Journal of Computational Analysis and Applications VOL. 33, NO. 7, 2024
Journal's ISSN: 1521-1398 (Paper),1572-9206 (Online)

Inversion Formula For Generalized Three Dimensional Fractional Cosine Transform

S. D. Kambale [1], S.B. Gaikwad [2]
[1]Department of Mathematics KET's V.G. Vaze College Mulund(E),Mumbai.
sandipkambale24@gmail.com
[2] Department of Mathematics New Arts College Ahmednagar.
sbgmathsnagar@gmail.com

Received: 18-07-2024 Revised: 22-07-2024 Accepted: 29-08-2024
Published: 22-11-2024

Abstract :

The fractional Fourier transform (FRFT) is an extension of the conventional Fourier transform. The fractional cosine transform is strongly associated with the fractional Fourier transform and is now used in optical and signal processing applications. Most often used in the field of image processing, particularly when dealing with compression. This paper describes an extension of the fractional cosine transform in three dimensions. This paper defines the test function space and generalises the 3D fractional cosine transform. Concurrently, the inverse formula and uniqueness theorem for the 3D fractional cosine transform are established.

ASM Mathematics Subject Classification (2010): 42A38, 42B10

Keywords: Fractional cosine transform,Fractional Fourier transform,Image processing.

1. Introduction:

V. Namias invented fractional operator of Fourier transform (FT) in 1980 [1]. He originally defined and mathematically framed Fractional Fourier Transform (FRFT).The phase plane angle α determines the Fractional Fourier transform (FRFT). This generalises signal processing notions like space (or time) and frequency domain.In signal and image processing, the Cosine and Sine transformations and their discrete equivalents are important for signal coding [2], watermarking [3], and de-focused picture restoration [4].In 1996, Lohmann [6] first defined FrFST and FrFCT with signal processing applications.Pei and Yeh[7] expanded the Cosine transform to the discrete fractional cosine transform (DFrCT) and the discrete fractional sine transform (DFrST). Both of them exhibit the DFrFT's angle additivity feature nicely. Furthermore, the DFrCT and DFrST are employed in the digital computation of FrFT to reduce the DFrFT's computing burden. Sharma and Khapre [8] established an inversion formula for generalized two-dimensional fraction cosine and sine transform.

Definition 1.1 Two-dimensional fractional Cosine transform with parameter α of $f(t_1, t_2)$ denoted by $F_C^\alpha(t_1, t_2)$ perform a linear operation given by the integral transform.

$$F_C^\alpha\{f(t_1,t_2)\}(l_1,l_2) = \int_0^\infty \int_0^\infty f(x,y)K_\alpha(t_1,t_2,l_1,l_2)dt_1dt_2 \tag{1.1}$$

Where the Kernel,

$$K_C^\alpha(t_1,t_2,l_1,l_2) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(t_1^2+t_2^2+l_1^2+l_2^2)cot\alpha}{2}}\cos(cosec\alpha.\, l_1t_1).\cos(cosec\alpha.\, l_2t_2)$$

Definition 1.2 (Three-Dimensional Generalized Fractional Cosine Transform)

Three-dimensional fractional Cosine transform with parameter α of $f(t_1,t_2,t_3)$ denoted by $F_C^\alpha(t_1,t_2,t_3)$ perform a linear operation given by the integral transform.

$$F_C^\alpha\{f(t_1,t_2,t_3)\}(l_1,l_2,l_3) = \int_0^\infty \int_0^\infty \int_0^\infty f(t_1,t_2,t_3)K_\alpha(t_1,t_2,t_3,l_1,l_2,l_3)dt_1dt_2dt_3 \tag{1.2}$$

where the Kernel,

$$K_C^\alpha(t_1,t_2,t_3,l_1,l_2,l_3) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}}\cos(cosec\alpha.\, l_1t_1).\cos(cosec\alpha.\, l_2t_2)\cos(cosec\alpha.\, l_3t_3) \tag{1.3}$$

2. The Test Function Space E

An infinitely differentiable complex-valued function ψ on R^n belong to $E(R^n)$ if for each compact set $I \subset S_{p,q,r}$,

where, $S_{p,q,r} = \{t_1,t_2,t_3 : t_1,t_2,t_3 \in R^n, |t_1| \le p, |t_2| \le q, |t_3| \le r, q,p,r > 0\}, I \in R^n$(2.1)

$\Upsilon_{E,a,b,c}(\psi) = Sup_{t_1,t_2,t_3}|D_{t_1,t_2,t_3}^{a,b,c}\psi(t_1,t_2,t_3)| < \infty$ *and*, $a,b,c = 1,2,3,4,5,\ldots..$

Thus $E(R^n)$ will denote the space of all $\psi \in E(R^n)$ with support contained in $S_{p,q,r}$. Note that the space E is complete and, therefore, a Frechet space. Moreover, we say that f is a fractional Cosine transformable if it is a member of E^*, the dual space of E.

