).$$

Similarly, we have

$$S(ð_{n+3}, ð_{n+4}, ð_{n+5}) \le r^3 S(ð_n, ð_{n+1}, ð_{n+2}).$$

By induction: for all $n \in \mathbb{N}$ and $k \ge 1$, then

$$S(ð_{n+k}, ð_{n+k+1}, ð_{n+k+2}) \le r^k S(ð_n, ð_{n+1}, ð_{n+2}).$$

We now sum these inequalities

$$S(ð_n, ð_{n+m}, ð_{n+m+k}) \le S(ð_n, ð_{n+1}, ð_{n+2}) + S(ð_{n+1}, ð_{n+2}, ð_{n+3}) + \cdots + S(ð_{n+m-1}, ð_{n+m}, ð_{n+m+1}).$$

Using the previous inequalities, we obtain

$$S(ð_n, ð_{n+m}, ð_{n+m+k}) \le S(ð_n, ð_{n+1}, ð_{n+2})(1 + r + r^2 + \cdots + r^{m-1}).$$

Since $r \in [0,1)$, the geometric series $1 + r + r^2 + \cdots + r^{m-1}$ converges to $\frac{1-r^m}{1-r}$. Since $r < 1$. Then,

$r^m \to 0$ as $m \to \infty$. This is implying

$$S(ð_n, ð_{n+m}, ð_{n+m+k}) \le \frac{S(ð_n, ð_{n+m}, ð_{n+2})}{1-r}.$$

We can pick m and n so that $S(ð_n, ð_{n+m}, ð_{n+m+k})$ is smaller than any $r > 0$. Thus, it is proved that $\{ð_n\}$ is a Cauchy sequence.

We provide a considerable improvement over Theorem 3.1 from [17] to improve our findings in Bi-S metric spaces. This new theorem strengthens our paper's basic notions with deeper insights and more durable answers.

Theorem 3.4. Consider that (Ξ, S_1, S_2) is a Bi-S-metric spaces, in which

i. $S_1(\lambda, \lambda, \varrho) \leq S_2(\lambda, \lambda, \varrho)$.

(ii) Consider that $F, T: \Xi \to \Xi$ be two self-maps on Ξ that satisfy

$$S_2(F\lambda, F\lambda, T\varrho) \leq \alpha \frac{S_2(\lambda, \lambda, F\lambda) \cdot S_2(\varrho, \varrho, T\varrho)}{S_2(\varrho, \varrho, T\varrho) + S_2(\varrho, \varrho, F\lambda)}$$

$$+\beta \frac{S_2(\lambda,\lambda,\varrho)[1+ S_2(\lambda,\lambda,F\lambda)+ S_2(\varrho,\varrho,F\lambda)]}{1+S_2(\lambda,\lambda,\varrho)+ S_2(\lambda,\lambda,F\lambda).S_2(\lambda,\lambda,T\varrho).S_2(\varrho,\varrho,F\lambda).S_2(\varrho,\varrho,T\varrho)} + \gamma \frac{[S_2(\lambda,\lambda,F\lambda).S_2(\lambda,\lambda,T\lambda)]}{S_2(\lambda,\lambda,\varrho)}. \quad (3.1)$$

Where, $\alpha, \beta, \gamma \epsilon$ [0,1) and $\alpha + \beta + 3\gamma < 1$.

iii. There is a point $\lambda_0 \in \Xi$ in which the sequence $\{\lambda_n\}$ of iterates defined as $\lambda_1 = F\lambda_0$, $\lambda_2 = T\lambda_1, \dots$, $\lambda_{2n} = T\lambda_{2n-1}$, $\lambda_{2n+1} = \lambda_{2n}$ for each $n \in \mathbb{N}$ has a convergent subsequence λ_{n_k}converging to λ^* in (Ξ, S_1).

iv. In (Ξ, S_1) both mappings F and T are continuous.

Subsequently, F also T possess a unique common fixed point in Ξ.

