COEFFICIENT ESTIMATES AND SECOND HANKEL INEQUALITY FOR A SUBCLASS OF STARLIKE FUNCTIONS ASSOCIATED WITH COMPOSITION OF FUNCTIONS

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Abstract

In this paper we will define a new class $S^*_\psi(a)$ which is subordinate to function $e^{az}p(z)$, we will find the Fekete - Szegö inequality for this class, Further we have solved the second Hankel determinant of this new class.

Keywords and phrases. Univalent functions, subordination, Fekete - Szegö inequality, Hankel determinant, coefficient inequalities

1 Introduction

The class of all the analytic functions in a unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, whose Taylor's series expansion is of the form

$$h(z) := z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + a_4 z^4 \cdots \ \forall z \in \mathbb{D} \tag{1.1}$$

and normalized by the conditions: $h(0) = 0, h'(0) = 1$, is denoted by $\mathcal{A}$.
Let $\mathcal{S}$ denotes a subclass of $\mathcal{A}$ of all univalent analytic functions In the unit disk $\mathbb{D}$. The class of Analytic -Univalent functions, with Taylor's series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.2}$$

in $\mathbb{D}$, such that $\Re e(P(z)) > 0$ is denoted by $\mathcal{P}$

In 1916 a German Mathematician Ludwig Bieberbach proposed a conjecture on the coefficients of analytic functions of from (1.1) in $\mathcal{S}$, i.e

$$|a_n| \le n, n \in \mathbb{N}$$

This conjecture was known by the name of Bieberbach conjecture (1916) [1]. Until 1985 this conjecture was considered as very challenging problem in Geometric function theory of Complex Analysis.
after 69 years of this conjecture, a French-American Mathematician Louis de Branges de Bourcia(1985) solved this conjecture [2]. Before de Branges's proof many scholars around the world tried to prove or disprove this conjecture, as a consequences of their these efforts they found multiple subfamilies of class $\mathcal{S}$.

The most common subfamilies of $\mathcal{S}$ are convex, star-like and close-to-convex functions whose set builder form is given by

$$C := \left\{ h \in \mathcal{S}: \Re e \left(\frac{(z(h'(z))'}{h'(z)} \right) > 0, \forall z \in \mathbb{D} \right\}$$

$$S^* := \left\{ h \in \mathcal{S}: \Re e \left(\frac{zh'(z)}{h(z)} \right) > 0, \forall z \in \mathbb{D} \right\}$$

$$R := \{ h \in \mathcal{S}: \Re e[h'(z)] > 0, \forall z \in \mathbb{D} \}$$

Subordination

Two functions h and g in $\mathcal{A}, h$ is said to be subordinated to g, or written as $h(z) \prec g(z)$, if we have a Schwarz functions $\omega(z)$ analytic over $\mathbb{D}$ with $\omega(0) = 0$ and also $|\omega(z)| < 1$, such that

$$h(z) = g(\omega(z)) \ \forall z \in \mathbb{D}$$

, but if function $g(z)$ is univalent in $\mathbb{D}$, then

$$h(z) \prec g(z) \text{ iff } h(0) = g(0) \text{ and } h(\mathbb{D}) \subset g(\mathbb{D})$$

Ma and Minda [3] introduced two classes of analytic functions which are.

$$S^*(\phi) := \left\{ h \in \mathcal{A}: \frac{zh'(z)}{h(z)} \prec \phi(z), \forall z \in \mathbb{D} \right\}$$

and

$$C(\phi) := \left\{ h \in \mathcal{A}: 1 + \frac{zh''(z)}{h'(z)} \prec \phi(z), \forall z \in \mathbb{D} \right\}$$

The function $\phi(z)$ is an univalent analytic function with positive real part in the unit disk $\mathbb{D}$ such that $\phi(0) = 1, \phi'(0) > 0$ where ϕ maps the open unit disk onto a region starlike with respect to 1 and symmetric with respect to the real axis, several other classes can be formed by varying the function ϕ, some of the examples are as follows

- When $\phi = e^z$, this class is denoted by S_e^* check out [5, 6] for more Details.
- When $\phi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ we get a new class, for further details see [7]
- When $\phi = \frac{1+Cz}{1+Dz}(-1 \leq D < C \leq 1)$ we get the class $S^*(C, D)$. See [8] for more Details.

