Nonlinear differential equations associated with degenerate (h, q)-tangent numbers

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Abstract : In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h, q)-tangent numbers. We give explicit identities for the degenerate (h, q)-tangent numbers.

Key words : Nonlinear differential equations, (h, q)-tangent numbers and polynomials, degenerate tangent numbers, degenerate (h, q)-tangent numbers, higher-order degenerate tangent numbers.

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1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Euler numbers and polynomials, degenerate Bernoulli numbers and polynomials, degenerate Genocchi numbers and polynomials, and degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 5, 6, 7]). In [1], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[2] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. The degenerate (h, q)-tangent numbers $\mathcal{T}_{n,q}^{(h)}(\lambda)$ are defined by the generating function:

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!} = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}. \tag{1.1}$$

The degenerate (h, q)-tangent numbers of higher order, $\mathcal{T}_{n,\lambda,q}^{(k,h)}$ are defined by means of the following generating function

$$\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k,h)}(\lambda)\frac{t^n}{n!}. \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.5}$$

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special numbers and polynomials in order to give explicit identities for special polynomials. In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h,q)-tangent numbers . We give explicit identities for the degenerate (h,q)-tangent numbers .

2. Nonlinear differential equations associated with degenerate (h,q)-tangent numbers

In this section, we study nonlinear differential equations arising from the generating functions of degenerate twisted (h,q)-tangent numbers. Let

$$F = F(t,\lambda,q,h) = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{\partial}{\partial t}F(t,\lambda,q,h) = \frac{\partial}{\partial t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right) \\ &= \frac{1}{1+\lambda t}\left(\frac{-4}{q^h(1+\lambda t)^{2/\lambda}+1}\right) + \frac{1}{1+\lambda t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^2 \\ &= \frac{-2F+F^2}{1+\lambda t}. \end{aligned} \tag{2.2}$$

By (2.2), we have

$$F^2 = 2F + (1+\lambda t)F^{(1)}. \tag{2.3}$$

Taking the derivative with respect to t in (2.3), we obtain

$$\begin{aligned} 2FF^{(1)} &= 2F^{(1)} + \lambda F^{(1)} + (1+\lambda t)F^{(2)} \\ &= (\lambda+2)F^{(1)} + (1+\lambda t)F^{(2)}. \end{aligned} \tag{2.4}$$

From (2.2), (2.3), and (2.4), we have

$$2F^3 = 4F + (1+\lambda)(1+\lambda t)F^{(1)} + (1+\lambda t)^2F^{(2)}.$$

Continuing this process, we can guess that

$$N!F^{N+1} = \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}, \quad (N=0,1,2,\ldots), \tag{2.5}$$

where $F^{(i)} = \left(\frac{\partial}{\partial t}\right)^i F(t,\lambda,q,h)$. Differentiating (2.5) with respect to t, we have

$$(N+1)!F^NF^{(1)} = \sum_{i=0}^{N} i\lambda a_i(N,\lambda,q,h)(1+\lambda t)^{i-1}F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i+1)} \tag{2.6}$$

and

$$(N+1)!F^NF^{(1)} = (N+1)!F^N\left(\frac{-2F+F^2}{1+\lambda t}\right) = (N+1)!\left(\frac{F^{N+2}-2F^{N+1}}{1+\lambda t}\right). \tag{2.7}$$

By (2.5), (2.6), and (2.7), we have

$$\begin{aligned}(N+1)!F^{N+2} &= 2(N+1)!F^{N+1} \\ &\quad + \sum_{i=0}^{N} \lambda i a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^{i+1} F^{(i+1)} \\ &= 2(N+1)\sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} \\ &\quad + \sum_{i=0}^{N} \lambda i a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^{i+1} F^{(i+1)} \\ &= \sum_{i=0}^{N} (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} + \sum_{i=1}^{N+1} a_{i-1}(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}.\end{aligned} \tag{2.8}$$

Now replacing N by $N+1$ in (2.5), we find

$$(N+1)!F^{N+2} = \sum_{i=0}^{N+1} a_i(N+1,\lambda,q,h)(1+\lambda t)^i F^{(i)}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned}\sum_{i=0}^{N+1} a_i(N+1,\lambda,q,h)(1+\lambda t)^i F^{(i)} &= \sum_{i=0}^{N} (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} \\ &\quad + \sum_{i=1}^{N+1} a_{i-1}(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}.\end{aligned} \tag{2.10}$$

Comparing the coefficients on both sides of (2.10), we obtain

$$\begin{aligned}2(N+1)a_0(N,\lambda,q,h) &= a_0(N+1,\lambda,q,h), \\ a_{N+1}(N+1,\lambda,q,h) &= a_N(N,\lambda,q,h),\end{aligned} \tag{2.11}$$

and

$$a_i(N+1,\lambda,q,h) = (2(N+1)+\lambda i)\, a_i(N,\lambda,q,h) + a_{i-1}(N,\lambda,q,h), (1 \le i \le N). \tag{2.12}$$

In addition, by (2.5), we have

$$F = a_0(0,\lambda,q,h)F, \tag{2.13}$$

which gives

$$a_0(0,\lambda,q,h) = 1. \tag{2.14}$$

It is not difficult to show that

$$F^2 = a_0(1,\lambda,q,h)F + a_1(1,\lambda,q,h)(1+\lambda t)F^{(1)} = 2F + (1+\lambda t)F^{(1)}. \tag{2.15}$$

