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On the symmetries of the second kind (h, q)-Bernoulli polynomials

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Abstract : In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Key words : Bernoulli numbers and polynomials, the second kind Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials, the second kind (h, q)-Bernoulli numbers and polynomials.

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1. Introduction

Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number and the second kind Bernoulli polynomials were studied by many authors(see [1-8]). Bernoulli numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In [5], by using the second kind Bernoulli numbers B_j and polynomials $B_j(x)$, we investigated the q-analogue of sums of powers of consecutive odd integers(see [6]). Let k be a positive integer. Then we obtain

$$O_k(n-1) = \sum_{i=0}^{n-1} (2i+1)^{k-1} = \frac{B_k(2n) - B_k}{2k}.$$

In [4], we introduced the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$. By using computer, we observed an interesting phenomenon of 'scattering' of the zeros of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ in complex plane. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$. In this paper, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [2, 5]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x) q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [2])}. \tag{1.1}$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \tag{1.2}$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}|_{x=0}$.

First, we introduce the second kind Bernoulli numbers B_n and polynomials $B_n(x)$. The second kind Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by means of the following generating functions (see [3]):

$$\frac{2te^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

and

$$\left(\frac{2te^t}{e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

respectively.

The second kind (h,q)-Bernoulli polynomials, $B_{n,q}^{(h)}(x)$ are defined by means of the generating function:

$$\left(\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind (h,q)-Bernoulli numbers $E_{n,q}^{(h)}$ are defined by means of the generating function:

$$\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!}. \tag{1.4}$$

In (1.2), if we take $g(x) = q^{hx}e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) = \frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}. \tag{1.5}$$

for $|t| \le p^{-\frac{1}{p-1}}, h \in \mathbb{Z}$. In (1.2), if we take $g(x) = e^{2nxt}$, then we also have

$$\int_{\mathbb{Z}_p} e^{2nxt} d\mu_1(x) = \frac{2nt}{e^{2nt}-1}. \tag{1.6}$$

for $|t| \le p^{-\frac{1}{p-1}}$. It will be more convenient to write (1.2) as the equivalent bosonic integral form

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + g'(0), \quad \text{(see [2])}. \tag{1.7}$$

For $n \in \mathbb{N}$, we also derive the following bosonic integral form by (1.7),

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + \sum_{k=0}^{n-1} g'(k), \text{ where } g'(k) = \frac{dg(x)}{dx}|_{x=k}. \tag{1.8}$$

In [4], we introduced the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ and investigate their properties. The following elementary properties of the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1. For $h \in \mathbb{Z}, q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$B_{n,q}^{(h)} = \int_{\mathbb{Z}_p} q^{hx}(2x+1)^n d\mu_1(x),$$
$$B_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{hy}(x+2y+1)^n d\mu_1(y).$$

Theorem 2. For any positive integer n, we have

$$B_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,q}^{(h)} x^{n-k}.$$

Theorem 3. For any positive integer m, we obtain

$$B_{n,q}^{(h)}(x) = m^{n-1} \sum_{i=0}^{m-1} q^{hi} B_{n,q^m}^{(h)} \left(\frac{2i+x+1-m}{m} \right) \text{ for } n \geq 0.$$

2. On the symmetries of the second kind (h, q)-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. We investigate interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for the second kind (h, q)-Bernoulli polynomials. W also obtain recurrence identities the second kind (h, q)-Bernoulli polynomials.

By (1.7), we obtain

$$\begin{aligned} &\frac{1}{h \log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \\ &= \frac{n \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{hnx} e^{2ntx} d\mu_1(x)} \end{aligned} \tag{2.1}$$

By (1.8), we obtain

$$\begin{aligned} &\frac{1}{h \log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} q^{hi}(2i+1)^k \right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

For each integer $k \geq 0$, let

$$O_{k,q}^{(h)}(n) = 1^k + q^h 3^k + q^{2h} 5^k + q^{3h} 7^k + \cdots + q^{nh}(2n+1)^k.$$

The above sum $O_{k,q}^{(h)}(n)$ is called the sums of powers of consecutive (h, q)-odd integers. From the above and (2.2), we obtain

$$\begin{aligned} &\frac{1}{h \log q + 2t} \left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) \right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} O_{k,q}^{(h)}(n-1) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

Thus, we have

$$\sum_{k=0}^{\infty} \left(q^{hn} \int_{\mathbb{Z}_p} q^{hx}(2x+2n+1)^k d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx}(2x+1)^k d\mu_1(x) \right) \frac{t^k}{k!} = \sum_{k=0}^{\infty} (h \log q + 2t) O_{k,q}^{(h)}(n-1) \frac{t^k}{k!}$$