Proof. Consider that, $\lambda_0 \in \Xi$ and $\{\lambda_n\}$ represent sequences of iterates over Ξ, indicated by

$$F(\lambda_{2n}) = \lambda_{2n+1} \text{ and } T(\lambda_{2n-1}) = \lambda_{2n}, \text{ for all } \forall n \in \mathbb{N}. \quad (3.2)$$

Using the equation (3.1) and (3.2), we conclude that,

$$S_2(\lambda_{2n+1}, \lambda_{2n+1}, \lambda_{2n+2})$$

$$= S_2(F\lambda_{2n}, F\lambda_{2n}, T\lambda_{2n+1})$$

$$\leq \alpha \frac{S_2(\lambda_{2n}, \lambda_{2n}, F\lambda_{2n}).S_2(\lambda_{2n+1}, \lambda_{2n+1} T\lambda_{2n+1},)}{S_2(\lambda_{2n+1}, \lambda_{2n+1}, T\lambda_{2n+1}) + S_2(\lambda_{2n+1}, \lambda_{2n+1}, F\lambda_{2n})}$$

$$+\beta \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})[1+S_2(\lambda_{2n},\lambda_{2n},F\lambda_{2n})+S_2(\lambda_{2n+1},\lambda_{2n+1},F\lambda_{2n})]}{1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})+S_2(\lambda_{2n},\lambda_{2n},F\lambda_{2n})\cdot S_2(\lambda_{2n},\lambda_{2n},T\lambda_{2n+1})\cdot S_2(\lambda_{2n+1},\lambda_{2n+1},F\lambda_{2n})\cdot S_2(\lambda_{2n+1},\lambda_{2n+1},T\lambda_{2n+1})} +$$
$$\gamma \frac{S_2(\lambda_{2n},\lambda_{2n},F\lambda_{2n}).S_2(\lambda_{2n},\lambda_{2n},T\lambda_{2n+1})}{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$\leq \alpha \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1}).S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2})}{S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2})}$$

$$\leq \beta \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})[1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})]}{1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$+\gamma \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})\cdot S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+2})}{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$\leq \alpha S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1}) + \beta S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1}) + \gamma S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+2}).$$

Then,

$$S_2(\lambda_{2n+1}, \lambda_{2n+1}, \lambda_{2n+2})$$

$$\leq (\alpha + \beta) S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1}) + \gamma\ S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1}) + \gamma\ S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1})$$
$$+ \gamma\ S_2(\lambda_{2n+1}, \lambda_{2n+1}, \lambda_{2n+2}).$$

It means that

$$(1-\gamma)S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2}) \leq (\alpha+\beta+2\gamma)S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1}).$$

So,

$$S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2}) \leq \frac{(\alpha+\beta+2\gamma)}{1-\gamma}S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1}).$$

Therefore, In general for all $n \in \mathbb{N}$

$$S_2\ (\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2}) \leq k\, S_2\ (\lambda_{2n},\lambda_{2n},\lambda_{2n+1}). \tag{3.3}$$

Where $\mathrm{k} = \frac{(\alpha+\beta+2\gamma)}{1-\gamma} < 1.$

By virtue of Lemma 3.1, it can be determined that the sequence $\{\lambda_n\}$ is a Cauchy sequence in the set Ξ. The Cauchy sequence$\{\lambda_n\}$, which is defined by (3.2), possesses a convergent subsequence $\{\lambda_{n_k}\}$ in (Ξ,S_1) that is convergent to λ^* in (Ξ,S_1), and thus, the sequence $\{\lambda_n\}$ too convergent to λ^* in (Ξ,S_1), according to the Cauchy sequence. From this,

$$\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda_{2n} = \lim_{n\to\infty}\lambda_{2n-1} = \lim_{n\to\infty}\lambda_{2n-2} = \lambda^*.$$

As a result of the fact that both F and T are continuous in the space (Ξ,S_1) we are now able to demonstrate that λ^* is a fixed point of both mappings F and T. Therefore,

$$F(\lambda^*) = F[\lim_{n\to\infty}\lambda_n] = \lim_{n\to\infty}[F\lambda_{2n}] = \lambda^*.$$

Similarly,

$$T(\lambda^*) = T[\lim_{n\to\infty}\lambda_{n-1}] = \lim_{n\to\infty}[T\lambda_{2n-1}] = \lambda^*$$

Consequently, λ^* represents a common fixed point for the mappings F and T.

