SOME NEW FUZZY BEST PROXIMITY POINT THEOREMS IN NON-ARCHIMEDEAN FUZZY METRIC SPACES

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ABSTRACT. In this paper, we define fuzzy weak P-property. Then we prove a fuzzy best proximity point theorems for γ-contractions with condition fuzzy weak P-property. Later, we give definition of fuzzy isometric distance between two functions in non-Archimedean fuzzy metric spaces. Also, we introduce γ-proximal contraction type-1 and type-2 contraction respectively via functions preserving fuzzy isometric distance and providing fuzzy isometry. Then, we obtain some fuzzy best proximity results for γ-proximal contractions types in non-Archimedean fuzzy metric spaces. Finally, we present some examples to illustrate the validity of the definitions and results obtained in the paper.

1. Introduction and Preliminaries

The Banach contraction principle found by Banach has an important resonance in mathematics as well as in other fields [1]. Later, the subject of fixed point theory attracted the attention of many aouthors and caused this subject to be discussed in different areas of mathematics and different topological spaces. Then, authors intensively introduced many works regarding the fixed point theory. On the other hand, the concept of fuzzy metric space was introduced in different ways by some authors (see [2, 7]). Importantly, Gregori and Sapena [5] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. At the same time, there are presented by many authors by expanding the Banach's result in the literature (see [9–11, 14, 16, 20, 21]).

In this work, we prove some fuzzy best proximity point results for mappings providing γ-proximal contractions. Then, we give some examples are supplied in order to support the useability of our results. Also, we show that our main results are more general than known results in the existing literature.

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Definition 1. *[12] A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangular norm (in short, continuous* $t-$*norm) if it satisfies the following conditions:*

(TN-1) $*$ *is commutative and associative,*
(TN-2) $*$ *is continuous,*
(TN-3) $*(a,1) = a$ *for every* $a \in [0,1]$,
(TN-4) $*(a,b) \leq *(c,d)$ *whenever* $a \leq c$, $b \leq d$ *and* $a,b,c,d \in [0,1]$.

An arbitrary $t-$norm $*$ can be extended (by associativity) in a unique way to an nary operator taking for $(x_1, x_2, ..., x_n) \in [0,1]^n$, $n \in N$, the value $*(x_1, x_2, ..., x_n)$ is defined, in [4], by $*_{I=1}^{0} x_i = 1$, $*_{I=1}^{n} x_i = *(*_{I=1}^{n-1} x_i, x_n) = *(x_1, x_2, ..., x_n)$.

Definition 2. *[3] A fuzzy metric space is an ordered triple* $(X, M, *)$ *such that* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$, *satisfying the following conditions, for all* $x, y, z \in X,\ s, t > 0:$

(FM-1) $M(x,y,t) > 0$,
(FM-2) $M(x,y,t) = 1$ *iff* $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$,
(FM-4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,
(FM-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

If, in the above definition, the triangular inequality (FM-4) is replaced by
(NA) $M(x,z,max\{t,s\}) \geq M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X,\ s,t > 0$, or equivalently,
$M(x,z,t) \geq M(x,y,t) * M(y,z,t)$
then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space [6].

Definition 3. *Let* $(X, M, *)$ *be a fuzzy metric space (or non-Archimedean fuzzy metric space). Then*

(i) *A sequence* $\{x_n\}$ *in* X *is said to converge to* x *in* X, *denoted by* $x_n \to x$, *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$, *i.e. for each* $r \in (0,1)$ *and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - r$ *for all* $n \geq n_0$ *[7, 13].*
(ii) *A sequence* $\{x_n\}$ *is a M-Cauchy sequence if and only if for all* $\varepsilon \in (0,1)$ *and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) \geq 1 - \varepsilon$ *for all* $m > n \geq n_0$ *[3, 13]. A sequence* $\{x_n\}$ *is a G-Cauchy sequence if and only if* $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ *for any* $p > 0$ *and* $t > 0$ *[4, 5, 15].*

(iii) *The fuzzy metric space* $(X, M, *)$ *is called M-complete (G-complete) if every M-Cauchy (G-Cauchy)sequence is convergent.*

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Definition 4. *[18,19] Let A, B be a non-empty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. The mapping $g : A \to A$ is said to be a fuzzy isometric if*

$$M(gx_1, gx_2, t) = M(x_1, x_2, t)$$

for all $x_1, x_2 \in A$.