- When $\phi = \cosh(z)$, this new class is denoted by $S^*_{\cosh}$ see.[9]
- When $\phi = 1 + \sin(z)$, the class is denoted by $S^*_{\sin}$, For more details see [4,5]
  C. Pommerenke (1966-67), [6, 7] stated the p^{th} Hankel determinant for $p \geq 1$ and $n \geq 1$ where $p, n \in \mathbb{N}$ of functions h of form 1.1 is defined as

$$\mathcal{H}_{(p,n)}(h) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+p-1} & a_{n+p} & \cdots & a_{n+2p-2} \end{vmatrix}$$

In Geometric function theory of Complex analysis finding upper bounds of Hankel determinant of various subfamilies of $\mathcal{A}$ is a widely famous and an interesting problem. Noonan(1976) and Noor(1983)[8,9] studied the growth rate of $\mathcal{H}_{(p,n)}$ for fixed values of p and n, as $n \to \infty$ of different subfamilies of the univalent function of class $\mathcal{S}$.
where

$$\mathcal{H}_{(2,2)}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

From past many years, a huge collection of research papers have been dedicated for finding the upper bounds for various orders of Hankel determinant, some recent work on second, third and fourth - order Hankel determinants see [[10] - [11]], Recently, Cho et al. [4] introduced the following function class $S^*_{\sin}$

$$S^*_{\sin} := \left\{ h \in \mathcal{A} : 1 + \frac{zh'(z)}{h(z)} \prec 1 + \sin\, z, \forall z \in \mathbb{D} \right\}$$

Various researchers established Fekete - Szegö inequality for various classes afterwards ([12] - [13])

Definition 1(a)

Lets define a new subclass $S^*_\psi(a)$ of $\mathcal{S}^*$. This subclass contains all those Analytic univalent functions in $\mathcal{A}$ such that,

$$S^*_\psi(a) := \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec e^{az}\psi(z) \right\}$$

where $\psi(z)$ is a univalent function such that $\psi(z) = 1 + \sum_{n=1}^{\infty} A_n z^n$ with positive real part and $\psi'(0) > 0$

2 Preliminary Lemmas.

Let $\mathcal{P}$ denotes the class of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in \mathbb{D} \tag{2.1}$$

where $\Re\mathrm{e}(p(z)) > 0$ in $\mathbb{D}$
Let $p(z)$ belongs to $\mathcal{P}$, such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)}$$

solving above for $w(z)$ we have

$$w(z) = \frac{c_1 z}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{4}(-c_1)c_2 + \frac{1}{8}c_1(c_1^2 - 2c_2) + \frac{c_3}{2}\right) z^3 + \cdots$$

Lemma 2.1. (see [13]) If $p(z) \in \mathcal{P}$, and c_n be the n^{th} cofficients of $P(z)$, then

$$|c_n| \leq 2 \;\; for\; all\; n \in \mathbb{N} \tag{2.2}$$

$$\left|c_2 - \gamma c_1^2\right| \leq 2max\{1, |2\gamma - 1|\}\; where\; \gamma \in \mathbb{C} \tag{2.3}$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2} \tag{2.4}$$

$$|c_{n+m} - \gamma c_n c_m| \leq 2, \gamma \in [0,1],\; Where\; n, m \in \mathbb{N} \tag{2.5}$$

$$\left|c_2 - \delta c_1^2\right| \leq \begin{cases} -4\delta + 2, & if\; \delta \leq 0, \\ 2, & if\; 0 \leq \delta \leq 1, \\ 4\delta - 2, & if\; 1 \leq \delta, \end{cases} \tag{2.6}$$

Lemma 2.2. (see[17]) If $P(z) \in \mathcal{P}$ then there exists $x, z \in \overline{\mathbb{D}}$ with $|x| \leq 1, |y| \leq 1$, such that

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.7}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{2.8}$$

3 Important Theorems

Theorem 3.1. If the function $h(z) \in S_{\psi}^{*}(\alpha)$ and is of the form (1.1), then

$$|a_2| \leq |a + A_1|$$
$$|a_3| \leq \frac{(a + A_1)}{2} \max\left\{1, \left|-\frac{3a^2 + 2A_1^2 + 2A_2 + 6\mathrm{aA}_1}{2(a + A_1)}\right|\right\}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{a + A_1}{2} \max\left\{1, \left|\frac{4a^2\mu - 3a^2 + 8aA_1\mu + A_1^2(4\mu - 2) - 2A_2 - 6\mathrm{aA}_1}{2(a + A_1)}\right|\right\}$$