Consider that λ^* and ϱ^* is fixed points of F and T, respectively. Therefore, $F\lambda^* = T\lambda^* = \lambda^*$ and $F\varrho^* = T\varrho^* = \varrho^*$ and

$$S_2(\lambda^*,\lambda^*,\varrho^*) = S_2(F\lambda^*.F\lambda^*.T\varrho^*)$$

$$\leq \alpha\frac{S_2(\lambda^*,\lambda^*,F\lambda^*)\quad S_2(\varrho^*,\varrho^*,T\varrho^*)}{S_2(\varrho^*,\varrho^*,T\varrho^*)+S_2(\varrho^*,\varrho^*,F\lambda^*)} + \beta\frac{S_2(\lambda^*,\lambda^*,\varrho^*)[1+S_2(\lambda^*,\lambda^*,F\lambda^*)+S_2(\varrho^*,\varrho^*,F\lambda^*)}{1+S_2(\lambda^*,\lambda^*,\varrho^*)+S_2(\lambda^*,\lambda^*,F\lambda^*).S_2(\lambda^*,\lambda^*,T\varrho^*)\cdot S_2(\varrho^*\varrho^*F\lambda^*).S_2(\varrho^*,\varrho^*,T\varrho^*)}$$
$$+\gamma\frac{[S_2(\lambda^*,\lambda^*,F\lambda^*)\quad S_2(\lambda^*,\lambda^*,T\varrho^*)]}{S_2(\lambda^*,\lambda^*,\varrho^*)}.$$

Then,

$$S_2(\lambda^*,\lambda^*,\varrho^*) \leq \alpha\frac{S_2(\lambda^*,\lambda^*,\lambda^*)\ S_2(\varrho^*,\varrho^*,\varrho^*)}{S_2(\varrho^*,\varrho^*,\varrho^*)+S_2(\varrho^*,\varrho^*,\lambda^*)} + \beta\frac{S_2(\lambda^*,\lambda^*,\varrho^*)[1+S_2(\lambda^*,\lambda^*,\lambda^*)+S_2(\varrho^*,\varrho^*,\lambda^*)]}{1+S_2(\lambda^*,\lambda^*,\varrho^*)+S_2(\lambda^*,\lambda^*,\lambda^*).S_2(\lambda^*,\lambda^*,\varrho^*).S_2(\varrho^*,\varrho^*,\lambda^*).S_2(\varrho^*,\varrho^*\varrho^*)}$$

$$+\gamma\frac{[S_2(\lambda^*,\lambda^*,\lambda^*)\,S_2(\lambda^*,\lambda^*,\varrho^*)]}{S_2(\lambda^*,\lambda^*,\varrho^*)}$$

$$\leq \beta\frac{S_2(\lambda^*,\lambda^*,\varrho^*)[1+S_2(\lambda^*,\lambda^*,\varrho^*)]}{[1+S_2(\lambda^*,\lambda^*,\varrho^*)]}.$$

Then,

$$S_2\left(\lambda^*, \lambda^*, \varrho^*\right) \leq \beta\, S_2\left(\lambda^*, \lambda^*, \varrho^*\right).$$

But $\beta < 1$, This indicates a contradiction. Therefore, F and T possess a unique common fixed point in Ξ.

Corollary 3.5. Given that (Ξ, S_1) is a complete Bi-S-metric space and that conditions (i), (ii), and (iv) of Theorem (3.4) are met, we can say that the mappings T, S: $\Xi \rightarrow \Xi$ a unique common fixed point in the space Ξ.