Thus, by (2.15), we also find

$$a_0(1,\lambda,q,h) = 2, \quad a_1(1,\lambda,q,h) = 1. \tag{2.16}$$

From (2.11), we note that

$$\begin{aligned}a_0(N+1,\lambda,q,h) &= 2(N+1)a_0(N,\lambda,q,h) = 4(N+1)Na_0(N-1,\lambda,q,h) \\ &= \cdots = 2^{N+1}(N+1)!,\end{aligned} \tag{2.17}$$

and

$$a_{N+1}(N+1,\lambda,q,h) = a_N(N,\lambda,q,h) = \cdots = 1. \tag{2.18}$$

For $i = 1,2,3$ in (2.11), then we find that

$$a_1(N+1,\lambda,q,h) = \sum_{k=0}^{N} 2^k \left(N+1+\frac{\lambda}{2}\right)_k a_0(N-k,\lambda,q,h),$$

$$a_2(N+1,\lambda,q,h) = \sum_{k=0}^{N-1} 2^k \left(N+1+\frac{\lambda}{2}\times 2\right)_k a_1(N-k,\lambda,q,h),$$

$$a_3(N+1,\lambda,q,h) = \sum_{k=0}^{N-2} 2^k \left(N+1+\frac{\lambda}{2}\times 3\right)_k a_2(N-k,\lambda,q,h).$$

Continuing this process, we can deduce that, for $1 \leq i \leq N$,

$$a_i(N+1,\lambda,q,h) = \sum_{k=0}^{N-i+1} 2^k \left(N+1+\frac{\lambda}{2}\times i\right)_k a_{i-1}(N-k,\lambda,q,h). \tag{2.19}$$

Note that, here the matrix $a_i(j,\lambda,q,h)_{0\leq i,j\leq N+1}$ is given by

$$\begin{pmatrix} 1 & 2 & 2!2^2 & 3!2^3 & \cdots & (N+1)!2^{N+1} \\ 0 & 1 & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 1 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,\lambda,q,h)$. By (2.17), (2.18), and (2.19), we have

$$\begin{aligned} a_1(N+1,\lambda,q,h) &= \sum_{k_1=0}^{N} 2^{k_1} \left(N+1+\frac{\lambda}{2}\right)_{k_1} a_0(N-k_1,\lambda,q,h) \\ &= \sum_{k_1=0}^{N} 2^{N}(N-k_1)! \left(N+1+\frac{\lambda}{2}\right)_{k_1}, \end{aligned}$$

$$a_2(N+1,\lambda,q,h) = \sum_{k_2=0}^{N-1} 2^{k_2} \left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2} a_1(N-k_2,\lambda,q,h)$$
$$= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-k_2-1} 2^{N-1}(N-k_2-k_1-1)! \left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2} \left(N-k_2+\frac{\lambda}{2}\right)_{k_1},$$

and

$$a_3(N+1,\lambda,q,h) = \sum_{k_3=0}^{N-2} 2^{k_3} \left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3} a_2(N-k_3,\lambda,q,h)$$
$$= \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-k_3-2} \sum_{k_1=0}^{N-k_3-k_2-2} 2^{N-2}(N-k_3-k_2-k_1-2)! \left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3}$$
$$\times \cdots \times \left(N-k_3+\frac{\lambda}{2}\times 2\right)_{k_2} \left(N-k_3-k_2-1+\frac{\lambda}{2}\right)_{k_1}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1,\lambda,q,h) = & \sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} 2^{N-i+1} \\ & \times (N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1)! \\ & \times \left(N+1+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}} \\ & \times \left(N-k_i-k_{i-1}-1+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}} \\ & \times \left(N-k_i-k_{i-1}-k_{i-2}-2+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\ & \times \left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+2+\frac{\lambda}{2}\right)_{k_1}. \end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 1. For $N=0,1,2,\ldots$, the nonlinear functional equation

$$N!F^{N+1} = \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}$$

has a solution

$$F = F(t,\lambda,q,h) = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1},$$

where

$$\begin{aligned} & a_0(N,\lambda,q,h) = 2^N N!, \\ & a_N(N,\lambda,q,h) = 1, \\ & a_i(N,\lambda,q,h) = \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(2q^h-x)^{N-i} \\ & \times (N-k_i-k_{i-1}-\cdots-k_2-k_1-i)! \\ & \times \left(N+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i-1+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}} \\ & \times \left(N-k_i-k_{i-1}-2+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}} \\ & \times \left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\ & \times \left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}. \end{aligned}$$

From (1.1) and (1.2), we note that

$$N!F^{N+1} = N!\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^{N+1} = N!\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!}. \tag{2.21}$$

From (2.5), we note that

$$F^{(i)} = \left(\frac{\partial}{\partial t}\right)^i F(t,\lambda,q,h) = \sum_{l=0}^{\infty}\mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!}. \tag{2.22}$$

From Theorem 1, (1.5), (2.21), and (2.22), we can derive the following equation:

$$\begin{aligned} N!F^N \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!} &= \sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)} \\ &= \sum_{i=0}^{N} a_i(N,\lambda,q,h) \sum_{k=0}^{\infty} (i)_k \lambda^k \frac{t^k}{k!} \sum_{l=0}^{\infty} \mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k} a_i(N,\lambda,q,h)(i)_k\lambda^k \mathcal{T}_{n-k+i,q}^{(h)}(\lambda)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.23}$$

By comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \ldots$, we have

$$N!\mathcal{T}_{n,q}^{(N+1,h)}(\lambda) = \sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k} a_i(N,\lambda,q,h)(i)_k\lambda^k \mathcal{T}_{n-k+i,q}^{(h)}(\lambda),$$

where

$$\begin{aligned} & a_0(N,\lambda) = N!2^N, \quad a_N(N,\lambda) = 1, \\ & a_i(N,\lambda) = \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i} 2^{N-i} \\ & \quad \times (N-k_i-k_{i-1}-\cdots-k_2-k_1-i)! \\ & \quad \times \left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\ & \quad \times \left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}. \end{aligned}$$

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