Proof. From definition 1(a) we have

$$S_{\psi}^{*}(\alpha) = \left\{h \in \mathcal{A}: \frac{zh'(z)}{h(z)} \prec e^{\alpha z}\psi(z)\right\}$$

After simplifying (??), we Have

$$\frac{zh'(z)}{h(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_3 a_2 + 3a_4)z^3 + (-a_2^4 + 4a_3 a_2^2 - 4a_4 a_2 - 2a_3^2 + 4a_5)z^4 + \dots \quad (3.1)$$

Using principal of subordination and expending $e^{aw(z)}\psi(w(z))$

$$1 + \frac{1}{2}z(ac_1 + A_1c_1) + \frac{1}{8}z^2(a^2c_1^2 + c_1^2((2a)A_1) - (2a)c_1^2 + (4a)c_2 - (2A_1)c_1^2 + (2A_2)c_1^2 + (4A_1)c_2)$$
$$+z^3\left(\frac{1}{4}\left(\frac{1}{4}a^2c_1^2 + (2a)\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\right)(A_1c_1) + \frac{1}{3}\left[\frac{1}{4}(ac_1)\left(\frac{1}{4}a^2c_1^2 + (2a)\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\right) + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)(a^2c_1)\right.\right.$$
$$\left.\left. +(3a)\left(\frac{1}{4}(-c_1)c_2 + \frac{1}{8}c_1(c_1^2 - 2c_2) + \frac{c_3}{2}\right)\right] + \frac{1}{2}(ac_1)\left(\frac{1}{4}A_2c_1^2 - \frac{1}{4}A_1(c_1^2 - 2c_2)\right) + \frac{1}{8}A_3c_1^3 - \frac{1}{4}(c_1^2 - 2c_2)(A_2c_1)\right.$$
$$+\dots \quad (3.2)$$

Comparing (3.1) and (3.2), we will Get

$$a_2 = \frac{1}{2}c_1(a + A_1) \quad (3.3)$$

$$a_3 = \frac{1}{16}c_1^2(3a^2 - 2a + 2A_1^2 - 2A_1 + 2A_2 + 6a\,\mathrm{A}_1) + \frac{1}{16}c_2(4a + 4A_1) \quad (3.4)$$

$$a_4 = \frac{1}{288}c_2c_1(60a^2 + 120aA_1 - 48a + 36A_1^2 - 48A_1 + 48A_2) + \frac{1}{288}c_1^3(17a^3 + 51a^2A_1 - 30a^2 + 36aA_1^2 - 60aA_1 + 30aA_2 + 12a + 6A_1^3 - 18A_1^2 + 12A_1 + 18A_1A_2 - 24A_2 + 12A_3) + \frac{1}{288}c_3(48a + 48A_1) \quad (3.5)$$

From equation (3.3) and (3.4)

$$|a_2| \le |a + A_1|$$
$$a_3 = \left[\frac{a + A_1}{4}\left(c_2 - \frac{c_1^2(-3a^2 + 2a - 2A_1^2 + 2A_1 - 2A_2 - 6aA_1)}{4(a + A_1)}\right)\right]$$

Now from lemma we have

$$|a_3| \le \frac{(a+A_1)}{2}\max\left\{1, \left|-\frac{3a^2+2A_1^2+2A_2+6aA_1}{2(a+A_1)}\right|\right\} \quad (3.6)$$

$$a_3 - \mu a_2^2 = \frac{(a+A_1)}{4}\left[c_2 - \left(\frac{4a^2\mu-3a^2+A_1(8a\mu+2)+2a+A_1^2(4\mu-2)-2A_2-6aA_1}{4(a+A_1)}\right)c_1^2\right] \quad (3.7)$$

Again using lemma we will Have

$$\left|a_3 - \mu a_2^2\right| \le \frac{a + A_1}{2}\max\left\{1, \left|\frac{4a^2\mu - 3a^2 + 8aA_1\mu + A_1^2(4\mu - 2) - 2A_2 - 6aA_1}{2(a + A_1)}\right|\right\} \quad (3.8)$$