Theorem 3.6. Consider (Ξ, S_1) and (Ξ, S_2) as Bi-S-metric spaces. Given that p = {Ti : i ∈ I, the collection of positive integers} is a set of mappings on Ξ in which the next situations hold.

i . $S_1(\lambda, \lambda, \varrho) \leq S_2(\lambda, \lambda, \varrho)$ for all $\lambda, \varrho \in \Xi$.

ii. Ξ completed processing with regard to S_1.

iii. for all $T_j: \Xi \rightarrow \Xi \in p$ there is $T_i: \Xi \rightarrow \Xi \in p$ such that

$$S_2\left(T_i{}^m\lambda, T_i{}^m\lambda, T_j{}^n\varrho\right) \leq \alpha \frac{S_2(\lambda,\lambda,\varrho)\, S_2(\varrho,\varrho,T_j{}^n\varrho)}{S_2(\lambda,\lambda,T_i{}^m\lambda)+S_2(\varrho,\varrho,T_i{}^m\lambda)} + \beta \frac{S_2(\lambda,\lambda,\varrho)[1+S_2(\lambda,\lambda,T_i{}^m\lambda)+S_2(\varrho,\varrho,T_i{}^m\lambda)]}{1+S_2(\lambda,\lambda,\varrho)} + \gamma \frac{[S_2(\lambda,\lambda,T_i{}^m\lambda).S_2(\lambda,\lambda,T_j{}^n\varrho)]}{S_2(\lambda,\lambda,\varrho)}. \tag{3.4}$$

With m and n being positive integers and α, β, and γ being integers in $[0,1)$, where $\alpha + \beta + 3\gamma < 1$.

iv. If the mapping g_i is continuous in (Ξ, S_1) for every i ∈ I, therefore, there is a unique shared fixed point for P.

Proof. Presuming λ_0 is an element of Ξ, we establish a sequence of iterates $\{\lambda_n\}$ in Ξ while

$$\lambda_{2n-1} = T_i{}^m(u_{2n-2}) \text{ and } \lambda_{2n} = T_j{}^n(\lambda_{2n-1}) \quad, \forall n \in \mathbb{N}. \tag{3.5}$$

Using equation (3.4) and (3.5), we obtain that,

$$S_2(\lambda_{2n+1}, \lambda_{2n+1}, \lambda_{2n+2}) = S_2\left(T_i{}^m\lambda_{2n}, T_i{}^m\lambda_{2n}, T_j{}^n\lambda_{2n+1}\right)$$

$$\leq \alpha \frac{S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1})\; S_2\left(\lambda_{2n+1}, \lambda_{2n+1}, T_j{}^n \lambda_{2n+1},\right)}{S_2(\lambda_{2n}, \lambda_{2n}, T_i{}^m\lambda_{2n}) + S_2(\lambda_{2n+1}, \lambda_{2n+1}, T_i{}^m\lambda_{2n})}$$

$$+ \beta \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})[1+S_2(\lambda_{2n},\lambda_{2n}\, T_i{}^m\lambda_{2n},)+S_2(\lambda_{2n+1},\lambda_{2n+1},T_i{}^m\lambda_{2n})]}{1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$+\gamma \frac{S_2(\lambda_{2n},\lambda_{2n}\, T_i{}^m\lambda_{2n},)\; S_2(\lambda_{2n},\lambda_{2n},T_j{}^n\lambda_{2n+1})}{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$\leq \alpha \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})\quad S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+2})}{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})+S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+1})}$$

$$+\beta \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})[1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})+S_2(\lambda_{2n+1},\lambda_{2n+1},\lambda_{2n+1})]}{1+S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$+\gamma \frac{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})\; S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+2})}{S_2(\lambda_{2n},\lambda_{2n},\lambda_{2n+1})}$$

$$\leq \alpha\, S_2(\lambda_{2n+1}, \lambda_{2n+1}, \lambda_{2n+2}) + \beta \frac{S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1})[1 + S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1})]}{1 + S_2(\lambda_{2n}, \lambda_{2n}, \lambda_{2n+1})}$$

$$+ \gamma\, S_2\,(ƛ_{2n}, ƛ_{2n}\,, ƛ_{2n+2}\,)$$

$$\leq \alpha\, S_2\,(ƛ_{2n+1}, ƛ_{2n+1}\,, ƛ_{2n+2}\,) + \beta S_2\,(ƛ_{2n}\,\,, ƛ_{2n}\,\,, ƛ_{2n+1}\,) + \,\gamma\, S_2\,(ƛ_{2n}, ƛ_{2n}\,, ƛ_{2n+2}\,)$$