By comparing coefficients $\frac{t^k}{k!}$ in the above equation, we have

$$\begin{aligned}&(h\log q+2t)O_{k,q}^{(h)}(n-1)\\&=\int_{\mathbb{Z}_p} q^{hx}(2x+2n+1)^k d\mu_1(x)-\int_{\mathbb{Z}_p} q^{hx}(2x+1)^k d\mu_1(x)^{\cdot}\end{aligned}$$

By using the above equation we arrive at the following theorem:

Theorem 4. Let k be a positive integer. Then we obtain

$$q^{hn}B_{n,q}^{(h)}(2n)-B_{n,q}^{(h)}=h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1). \tag{2.4}$$

Remark 5. For the alternating sums of powers of consecutive integers, we have

$$\begin{aligned}\lim_{q\to 1}\left(h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1)\right)&=\sum_{i=0}^{n-1}(2i+1)^{k-1}\\&=\frac{B_k(2n)-B_k}{2k},\ \text{ for } k\in\mathbb{N}.\end{aligned}$$

By using (2.1) and (2.3), we arrive at the following theorem:

Theorem 6. Let n be positive integer. Then we have

$$\frac{n\int_{\mathbb{Z}_p} q^{hx}e^{(2x+1)t}d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{hnx}e^{2ntx}d\mu_1(x)}=\sum_{m=0}^{\infty}\left(O_{m,q}^{(h)}(n-1)\right)\frac{t^m}{m!}. \tag{2.5}$$

Let w_1 and w_2 be positive integers. By using (1.5) and (1.6), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{h(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x}e^{2w_1w_2xt}d\mu_1(x)}\\&=\frac{(h\log q+2t)e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{hw_1w_2}e^{2w_1w_2t}-1)}{(q^{hw_1}e^{2w_1t}-1)(q^{hw_2}e^{2w_2t}-1)}\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), after calculations, we obtain

$$\begin{aligned}S&=\left(\frac{1}{w_1}\int_{\mathbb{Z}_p} q^{hw_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p} q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}\sum_{m=0}^{\infty}B_{m,q^{w_1}}^{(h)}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_2}}^{(h)}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.7}$$

By using Cauchy product in the above, we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}^{(h)}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}^{(h)}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!} \tag{2.8}$$

By using the symmetry in (2.7), we have

$$\begin{aligned}S&=\left(\frac{1}{w_2}\int_{\mathbb{Z}_p} q^{hw_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p} q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}\sum_{m=0}^{\infty}B_{m,q^{w_2}}^{(h)}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_1}}^{(h)}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}^{(h)}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}^{(h)}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.9}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}^{(h)}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}^{(h)}(w_1-1)w_2^{m-j}\\&=\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}^{(h)}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}^{(h)}(w_2-1)w_1^{m-j},\end{aligned}$$

where $B_{k,q}^{(h)}(x)$ and $O_{m,q}^{(h)}(k)$ denote the second kind (h,q)-Bernoulli polynomials and the sums of powers of consecutive (h,q)-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be positive integers. Then we have

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j-1}x^{j-k}B_{k,q^{w_2}}^{(h)}O_{m-j,q^{w_1}}^{(h)}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j-1}w_2^{m-k}x^{j-k}B_{k,q^{w_1}}^{(h)}O_{m-j,q^{w_2}}^{(h)}(w_1-1),\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}S&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\sum_{j=0}^{w_1-1}q^{w_2hj}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}q^{w_2hj}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_1(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}q^{w_2hj}B_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

By using the symmetry property in (2.10), we also have

$$\begin{aligned}S&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\sum_{j=0}^{w_2-1}q^{w_1hj}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}q^{w_1hj}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_1(x_2)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}q^{w_1hj}B_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 9. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1} q^{w_2hj} B_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\\ =&\sum_{j=0}^{w_2-1} q^{w_1hj} B_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}.\end{aligned} \tag{2.12}$$

Observe that if $h=1$, then (2.12) reduces to Theorem 5 in [9](see [5, 9]). Substituting $w_1=1$ into (2.12), we arrive at the following corollary.

Corollary 10. Let w_2 be positive integer. Then we obtain

$$B_{n,q}^{(h)}(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} q^{hj} B_{n,q^{w_2}}^{(h)}\left(\frac{x-w_2+2j+1}{w_2}\right). \tag{2.13}$$

The Corollary 10 is shown to yield the known distribution relation of the second kind (h,q)-Bernoulli polynomials(see Theorem 3). Note that if $q\to 1$, then (2.13) reduces to distribution relation of the second kind Bernoulli polynomials(see [8]).

Corollary 11. Let w_2 be positive integer. Then we have

$$B_n(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} B_n\left(\frac{x-w_2+2j+1}{w_2}\right).$$

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