2.1 Distributional Three-Dimensional Fractional Cosine Transform

The three-dimensional distributional fractional cosine transform of $f(t_1,t_2,t_3) \in E^*(R^n)$ defined by $F_C^\alpha\{f(t_1,t_2,t_3)\} = F^\alpha(l_1,l_2,l_3) =< f(t_1,t_2,t_3), K_\alpha(t_1,t_2,t_3,l_1,l_2,l_3) >$

$K_C^\alpha(t_1,t_2,t_3,l_1,l_2,l_3) =$

$$\sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}}\cos(cosec\alpha.l_1t_1)\cos(cosec\alpha.l_2t_2)\cos(cosec\alpha.l_3t_3) \quad (2.2)$$

Where, RHS of the above equation has a meaning as the application of $f \in E^*$ to $K_\alpha(t_1,t_2,t_3,l_1,l_2,l_3) \in E^*$

3. Inversion Formula for Generalized 3Dimensional Fractional Cosine Transform (3DFrCT):

Theorem 3.1: If the three-dimensional fractional cosine transform is given by

$$F_C^\alpha\{f(t_1,t_2,t_3)\}(l_1,l_2,l_3) = \int_0^\infty \int_0^\infty \int_0^\infty f(t_1,t_2,t_3)K_\alpha(t_1,t_2,t_3,l_1,l_2,l_3)dt_1dt_2dt_3 \quad (3.1)$$

Then its inverse $f(t_1,t_2,t_3)$ is given by

$$f(t_1,t_2,t_3) = \frac{8}{\pi^3}\int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(l_1,l_2,l_3)\overline{K_\alpha}(t_1,t_2,t_3,l_1,l_2,l_3)dl_1dl_2dl_3 \quad (3.2)$$

Where,

$$\overline{K_\alpha} = e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}}\left(\sqrt{\frac{1-icot\alpha}{2\pi}}\right)^{-1} cosec^3\alpha\cos(cosec\alpha.l_1t_1)\cos(cosec\alpha.l_2t_2)\cos(cosec\alpha.l_3t_3)$$

Proof:

$$F_C^\alpha(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}}$$
$$= C_k\int_0^\infty \int_0^\infty \int_0^\infty f(t_1,t_2,t_3)e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}}\cos(cosec\alpha.l_1t_1)\cos(cosec\alpha.l_2t_2)$$
$$\cos(cosec\alpha.l_3t_3)dt_1dt_2dt_3, \quad (3.3)$$

where $C_k = \sqrt{\frac{1-icot\alpha}{2\pi}}$.

$$F_C^\alpha(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} =$$
$$C_k\int_0^\infty \int_0^\infty \int_0^\infty g(t_1,t_2,t_3)\cos(cosec\alpha.l_1t_1)\cos(cosec\alpha.l_2t_2)\cos(cosec\alpha.l_3t_3)dt_1dt_2dt_3 \quad (3.4)$$

where , $g(t_1,t_2,t_3) = C_ke^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}}f(t_1,t_2,t_3)$.

$$F_C^\alpha(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = [Cg(t_1,t_2,t_3)]\cos(cosec\alpha.l_1)\cos(cosec\alpha.l_2)\cos(cosec\alpha.l_3) \quad (3.5)$$

Let $cosec\alpha \cdot l_1 = \eta, \quad cosec\alpha \cdot l_2 = \zeta, \quad cosec\alpha \cdot l_3 = \mu$

$\therefore d\eta = cosec\alpha \cdot dl_1, \quad d\zeta = cosec\alpha \cdot dl_2, \quad d\mu = cosec\alpha \cdot dl_3$

$$F_C^\alpha(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = [Cg(t_1,t_2,t_3)](\eta,\zeta,\mu) \quad (3.6)$$

$$F_C^{\alpha}(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = G(\eta,\zeta,\mu) \quad (3.7)$$

The R.H.S is the cosine transform of $g(t_1,t_2,t_3)$ with argument η,ζ,μ.