Then,

$$\begin{aligned} S_2(ƛ_{2n+1}, ƛ_{2n+1}\,, ƛ_{2n+2}\,) &\\ &\leq \alpha\, S_2\,(ƛ_{2n+1}, ƛ_{2n+1}\,, ƛ_{2n+2}\,) + \beta S_2\,(ƛ_{2n}\,\,, ƛ_{2n}\,\,, ƛ_{2n+1}\,) + \,\gamma\, S_2\,(ƛ_{2n}, ƛ_{2n}\,, ƛ_{2n+1}\,) \\ &+ \gamma\, S_2\,(ƛ_{2n}, ƛ_{2n}\,, ƛ_{2n+1}\,) + \gamma\, S_2(\,ƛ_{2n+1}, ƛ_{2n+1}, ƛ_{2n+2}). \end{aligned}$$

It means that,

$$(\,1 - \alpha - \gamma) S_2\,(\,ƛ_{2n+1}\,, ƛ_{2n+1}\,, ƛ_{2n+2}\,) \leq (\beta + 2\gamma\,) S_2(\,ƛ_{2n}, ƛ_{2n}, ƛ_{2n+1}\,).$$

So,

$$S_2\,(ƛ_{2n+1}, ƛ_{2n+1}\,, ƛ_{2n+2}\,) \leq \frac{(\,\beta+2\gamma)}{(1-\alpha-\gamma)} S_2(\,ƛ_{2n}, ƛ_{2n}, ƛ_{2n+1}\,).$$

Then, in general for all $n \in \mathbb{N}$, we have

$$S_2\,\,(ƛ_{2n+1}, ƛ_{2n+1}\,, ƛ_{2n+2}\,) \leq r.\, S_2\,\,(ƛ_{2n}, ƛ_{2n}\,, ƛ_{2n+1}\,),$$

where $r\; = \frac{(\beta+2\gamma)}{1-\alpha-\gamma} < 1$.

According to Lemma 3.3, the sequence $\{ƛ_{\mathrm{n}}\}$ is qualifies as a Cauchy sequence within the set Ξ. Considering that the sequence $\{ƛ_{\mathrm{n}}\}$ distinct by (3.2) is a Cauchy sequence in Ξ and that $(\,\Xi, \mathrm{S}_1)$ is complete, it follows that the sequence$\{ƛ_n\}$ is convergent in $(\,\Xi, S_1)$ because of the completeness of the set. As a result, $\lim_{n\to\infty} ƛ_n = \lim_{n\to\infty} ƛ_{2n} = \lim_{n\to\infty} ƛ_{2n-1} = \lim_{n\to\infty} ƛ_{2n+1} = ƛ^*$.

Then, we demonstrate that $ƛ^*$ is a fixed point in both the ${T_i}^m$ and ${T_j}^n$ mappings. Because ${T_i}^m$ and ${T_j}^n$ are continuous in $(\,\Xi, S_1)$ it follows that

$${T_i}^m\,(ƛ^*) = {T_i}^m[\lim_{n\to\infty} ƛ_n] = \lim_{n\to\infty}[{T_i}^m\,ƛ_{2n}] = \,ƛ^*.$$

Similarly,

$${T_j}^n\,(ƛ^*\,) = {T_j}^n[\lim_{n\to\infty} ƛ_{n-1}] = \lim_{n\to\infty}\left[{T_j}^n ƛ_{2n-1}\right] = \,ƛ^*.$$

Therefore, $ƛ^*$provides a common fixed point of the mappings ${T_i}^m$ and ${T_j}^n$.

Considering $ƛ^*$ and ϱ^* denote both of the fixed points of the mappings ${T_i}^m$ and ${T_j}^n$. Consequently

