Nonlinear Discrete Inequalities Method for the Ulam Stability of First Order Nonlinear Difference Equations

R.Dhanasekaran[1], E.Thandapani[2] and J.M.Rassias[3]

[1] Department of Mathematics,
Vel Tech Rangarajan Dr.Sagunthala,
R and D Institute of Science and Technology,
Chennai - 600 062, India.
e-mail: drrdsekar@yahoo.com

[2] Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai - 600 005, India.
e-mail: ethandapani@yahoo.co.in

[3] Pedagogical Department E.E.,Section of Mathematics and Informatics,
National and Capodistrian University of Athens,
Athens 15342,GREECE.
e-mail:jrassias@primedu.uoa.gr

Abstract

In this paper, first we derive some nonlinear discrete inequalities, and then as an application, we study the Ulam stability of the first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq n_0,$$

where f is a given function. The obtained result on Ulam stability is new to the literature in the sense that our approach does not require the explicit form of solutions of the investigated equations.

2010 Mathematics Subject Classification: 39A30,39B82

Keywords and Phrases: Ulam stability, discrete inequality, nonlinear difference equation.

1. Introduction

In the passed years, the Ulam stability of functional equations received a great attention.In general, we say that an equation is stable in the sense of Ulam if for every approximate solution of that equation there exists an exact solution of the equation near it. For more details on Ulam stability, one can refer to [13].

The problem of the Ulam stability of difference equations is related to the notion of the perturbation of discrete dynamical systems. In [2–5, 7–9, 11, 12, 14, 17], the authors studied Ulam stability of linear difference equations and in [16], the authors obtained some results on Ulam stability for some second order linear difference equations. In all these papers, the authors studied the Ulam stability of first and second order linear difference equations and it seems that no results dealing with Ulam stability for the nonlinear difference equations are available in the literature.

Therefore the purpose of this paper is to study that Ulam stability of the following first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq \mathbb{N}, \tag{1.1}$$

where $f \in C(\mathbb{N}, \mathbb{R})$ and $\mathbb{N}$ denotes the set of all non-negative integers, without using the explicit form of the solutions.

Next, we present the definition of the Ulam stability for difference equations.

Definition 1.1. *The equation* (1.1) *is called stable in Ulam sense if there exists a constant* $L \geq 0$ *such that for every* $\epsilon > 0$ *and every* $\{y(n)\}$ *in* $\mathbb{R}$ *satisfying*

$$|\Delta y(n) - f(n, y(n))| \leq \epsilon,\ n \geq 0 \tag{1.2}$$

there exists a sequence $\{x(n)\}$ *in* $\mathbb{R}$ *with the properties*

$$\Delta x(n) = f(n, x(n)),\ n \geq 0 \tag{1.3}$$

and

$$|y(n) - x(n)| \leq L\epsilon,\ n \geq 0. \tag{1.4}$$

A sequence $\{y(n)\}$ which satisfies (1.2) for some $\epsilon > 0$ is called an approximate solution of the nonlinear difference equation (1.1), and we reformulate the above definition as: the equation (1.1) is called Ulam stable if for every approximate solution of it there exists an exact solutions close to it. If in Definition 1.1, the number ϵ is replaced by a sequence of positive numbers $\{\epsilon(n)\}$ and $L\epsilon$ from (1.4)by a sequence of positive numbers $\{\eta(n)\}$ the equation (1.1) is called genelized stable in the Ulam sense.

In this paper first we derive some nonlinear discrete inequalities, and as an application we investigate the Ulam stability of equations (1.1).

2. Nonlinear Discrete Inequalities

In this section, we present some nonlinear discrete inequalities which provide us a powerful tool for investigating the Ulam stability of a nonlinear first order difference equations.

We begin with the following results which can be found in: [[6], Theorem 41, pp.39].

Lemma 2.1. *If* $a > 0$ *and* $0 < \alpha \leq 1$, *then*

$$a^{\alpha} \leq \alpha a + (1 - \alpha)$$

and the equality holds if $\alpha = 1$.

Theorem 2.2. *Let* $\{u(n)\}$, $\{f(n)\}$, $\{g(n)\}$ *and* $\{h(n)\}$ *be nonnegative real sequences defined for all* $n \in \mathbb{N}$, *and*

$$u(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s) u^{\alpha}(s), \tag{2.1}$$

where $0 < \alpha \leq 1$. *Then*

$$u(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)(\alpha f(s) + (1 - \alpha)) \exp\left(\sum_{t=s+1}^{n-1} \alpha f(t) g(t) \right). \tag{2.2}$$

Proof. Defining a sequence $R(n)$ by

$$R(n) = \sum_{s=0}^{n-1} h(s) u^{\alpha}(s),$$

then $R(0) = 0$ and $u(n) \leq f(n) + g(n)R(n)$. Now using Lemma 2.1, one can obtain

$$\begin{aligned} \Delta R(n) &= h(n)u^{\alpha}(n) \leq h(n)(f(n) + g(n)R(n))^{\alpha} \\ &\leq (\alpha h(n) f(n) + (a - \alpha)h(n)) + \alpha h(n) g(n) R(n) \end{aligned}$$

or

$$R(n+1) - (1+\alpha h(n)g(n))R(n) \leq h(n)(\alpha f(n) + (1-\alpha)). \tag{2.3}$$