Definition 5. *[17] For $t > 0$, a non-empty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-approximatively compact if for each x in X and each sequence y_n in A with $M(y_n, x, t) \longrightarrow M(A, x, t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.*

Definition 6. *[22] Let $\gamma : [0, 1) \to \mathbb{R}$ be a strictly increasing, continuous mapping and for each sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 1$ if and only if $\lim_{n\to\infty} \gamma(a_n) = +\infty$. Let Γ is the family of all γ functions.*
A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0, 1)$ such that

$$M(Tx, Ty, t) < 1 \Rightarrow \gamma(M(Tx, Ty, t)) \geq \gamma(M(x, y, t)) + \delta \tag{1.1}$$

for all $x, y \in X$ and $\gamma \in \Gamma$.

2. Main Results

In this section, we present some definitions and deduce some best proximity point results in non-Archimedean fuzzy metric spaces.

Let $A_0(t)$ and $B_0(t)$ two nonempty subsets of a fuzzy metric space $(X, M, *)$. We will use the following notations:

$$M(A, B, t) = \sup \{M(x, y, t) : x \in A, y \in B\};$$
$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\};$$
$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}.$$

Now, let us state our main results.

Definition 7. *Let (A, B) be a pair of nonempty subsets of a non-Archimedean fuzzy metric space X with $A_0 \neq 0$. Then the pair (A, B) is said to have the fuzzy weak P-property if and ony if*

$$\begin{cases} M(x_1, y_1, t) = M(A, B, t) \\ M(x_2, y_2, t) = M(A, B, t) \end{cases} \Longrightarrow M(x_1, x_2, t) \geq M(y_1, y_2, t)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 8. *Let $X = R{\times}R$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0, \infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(0, 0)\}$, $B = \{(1, 0), (-1, 0)\}$. Then here, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let us consider

$$\begin{aligned} M(u_1, x_1, t) &= M(A, B, t) \\ M(u_2, x_2, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have
$(u_1, x_1) = ((0, 0), (1, 0))$ and $(u_2, x_2) = ((0, 0), (-1, 0))$

$$M(u_1, u_2, t) = M((0, 0), (0, 0), t) = 1 > \frac{t}{t + 2} = M(x_1, x_2, t).$$

Then it is easy to see that (A, B) is said to have the fuzzy weak P-property.

Definition 9. *Let A, B be a nonempty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. Given $T : A \to B$ and a fuzzy isometry $g : A \to A$, the mapping T is said to preserve fuzzy isometric distance with respect to g if*

$$M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$$

for all $x_1, x_2 \in A$.

Example 10. *Let $X = R\times [0, 1]$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0, \infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(x, 0) : \text{for all } x \in R\}$. Define $g : A \to A$ by $g(x, 0) = (-x, 0)$. Then $M(x, y, t) = \frac{t}{t+d(x,y)} = M(gx, gy, t)$, where $x = (x_1, 0)$ and $y = (y_1, 0) \in A$. Therefore, g is a fuzzy isometry.

Theorem 11. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Let $T : A \to B$ be γ-contraction such that $T(A_0(t)) \subseteq B_0(t)$. Suppose that the pair (A, B) has the fuzzy P-property. Then, there exists a unique x^* in A such that $M(x^*, Tx^*, t) = M(A, B, t)$.*

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Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, we can find $x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$, it follows that there is an element x_2 in $A_0(t)$ such that $M(x_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in N. \tag{2.1}$$

(A, B) satisfies the fuzzy weak P-property, therefore from (2.1) we obtain

$$M(x_n, x_{n+1}, t) \geq M(Tx_{n-1}, Tx_n, t), \quad \text{for all } n \in N. \tag{2.2}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(Tx_{n_0-1}, Tx_{n_0}, t) = 1$, then by (2.2) we get $M(x_{n_0}, x_{n_0+1}, t) = 1$ which implies $x_{n_0} = x_{n_0+1}$. Therefore, we get

$$Tx_{n_0} = Tx_{n_0+1} \Longrightarrow M(Tx_{n_0}, Tx_{n_0+1}, t) = 1. \tag{2.3}$$