Theorem 3.2 *: If function $h \in S_\psi^*(\alpha)$, then* $\left|a_3 - \mu a_2^2\right|$

$$\le \begin{cases} \dfrac{(-4a^2\mu + 3a^2 - 8aA_1\mu + A_1^2(2 - 4\mu) + 2A_2 + 6aA_1)}{4} \text{ if } \mu \le \dfrac{3a^2 - 2a + 2A_1^2 - 2A_1 + 2A_2 + 6aA_1}{4(a + A_1)^2} \\ \dfrac{a + A_1}{2} \text{ if } \dfrac{3a^2 - 2a + 2A_1^2 - 2A_1 + 2A_2 + 6\mathrm{a}\mathrm{A}_1}{4(a + A_1)^2} \le \mu \le \dfrac{3a^2 + 2a + 2A_1^2 + 2A_1 + 2A_2 + 6\mathrm{a}\mathrm{A}_1}{4(a + A_1)^2} \\ \dfrac{(4a^2\mu - 3a^2 + 8aA_1\mu + A_1^2(4\mu - 2) - 2A_2 - 6\mathrm{a}\mathrm{A}_1)}{4} \text{ if } \dfrac{3a^2 + 2a + 2A_1^2 + 2A_1 + 2A_2 + 6\mathrm{a}\mathrm{A}_1}{4(a + A_1)^2} \le \mu \end{cases}$$

Proof. From (3.7) we have

$$a_3 - \mu a_2^2 = \frac{(a + A_1)}{4}\left[c_2 - \left(\frac{4a^2\mu - 3a^2 + A_1(8a\mu + 2) + 2a + A_1^2(4\mu - 2) - 2A_2 - 6aA_1}{4(a + A_1)}\right)c_1^2\right] \quad (3.9)$$

this can be written as

$$a_3 - \mu a_2^2 = \frac{(a + A_1)}{4}[c_2 - \kappa c_1^2]$$

where $\kappa := \frac{4a^2\mu-3a^2+A_1(8a\mu+2)+2a+A_1^2(4\mu-2)-2A_2-6aA_1}{4(a+A_1)}$
Using lemma [2.1] in (3.8) we have $\left|a_3 - \mu a_2^2\right|$

$$\leq \begin{cases} \dfrac{(-4a^2\mu + 3a^2 - 8aA_1\mu + A_1^2(2-4\mu) + 2A_2 + 6aA_1)}{4} \text{ if } \mu \leq \dfrac{3a^2 - 2a + 2A_1^2 - 2A_1 + 2A_2 + 6aA_1}{4(a+A_1)^2} \\ \dfrac{a+A_1}{2} \text{ if } \dfrac{3a^2 - 2a + 2A_1^2 - 2A_1 + 2A_2 + 6aA_1}{4(a+A_1)^2} \leq \mu \leq \dfrac{3a^2 + 2a + 2A_1^2 + 2A_1 + 2A_2 + 6aA_1}{4(a+A_1)^2} \\ \dfrac{(4a^2\mu - 3a^2 + 8aA_1\mu + A_1^2(4\mu-2) - 2A_2 - 6aA_1)}{4} \text{ if } \dfrac{3a^2 + 2a + 2A_1^2 + 2A_1 + 2A_2 + 6aA_1}{4(a+A_1)^2} \leq \mu \end{cases}$$

Corollary: using (3.8) we can see that

$$\left|H_{(2,1)}(h)\right| = \left|a_3 - a_2^2\right| \leq \frac{a+A_1}{2}\max\left\{1, \left|\frac{a^2 + 2aA_1 + 2A_1^2 - 2A_2}{2(a+A_1)}\right|\right\}$$

When $a = 0$ and function $p(z) = \frac{2}{1+e^{-z}}$ this is the class S_{SG}^*[15] studied by the Khan, Muhammad Ghaffar and Ahmad, Bakhtiar et. al using the (3.8) we have

$$\left|H_{(2,1)}(h)\right| \leq \frac{1}{4}$$

4 Conclusions

In this paper we have introduced a new class $S_\psi^*(a)$, where ($0 \leq a \leq 1$), and have worked on the Fekete-Szegö inequality along with upper bound of second Hankel determinant.

Acknowledgment

The authors are very much thankful to the anonymous Editor and Referees for their valuable suggestions to improve the paper.

Conflict of interest

The authors declare that they have no competing interest.

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