Multiplying (2.3) by $\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}$, we have

$$\Delta\left(R(n)\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}\right) \leq h(n)(\alpha f(n)+(1-\alpha))\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}.$$

Summing up the last inequality from 0 to $n-1$, we obtain

$$\begin{aligned} R(n) &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha))\prod_{t=s+1}^{n-1}(1+\alpha h(t)g(t)) \\ &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha))\left(\exp\sum_{t=s+1}^{n-1}\alpha h(t)g(t)\right). \end{aligned} \tag{2.4}$$

Using (2.4) in $u(n) \leq f(n) + g(n)R(n)$, we have the desired inequality (2.2). This completes the proof. ■

Corollary 2.3. *Let* $u(n)$ *and* $p(n)$ *be non-negative real sequences defined for all* $n \in \mathbb{N}$ *such that*

$$u(n) \leq c + \sum_{s=0}^{n-1} p(s)u^{\alpha}(s) \tag{2.5}$$

where $c \geq 0$ *and* $0 < \alpha \leq 1$. *Then*

$$u(n) \leq \left(\frac{c\alpha + (1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \tag{2.6}$$

Proof. Let $f(n) = c \geq 0$, $g(n) = 1$ and $h(n) = p(n)$ in (2.2), we have

$$\begin{aligned} u(n) &\leq c + \sum_{s=0}^{n-1} p(s)(\alpha c + 1 - \alpha)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c + \frac{(\alpha c + (1-\alpha))}{\alpha}\sum_{s=0}^{n-1}\alpha p(s)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c + \frac{(\alpha c + (1-\alpha))}{\alpha}\left(\prod_{s=0}^{n-1}(1+\alpha p(s)) - 1\right) \\ &\leq \left(\frac{\alpha c + (1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \end{aligned}$$

The proof is now complete. ■

Theorem 2.4. *Let $u(n), p(n)$ and $h(n)$ be non-negative real sequences for all $n \in \mathbb{N}$ and*

$$u(n) \leq c + \sum_{s=0}^{n-1} p(s)u(s) + \sum_{s=0}^{n-1} h(s)u^{\alpha}(s), \tag{2.7}$$

where $c \geq 0$ and $0 < \alpha \leq 1$. Then

$$u(n) \leq \left(c + (1-\alpha) \sum_{s=0}^{n-1} h(s) \right) \exp \left(\sum_{s=0}^{n-1} (p(s) + \alpha h(s)) \right). \tag{2.8}$$

Proof. Let $R(n)$ be the right hand side of (2.7). Then $R(0) = c$ and $u(n) \leq R(n)$ and

$$\begin{aligned} \Delta R(n) &= p(n)u(n) + h(n)u^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)R^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)(\alpha R(n) + (1-\alpha)) \\ &= (p(n) + \alpha h(n))R(n) + (1-\alpha)h(n) \end{aligned} \tag{2.9}$$

where we have used Lemma 2.1. Now from (2.9), we have

$$R(n+1) - (1 + p(n) + \alpha h(n))R(n) \leq (1-\alpha)h(n).$$

Arguing as in the proof of Theorem 2.2, one can easily obtain the desired result and hence the details are omitted. ■

Remark 2.1. *(a) If $\alpha = 1$ in Theorem 2.2, then it reduced to the well-known Pachpatte inequality [10], in 2002. For $0 < \alpha < 1$ the estimate (2.2) of Theorem 2.2 is new to the literature.*

(b) If $\alpha = 1$ and $g(n) \equiv 1$, then Theorem 2.2 reduced to a well-known result due to Sugiyama [15], in 1969.

Remark 2.2. *If $\alpha = 1$ in Corollary 2.3, then it reduced to the discrete analogue of the well-known Gronwall-Bellman inequality [1].*

Remark 2.3. *The result obtained in Theorem 2.4 is different from that one by Willet and Wong [18] for the case $0 < \alpha < 1$.*

3. Ulam Stability

As an application of the discrete inequalities established in Section 2, we investigate the Ulam stability of equation (1.1).

Theorem 3.1. *Let $p(n)$ be a positive real sequence for all $n \in \mathbb{N}$ such that*

$$|f(n,u) - f(n,v)| \leq p(n)|u - v|^{\alpha} \tag{3.1}$$

where $0 < \alpha \leq 1$, and

$$\sum_{n=0}^{\infty} p(n) < \infty. \tag{3.2}$$

If for a positive real sequence $\phi(n)$ such that $\sum_{n=0}^{\infty} \phi(n) < \infty$, and

$$|\Delta y(n) - f(n, y(n))| \leq \phi(n) \tag{3.3}$$

then there exists a real sequence $x(n)$ and a constant $k > 0$ satisfying

$$\Delta x(n) = f(n, x(n)) \tag{3.4}$$

such that $|y(n) - x(n)| \leq k$; that is, equation (1.1) has the Ulam stability.