Applying the inverse cosine transform we can write ,

$$g(t_1,t_2,t_3) = \frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} G(\eta,\zeta,\mu)cos(\eta t_1)cos(\zeta t_2)cos(\mu t_3)d\eta d\zeta d\mu \quad (3.8)$$

$$C_k e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} f(t_1,t_2,t_3) =$$

$$\frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F_C^{\alpha}(l_1,l_2,l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} cos(\eta x)cos(\zeta y)cos(\mu z)d\eta d\zeta d\mu$$

$$C_k e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} f(t_1,t_2,t_3) = \frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F_C^{\alpha}(u,v,w)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(cosec\alpha.\, l_1t_1)$$

$$.\cos(cosec\alpha l_2t_2)\cos(cosec\alpha l_3t_3)cosec^3\alpha\, dl_1dl_2dl_3. \quad (3.9)$$

$$f(t_1,t_2,t_3) = \frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F_C^{\alpha}(u,v,w)C_k^{-1}e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(cosec\alpha.\, l_1t_1)$$

$$f(t_1,t_2,t_3) = \frac{8}{\pi^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} F_C^{\alpha}(l_1,l_2,l_3)\overline{K_\alpha}(t_1,t_2,t_3,l_1,l_2,l_3)dl_1dl_2dl_3, \quad (3.10)$$

where,

$$\overline{K_\alpha}(t_1,t_2,t_3,l_1,l_2,l_3) =$$

$$\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(cosec\alpha.\, l_1t_1)\cos(cosec\alpha.\, l_2t_2)\cos(cosec\alpha.\, l_3t_3)cosec^3\alpha$$

Theorem 3.2: Uniqueness Theorem for Generalized Three-Dimensional Fractional Cosine Transform :

If $[F_c^{\alpha}f(t_1,t_2,t_3)](l_1,l_2,l_3) = F(l_1,l_2,l_3)[F_C^{\alpha}g(t_1,t_2,t_3)](l_1,l_2,l_3) = G(l_1,l_2,l_3)$ (3.11)

for $0 < \alpha \le \frac{\pi}{2}$ and $Supf \subset S_{p,q,r}, Supg \subset S_{p,q,r}$, where $S_{p,q,r} = \{t_1,t_2,t_3 : t_1,t_2,t_3 \in R\,, |t_1| \le p\,, |t_2| \le q, |t_3| \le r, p,q,r > 0\}$. If $F_\alpha(l_1,l_2,l_3) = G_\alpha$ then $f = g$ in the sense of equality in $D^*(I)$.

Proof*.* By inversion theorem

$$f - g = \frac{8}{\pi^3}\lim_{x\to\infty}\int_{-x}^{x}\int_{-x}^{x}\int_{-x}^{x}\overline{K_\alpha}(t_1,t_2,t_3,l_1,l_2,l_3)dl_1dl_2dl_3 = 0. \quad (3.12)$$

Thus $f = g$ in $D^*(I)$.

4. Conclusion

This research focuses on developing the space of testing functions and the distributional generalized three-dimensional fractional cosine transform. The study establishes the foundational framework for these transformations and proves the inversion and uniqueness theorems associated with the generalized three-dimensional fractional cosine transform. These results demonstrate the robustness of the transform's mathematical properties and ensure its applicability to broader analytical contexts. The work lays a solid theoretical groundwork for future research and practical implementations in signal processing and related fields by providing a rigorous approach to inversion and uniqueness. This advancement opens new avenues for exploration in mathematical analysis.

References

1. Namias, V. "The Fractional Order Fourier Transform and Its Application to Quantum Mechanics." *J. Inst. Maths Applies*, 1980, pp. 241–265.
2. Bellifemine, F., and R. Picco. "Video Signal Coding with DCT and Vector Quantization." *IEEE Transactions on Communications*, vol. 42, 1994, p. 200.
3. Chang, H. T., and C. L. Tsan. "Image Watermarking by Use of Digital Holography Embedded in the Discrete-Cosine-Transform Domain." *Applied Optics*, vol. 44, 2005, p. 6211.
4. Lam, E. Y., and J. W. Goodman. "Discrete Cosine Transform Domain Restoration of Defocused Images." *Applied Optics*, vol. 37, 1998, p. 6213.
5. Lohmann, A. W., et al. "Some Important Fractional Transformation for Signal Processing." *Optics Communications*, vol. 125, 1996, pp. 18–20.
6. Pei, S. C., and M. H. Yeh. "The Discrete Fractional Cosine and Sine Transform." *IEEE Transactions on Signal Processing*, vol. 49, 2001, p. 1198.
7. Sharma, V. D., and S. A. Khapre. "Analyticity of the Generalized Two Dimensional Fractional Cosine Transforms." *Journal of Mathematics and Computer Science*. ISSN 1927-5307.
8. Sharma, V. D., and S. A. Khapre. "Inversion Formula for Generalized Two Dimensional Fractional Cosine Transforms." *American Journal of Mathematics and Mathematical*

Sciences, vol. 2, no. 1, Jan.–June 2013, pp. 83–86, ISSN 2278-0874.

9. Pathak, R. S. *A Course in Distribution Theory and Application*. Narosa Publishing House, 2001.
10. Zemanian, A. H. *Generalized Integral Transform*. Interscience Publishers, 1968.