${T_i}^m ƛ^* = {T_j}^n ƛ^* = ƛ^*$ and ${T_i}^m \varrho^* = {T_j}^n \varrho^* = \varrho^*$. Then,

$$S_2(ƛ^*, ƛ^*, \varrho^*) \;=\; S_2({T_i}^m ƛ^*, {T_i}^m ƛ^*, {T_j}^n \varrho^*)$$

$$\begin{aligned} &\leq \alpha \frac{S_2\,(\,ƛ^*, ƛ^*, \varrho^*)\;\,S_2\,(\varrho^*, \varrho^*, {T_j}^n \varrho^*)}{S_2\,(ƛ^*, ƛ^*, {T_i}^m ƛ^*) + S_2\,(\varrho^*\,, \varrho^*, {T_i}^m ƛ^*\,)} + \beta \frac{S_2\,(ƛ^*, ƛ^*, \varrho^*)[1 + S_2\,(ƛ^*, ƛ^* {T_i}^m ƛ^*) + S_2\,(\varrho^*\,, \varrho^* {T_i}^m ƛ^*\,)}{1 + S_2\,(\,ƛ^*, ƛ^*, \varrho^*)} \\ &+ \gamma \frac{[S_2\,(\,ƛ^*, ƛ^*, {T_i}^m ƛ^*)\;\,S_2\,(ƛ^*\,, ƛ^*, {T_j}^n\,\varrho^*\,)]}{S_2\,(ƛ^*, ƛ^*, \varrho^*)}. \end{aligned}$$

So,

$$S_2(\lambda^*,\lambda^*,\varrho^*) \leq \alpha \frac{S_2(\lambda^*,\lambda^*,\varrho^*)\ \ S_2(\varrho^*,\varrho^*,\varrho^*)}{S_2(\lambda^*,\lambda^*,\lambda^*)+\ S_2(\varrho^*,\varrho^*,\lambda^*)} + \beta \frac{S_2(\lambda^*,\lambda^*,\varrho^*)[1+S_2(\lambda^*,\lambda^*,\lambda^*)+S_2(\varrho^*,\varrho^*,\lambda^*)]}{1+S_2(\lambda^*,\lambda^*,\varrho^*)}$$

$$+\gamma \frac{[S_2(\lambda^*,\lambda^*,\lambda^*)\, S_2(\lambda^*,\lambda^*,\varrho^*)]}{S_2(\lambda^*,\lambda^*,\varrho^*)}$$

$$\leq \beta \frac{S_2(\lambda^*,\lambda^*,\varrho^*)[1+S_2(\varrho^*,\varrho^*,\lambda^*)]}{[1+S_2(\lambda^*,\lambda^*,\varrho^*)]}.$$

Then,

$$S_2(\lambda^*,\lambda^*,\varrho^*) \leq \beta\, S_2(\lambda^*,\lambda^*,\varrho^*).$$

But $\beta < 1$,Therefore, we have,

$$S(\lambda^*,\lambda^*,\varrho^*) < S(\lambda^*,\lambda^*,\varrho^*).$$

The statement is incongruous. Because of this, λ^* is unique common fixed point by $T_i{}^m$ and $T_J{}^n$. While the fixed point of $T_i{}^m$ is a fixed point of T_i, the fixed point of $T_j{}^n$ is a fixed point of T_j. Both of these fixed points are the same. In light of this, λ^* is unique common fixed point of T_i and T_j. As a result, λ^* is the only common fixed point that is unique to P. We have completed the proofing process.

4. Application of Integral Equations.

The most exciting applications of fixed-point theorems are in function spaces. It is integral equation theorems in S-metric spaces generalize conventional results. These theorems greatly expand mathematical analysis by establishing integral equation solutions exist and are unique. Discussions include Fredholm equation solutions' existence and uniqueness. 4.1. Integral equation of Fredholm.

At this point in time, we will look into the existence of a unique solution to Fredholm's nonlinear integral problem. We applied the fundamental result gained from Theorem (3.4).

Let us take into consideration the Fredholm integral equation of the second kind.

$$f(ð) = g(ð) + \lambda \int_0^1 K(ð;t), f(t))dt \qquad \text{for } t, ð \in [0,1]. \tag{4.1}$$

Where g and K are continuous functions and λ is a constant.

Define $T: \Xi \to \Xi$ is any self-operator, by

$$T(f)(ð) = g(ð) + \lambda \int_0^1 K(ð;t), f(t))dt \qquad \text{for } t, ð \in [0,1]. \tag{4.2}$$

Considering $g: [0,1] \to \mathbb{R}$ and $K: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous function,

To solve the integral equation so that it corresponds to the fixed point of T, we assume that f^* is found such that $T(f)(ð) = f(ð)$.