From (2.2) and (2.3), we have that

$$M(x_{n_0+2}, x_{n_0+1}, t) \geq M(Tx_{n_0+1}, Tx_{n_0}, t) = 1 \Longrightarrow x_{n_0+2} = x_{n_0+1}.$$

Therefore, $x_n = x_{n_0}$, for all $n \geq n_0$ and $\{x_n\}$ is convergent in $A_0(t)$. Also, we obtain

$$M(x_{n_0}, Tx_{n_0}, t) = M(x_{n_0+1}, Tx_{n_0}, t) = M(A, B, t).$$

This shows that x_{n_0} is a fuzzy best proximity point of T and the proof is completed. Due to this reason, we suppose that $M(Tx_{n-1}, Tx_n, t) \neq 1$, for all $n \in N$. In view of (1.1) and by (2.2), we get

$$\begin{aligned} \gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ &\geq \gamma(M(x_{n-2}, x_{n-1}, t)) + 2\delta \\ &\quad \ldots \\ &\geq \gamma(M(x_0, x_1, t)) + n\delta. \end{aligned} \tag{2.4}$$

Letting $n \to \infty$, from (2.4) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{2.5}$$

Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all

$k \in N$ there exist $n(k), m(k) \in N$ with $n(k) > m(k) > k$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \varepsilon \ . \tag{2.6}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the inequality (2.6). Then, we have

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \varepsilon$$

and so, for all $k \in N$, we get

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{n(k)}, x_{m(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * (1 - \varepsilon \ ). \end{aligned} \tag{2.7}$$

By taking $k \to \infty$ in (2.7) and using (2.5), we obtain

$$\lim_{k \to \infty} M(x_{n(k)}, x_{m(k)}, t_0) = 1 - \varepsilon. \tag{2.8}$$

From (FM-4), we get

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) &\geq M(x_{m(k)+1}, x_{m(k)}, t_0) \\ &\quad *M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad *M(x_{n(k)1}, x_{n(k)+1}, t_0). \end{aligned} \tag{2.9}$$

Taking the limit as $k \to \infty$ in (2.9), we obtain

$$\lim_{k \to \infty} M(x_{n(k)+1}, x_{m(k)+1}, t_0) = 1 - \varepsilon \ . \tag{2.10}$$

By applying the inequality (1.1) with $x = x_{m(k)}$ and $y = x_{n(k)}$

$$\gamma(M(x_{n(k)+1}, x_{m(k)+1}, t)) \geq \gamma(M(x_{n(k)}, x_{m(k)}, t)) + \delta. \tag{2.11}$$

Taking the limit as $k \to \infty$ in (2.11), applying (1.1), from (2.8), (2.10) and continuitiy of γ, we obtain

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta$$

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which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X. Since $A_0(t)$ is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x^* \in A_0(t)$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is continuous, we obtain $Tx_n \to Tx^*$. Also, from continuity of the fuzzy metric function M, we have $M(x_{n+1}, Tx_n, t) = M(x^*, Tx^*, t)$. From (2.1), $M(x^*, Tx^*, t) = M(A, B, t)$. So, we prove that x^* is a fuzzy best proximity point of T. The uniqueness of the best proximity point of T. From the condition that T is γ-contraction, we get

$$x_1, x_2 \in A \text{ such that } x_1 \neq x_2 \text{ and } M(x_1, Tx_1, t) = M(x_2, Tx_2, t) = M(A, B, t).$$

Then by the fuzzy weak P-property of (A, B), we have $M(x_1, x_2, t) \geq M(Tx_1, Tx_2, t)$. Also

$$x_1 \neq x_2 \Longrightarrow M(x_1, x_2, t) \neq 1.$$

Hence,

$$\gamma(M(x_1, x_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta > \gamma(M(x_1, x_2, t))$$

which is a contradiction. Therefore the fuzzy best proximity point is unique. □

Corollary 12. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and $A_0(t)$ a nonempty closed subsets of X. Let $T : A \to A$ be a γ-contraction. Then, there exists a unique x^* in A.*