Proof. From the inequality (3.3), we have

$$y(n) \leq y(0) + \sum_{s=0}^{n-1} f(s, y(s)) + \sum_{s=0}^{n-1} \phi(s) \tag{3.5}$$

and from the equation (3.4), we obtain

$$x(n) = x(0) + \sum_{s=0}^{n-1} f(s, x(s)). \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$|y(n) - x(n)| \leq |y(0) - x(0)| + \sum_{s=0}^{n-1} |f(s, y(s)) - f(s, x(s))| + \sum_{s=0}^{n-1} \phi(s).$$

Using the condition (3.1) in the above inequality, we have

$$|y(n) - x(n)| \leq M_1 + \sum_{s=0}^{n-1} p(s)|y(s) - x(s)|^{\alpha} + M_2 \tag{3.7}$$

where $M_1 = |y(0) - x(0)|$ and $\sum_{n=0}^{\infty} \phi(n) \le M_2$ by hypothesis. Now applying Corollary 2.3 in (3.7), we obtain

$$|y(n) - x(n)| \le \frac{((M_1 + M_2)\alpha + (1 - \alpha))}{\alpha} \exp\left(\sum_{s=0}^{n-1} \alpha p(s)\right). \tag{3.8}$$

It follows from (3.2) that there is a constant $M_3 > 0$ such that $\sum_{n=0}^{\infty} p(n) \le M_3$, and using this in (3.8), one obtains

$$|y(n) - x(n)| \le k$$

where $k = \frac{((M_1+M_2)\alpha+(1-\alpha))}{\alpha} \exp(\alpha M_3)$. This completes the proof. ■

4. Conclusion

In this paper, first we have obtained some new nonlinear discrete inequalities and then as an application we investigate the Ulam stability of a nonlinear first order difference equation. In this approach, we do not need to require the explicit form of the solution of the studied equation, where as in [3,4,7-9,11,12,14] the authors used the explicit form of the solutions to prove their established results.

References

[1] R.P.Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

[2] A.R.Baias, F.Blaga and D.Popa, Best Ulam constant for a linear difference equations, Carpathian J. Math., 35(2019), 13-21.

[3] J.Brzdek, D.Popa and B.Xu, The Hyers-Ulam stabiility of nonlinear recurrences, J. Math. Anal. Appl., 335(2007), 443-449.

[4] J.Brzdek, D.Popa and B.Xu, Remarks on stability of linear recurrence of higher order, Appl. Math.Lett., 23(2010), 1459-1463.

[5] J.Brzdek, D.Popa and B.Xu, On nonstability of the linear recurrence of order one, J. Math. Anal. Appl., 367(2010), 146-153.

[6] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities, Cambridge Univ. Press, Cambridge, 1934.

[7] S.M.Jung, Hyers-Ulam stability of the first order matrix differene equations, Adv. Diff. Equ., 2015(2015), Art. ID.2015:170, 13 pages.

[8] S.M.Jung and Y.W.Nam, On the Hyers-Ulam stability of the first order difference equation, J. Funct. Spaces, Vol.2016, Art. ID:6078298, 6 pages.

[9] M.Onitsuka, Hyers-Ulam stability of first order nonhomogeneous difference equations with a constant stepsize, Appl. Math. Comput., 330(2018), 143-151.

[10] B.G.Pachpatte, Inequalities For Finite Difference Equations, Marcel Dekker, New York, 2002.

[11] D.Popa, Hyers-Ulam stability of the linear recurrence with constant coefficients, Adv. Diff. Equ., 2(2005), 101-107.

[12] D.Popa, Hyers-Ulam-Rassias stability of a linear recurrence, J. Math. Anal. Appl., 309(2005), 591-597.

[13] J.M.Rassias, E.Thandapani, K.Ravi and B.V.Senthil kumar,Book: Functional Equations and Inequalities: Solutions and Stability Results,Series on Concrete and Applicable Mathematics:Vol.21, World Sci. Pub.Comp., Singapore, 2017,Pages 396.

[14] Y.Shen and Y.Li, The Z-transform method for the Ulam stability of linear difference equations with constant coefficients, Adv. Diff. Equ., (2018), 2018:396, 15 pages.

[15] S.Sugiyama, On the stability of difference equations, Bull.Sci.Engr.Research Lab., Waseda Univ., 45(1969), 140-144.

[16] A.K.Tripathy, Hyers-Ulam stability of second order linear difference equation, Inter. J. Diff. Equ. Appl., 16(2017), 53-65.

[17] A.K.Tripathy and P.Senapati, Hyers-Ulam stability of first order linear difference operators on Banach spaces, J. Adv. Math., 14(2018), 7475-7485.

[18] D.Willett and J.S.W.Wong, On the discrete analogues of some generalizations

of Gronwall's inequality, Monatsh. Math., 69(1965), 362-367.