Here, we consider the Bi-S- metric space (Ξ, S_1, S_2), where $\Xi = C([0,1])$ is the set of every continuous functions distinct on $[0,1]$. S_1 and S_2 are two S-metric spaces as follows

$$S_1(f,g,h) = \int_0^1 |f(ð) - g(ð)|\, dð + \int_0^1 |f(ð) - h(ð)|\, dð.$$

$$S_2(f,g,h) = \left(\int_0^1 |f(ð) - g(ð)|^p\, dð\right)^{\frac{1}{p}} + \left(\int_0^1 |f(ð) - h(ð)|^p\, dð\right)^{\frac{1}{p}},$$

Claim 1: verification of the condition (i) for Theorem 3.4. From Holder's inequality between L^1 −norm and L^p −norm, we have

$\int_0^1 |f(ð) - g(ð)|\, dð \leq \left(\int_0^1 |f(ð) - g(ð)|^p\, dð\right)^{\frac{1}{p}}.$

And similarity,

$$\int_0^1 |f(ð) - h(ð)|\, dð \leq \left(\int_0^1 |f(ð) - h(ð)|^p\, dð\right)^{\frac{1}{p}}.$$

So,

$$S_1(f,g,h) \leq S_2(f,g,h).$$

Claim 2: verification of the condition (ii) for Theorem 3.4. For F = T in condition (ii) is became

$$S_2(T(f),T(f),T(h)) \leq \alpha \frac{S_2(f,f,T(f)) \cdot S_2(h,h,T(h))}{S_2(h,h,T(h)) + S_2(h,h,T(f))}$$

$$+\beta \frac{S_2\,(f,f,h)[1+ S_2\,(f,f,T(f)) + S_2\,(h,h,T(f))]}{1 + S_2\,(f,f,h) + S_2\,(f,f,T(f)\,).S_2\,(h,h,T(h)).S_2(h,h,T(f)).S_2\,(h,h,T(h))}$$

$$+\gamma \frac{[S_2\,(f,f,T(f)).S_2\,(f,f,T((f))]}{S_2\,(f,f,h)},$$

where, $\alpha, \beta, \gamma \epsilon\, [0,1)$ and $\alpha + \beta + 3\gamma < 1$.

Since T is identity operator and with simple calculations, one can have

$$S_2(T(f),T(f),T(h)) \leq \beta\, S_2\,(f,f,h).$$

Since $\beta \epsilon\, [0,1)$ and K is continuous functions and bounded in Fredholm integral equation. At this time, T is sufficiently contractive to satisfy the following condition:

$$S_2(T(f),T(f),T(h)) < S_2\,(f,f,h).$$

Claim 3: $\{f_n\}$ is converge sequence to f^* in (Ξ, S_1).

Define the iterative sequence $f_{n+1}(ð) = T(f_n)(ð)$. Let us starting from $f_0\,(ð) \in \Xi$, from the first and second conditions we get that S_1 is contractive with repetition. So, $S_1\,(f_{n+1}, f_{n+1}, f_n) \to 0$ as $n \to \infty$. This means that the iterative series f_n converges in S_1-metric space to the fixed point f^*.
Claim 4: T and F are continuous functions. This is directly because from the continuity of the function g and the kernel function K. Then, all conditions are satisfied in the Bi- S -metric space, and hence by Theorem 3.4, F and T possess a unique common fixed point f^* which is the solution of the Fredholm integral equation.

5. Conclusions and Future works

5.1 Conclusions

The purpose of this study is to demonstrate that there are common fixed points for two contractive kind mappings in Bi-S-metric space, and that these fixed points are both unique and exist. In addition, the fact that there are solutions to the Fredholm problem, as well as the fact that those solutions are one of a kind. In mathematical model analysis and equilibrium theories, fixed points play a vital role. These problems encompass hemi-equilibrium, variational, and KKM issues. Further reading on these topics is available (see [18-21]).

5.2 Future works

Future results include studying and finding new definitions and theories for Bi-rectangular S-metric, Bi-order S-metric or Bi-partial S-metric spaces.

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