Example 13. *Let $X = [0, 1] \times R$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by as in Example 10. Let $A = \{(0, x) : \text{for all } x \in R\}$, $B = \{(1, y) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x}$ for all $x \in X$. Now, define $T : A \to B$ by $T(0, x) = (1, \frac{x}{6})$. Then, we get $T(A_0(t)) = B_0(t)$. Let us consider*

$$\begin{aligned} M(u_1, Tx_1, t) &= M(A, B, t) \\ M(u_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((0, -\frac{z_1}{6}), (0, -z_1))$ *or* $(u_2, x_2) = ((0, -\frac{z_2}{6}), (0, -z_2))$. *Then from (1.1), we obtain,*

$$\begin{aligned}\gamma(M(u_1, u_2, t)) &= \gamma(M((0, \tfrac{-z_1}{6}), (0, \tfrac{-z_2}{6}), t)) = \gamma(\frac{t}{t + \frac{|z_1 - z_2|}{6}}) \\ &= \frac{1}{1 - \frac{t}{t + \frac{|z_1 - z_2|}{6}}} > \frac{1}{1 - \frac{t}{t + |z_1 - z_2|}} = \gamma(\frac{t}{t + |z_1 - z_2|}) \\ &= \gamma(M(x_1, x_2, t)).\end{aligned}$$

That is,

$$\gamma(M(u_1, u_2, t)) > \gamma(M(x_1, x_2, t)).$$

Therefore, there exixts a $\delta \in (0, 1)$ *such that*

$$\gamma(M(u_1, u_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta$$

Then it is easy to see that T *is a* γ*-contraction and* $(0, 0)$ *is a unique fuzzy best proximity point of* T.

Definition 14. *(γ-proximal contraction of Type-1) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0, 1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\begin{cases} M(u_1, Tx_1, t) = M(A, B, t) \\ M(u_2, Tx_2, t) = M(A, B, t) \\ M(u_1, u_2, t), M(x_1, x_2, t) < 1 \end{cases} \Longrightarrow \gamma(M(u_1, u_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta. \quad (2.12)$$

Definition 15. *(γ-proximal contraction of Type-2) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0, 1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\begin{cases} M(u_1, Tx_1, t) = M(A, B, t) \\ M(u_2, Tx_2, t) = M(A, B, t) \\ M(Tu_1, Tu_2, t), M(Tx_1, Tx_2, t) < 1 \end{cases} \Longrightarrow \gamma(M(Tu_1, Tu_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) + \delta. \quad (2.13)$$

Theorem 16. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that* $T : A \to B$ *and* $g : A \to A$ *satisfy the following conditions:*

(*i*) $T(A_0(t)) \subseteq B_0(t)$,
(*ii*) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-1,*
(*iii*) g *is a fuzzy isometry,*
(*iv*) $A_0(t) \subseteq g(A_0(t))$.

Then, there exists a unique element x *in* A *such that* $M(gx, Tx, t) = M(A, B, t)$.

Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \qquad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(gx_{n_0}, Tx_{n_0+1}, t) = 1$, then it is clear that sequence $\{x_n\}$ is convergent. Hence, let $M(gx_{n_0}, gx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-1 and (2.14), we have

$$\begin{aligned} \gamma(M(gx_n, gx_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ \Rightarrow \gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ &\quad \cdots \\ &\geq \gamma(M(x_0, x_1, t)) + n\delta. \end{aligned} \tag{2.15}$$

Letting $n \to \infty$, from (2.15) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{x_n\}$ is a Cauchy sequence.

Since is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x \in A_0(t)$ such that $\lim_{n\to\infty} x_n = x$.

Since T,g and M are continuous, passing to the limit $n \to \infty$, we have

$$M(gx, Tx, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $x = x^*$. Suppose to the contrary, let $x \neq x^*$. Therefore, $M(x, x^*, t) \neq 1$. Since T is a γ-proximal contraction of type-1 and g is an isometry, we have

$$\gamma(M(x, x^*, t)) = \gamma(M(gx, gx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $x = x^*$. Therefore, the proof of Theorem 16 is completed. □

If we take g is the identity mapping, we obtain the following result.

Corollary 17. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ is a continuous γ-proximal contraction of type-1,

Then, T has a unique fuzzy best proximity point in A.

Example 18. *Let $X = R\times[-2, 2]$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0, \infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(x, -2) : \text{for all } x \in R\}$, $B = \{(y, 2) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 4$ and $M(A, B, t) = \frac{t}{t+4}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x^2}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(x, -2) = (\frac{x}{2}, 2) \text{ and } g(x, -2) = (-x, -2)$$

Clearly, g is fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx, t) &= M(A, B, t) \\ M(gu_2, Tx, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((-\frac{z_1}{2}, -2), (z_1, -2))$ or $(u_2, x_2) = ((-\frac{z_2}{2}, -2), (z_2, -2))$. We claim that T is a γ-proximal contraction type-1. Now, putting $u_1 = (-\frac{z_1}{2}, -2), x_1 = (z_1, -2)$, $u_2 = (-\frac{z_2}{2}, -2)$ and $x_2 = (z_2, -2)$ in (2.12), we have

$$\begin{aligned} \gamma(M(gu_1, gu_2, t)) &= \gamma(M((\tfrac{z_1}{2}, -2), (\tfrac{z_2}{2}, -2), t) = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{2}}) \\ &= \frac{1}{1 - (\frac{t}{t+\frac{|z_1-z_2|}{2}})^2} > \frac{1}{1 - (\frac{t}{t+|z_1-z_2|})^2} = \gamma(\frac{t}{t + |z_1 - z_2|}) \\ &= \gamma(M(x_1, x_2, t)). \end{aligned}$$

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That is, we have

$$\gamma(M(u_1,u_2,t)) > \gamma(M(x_1,x_2,t)).$$

Therefore, there exixts a $\delta \in (0,1)$ *such that*

$$\gamma(M(u_1,u_2,t) \geq \gamma(M(x_1,x_2,t)) + \delta.$$

Then it is easy to see that T *is a* γ*-proximal contraction type-1. It now follows from Theorem 16 that* $(0,-2)$ *is a unique fuzzy best proximity point of* T*.*

Theorem 19. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Assume that* A *is approximatively compact with respect to* B*. Also, suppose that* $T : A \to B$ *and* $g : A \to A$*satisfy the following conditions:*

(*i*) $T(A_0(t)) \subseteq B_0(t)$,
(*ii*) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-2,*
(*iii*) g *is a fuzzy isometry,*
(*iv*) $A_0(t) \subseteq g(A_0(t))$,
(*v*) T *preserves fuzzy isometric distance with respect to* g*.*

Then, there exists an element x *in* A *such that* $M(gx, Tx, t) = M(A, B, t)$*. Moreover, if* x^* *is another element of* A *such that* $M(gx^*, Tx^*, t) = M(A, B, t)$*.*

Proof. Let we choose an element Tx_0 in $T(A_0(t))$. Since $Tx_0 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in N. \tag{2.16}$$

Now we will prove that the sequence $\{Tx_n\}$ is convergent in B. If there exists $n_0 \in N$ such that $M(Tgx_{n_0}, Tgx_{n_0+1}, t) = 1$, then it is clear that sequence $\{Tx_n\}$ is convergent. Hence, let $M(Tgx_{n_0}, Tgx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-2, T preserves fuzzy isometric distance with respect to g and (2.16), we have

$$\begin{aligned}
\gamma(M(Tgx_n, Tgx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\
\Rightarrow \gamma(M(Tx_n, Tx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\
&\cdots \\
&\geq \gamma(M(Tx_0, Tx_1, t)) + n\delta. \qquad (2.17)
\end{aligned}$$

Letting $n \to \infty$, from (2.17) we get

$$\lim_{n\to\infty} \gamma(M(Tx_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{Tx_n\}$ is a Cauchy sequence in B.

Since B is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$. From the triangular inequality, we obtain

$$\begin{aligned} M(y, A, t) \geq M(y, gx_n, t) \geq M(y, Tx_{n-1}, t) * M(Tx_{n-1}, gx_n, t) \\ = M(y, Tx_{n-1}, t) * M(A, B, t) \\ \geq M(y, Tx_{n-1}, t) * M(y, A, t). \end{aligned} \tag{2.18}$$

Passing to the limit as $n \to \infty$ in (2.18), we have

$$\lim_{n\to\infty} M(y, gx_n, t) = M(y, A, t).$$

Since $A_0(t)$ is approximatively compact with respect to B, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that converges to some z in $A_0(t)$. Therefore, we have

$$M(z, y, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k-1}, t) = M(y, A, t).$$

Hence, it implies that $z \in A_0(t)$. Since $A_0(t) \subseteq g(A_0(t))$, there exists $x \in A_0(t)$ such that $z = gx$. Taking to the limit as $\lim_{k\to\infty} gx_{n_k} = gx$ and g is a fuzzy isometry, we obtain

$$\lim_{k\to\infty} x_{n_k} = x.$$

Since T is continuous and $\{Tx_n\}$ is convergent to y, we have

$$\lim_{k\to\infty} Tx_{n_k} = Tx = y.$$

Hence, it follows that

$$M(gx, Tx, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k}, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $Tx = Tx^*$. Suppose to the contrary, let $Tx \neq Tx^*$. Therefore, $M(x, Tx^*, t) \neq 1$. Since T is a γ-proximal contraction of type-2 and T preserves fuzzy isometric distance with respect to g, we have

$$\gamma(M(Tx, Tx^*, t)) = \gamma(M(Tgx, Tgx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $Tx = Tx^*$. Therefore, the proof of Theorem 19 is completed. □

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If we take g is the identity mapping, we obtain the following result.

Corollary 20. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ is a continuous γ-proximal contraction of type-2,

Then, T has a unique fuzzy best proximity point in A. Moreover, if x^* is another fuzzy best proximity point T, then $Tx = Tx^*$.

Example 21. *Let $X = [0, 1]\times R$ and $M : X\times X\times(0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0, \infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(0, x) : \text{for all } x \in R\}$, $B = \{(1, y) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{\sqrt{1-x}}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(0, x) = (1, \frac{x}{3}) \text{ and } g(0, x) = (0, -x)$$

Clearly, g is a fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx_1, t) &= M(A, B, t) \\ M(gu_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

. Clearly, T is preserve isometric distance with respect to g. That is $M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$. We claim that T is a $\gamma-$proximal contraction type-2. Now, putting $u_1 = (0, -\frac{z_1}{3}), x_1 = (0, z_1)$, $u_2 = (0, -\frac{z_2}{3},)$ and $x_2 = (0, z_2)$ in (2.13), we have

$$\begin{aligned} \gamma(M(Tgu_1, Tgu_2, t) &= \gamma(M(1, \tfrac{z_1}{9}), (1, \tfrac{z_2}{9}), t) = \gamma(\frac{t}{t + \frac{|z_1 - z_2|}{9}}) \\ &= \frac{1}{\sqrt{1 - \frac{t}{t+\frac{|z_1-z_2|}{9}}}} > \frac{1}{\sqrt{1 - \frac{t}{t+\frac{|z_1-z_2|}{3}}}} = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{3}}) \\ &= \gamma(M(Tx_1, Tx_2, t)). \end{aligned}$$

Since, T preserves isometric distance with respect to g, we have

$$\gamma(M(Tu_1, Tu_2, t)) > \gamma(M(Tx_1, Tx_2, t)).$$

Therefore, there exixts a $\delta \in (0,1)$ such that

$$\gamma(M(Tu_1, Tu_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) + \delta.$$

Then it is easy to see that T is a γ-proximal contraction type-2. It now follows from Theorem 19 that $(0,0)$ is a unique fuzzy best proximity point of T.

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