Some convergence results using K^* iteration process in $CAT(0)$ spaces

Kifayat Ullah, Dong Yun Shin, Choonkil Park and Bakhat Ayaz Khan

Abstract. In this paper, some strong and Δ-convergence results for Suzuki generalized nonexpansive mappings in the setting of complete $CAT(0)$ spaces are proved. We are using newly introduced K^* iteration process for approximation of fixed point. We also give an example to show the efficiency of the K^* iteration process. Our results are extension, improvement and generalization of many well known results in the literature of fixed point theory in $CAT(0)$ spaces.

Mathematics Subject Classification (2010). Primary 47H09, 47H10.

Keywords. Suzuki generalized nonexpansive mapping; $CAT(0)$ space; K^* iterative process; Δ-convergence; strong convergence.

1. Introduction

It is well-known that several mathematics problems are naturally formulated as fixed point problem $Tx = x$, where T is some suitable mapping, may be nonlinear. For example, for given functions $\zeta : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ and $\xi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$, the solution of following nonlinear integral equation

$$x(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr,$$

where $x \in C[a, b]$ (the set of all continuous real-valued functions defined on $[a, b] \subseteq \mathbb{R}$), is equivalently to fixed point problems for the following mapping $T : C[a, b] \to C[a, b]$ defined by

[0]*Corresponding author: Dong Yun Shin (email: dyshin@uos.ac.kr).
This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937). We would like to thank Prof. Balwant Singh Thakur for technical assistance.

$$(Tx)(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr$$

for all $x \in C[a, b]$.

The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed point. Many iterative processes have been developed to approximate fixed points of contraction type of mapping in $CAT(0)$ type spaces of ground spaces. Some of the other well-known iterative processes are those of Mann [17], Ishikawa [10], Noor [8], Abbas [1], Agarwal [2], Phuengrattana and Suantai [19], Karahan and Ozdemir [11], Chugh, Kumar and Kumar [6], Sahu and Petrusel [20], Khan [14], Gursoy and Karakaya [9], Thakur, Thakur and Postolache [22] and so on. See also [13, 23, 25] for more information on $CAT(0)$ spaces and applications. Recently, Ullah and Arshad [24] introduced a new three steps iteration process as the K^* iteration process and proved that it is strong and converges fast as compared to all above mentioned iteration processes. They use uniformly convex Banach space as a ground space.

Motivated by above, in this paper, first we develop an example of Suzuki generalized nonexpansive mappings is given which is not nonexpansive. We compare the speed of convergence of the K^* iteration process with the leading two steps S-iteration process and leading three steps Picard-S-iteration process for Suzuki generalized nonexpansive mappings, and graphic representation is also given.

Finally, we prove some strong and Δ-convergence theorems for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ spaces.

2. Preliminaries

Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a geodesic space if every two points of X is joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we denote by $[x, y]$, called the segment joining x to y.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If $x, y_{1,} y_2$ are points in $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \qquad \text{(CN)}$$

This is the (CN) inequality of Burhat and Tits [5].

We recall the following result from Dhompongsa and Panyanak [8].

Lemma 2.1. ([8]) *For $x, y \in X$ and $\alpha \in [0, 1]$, there exists a unique point $z \in [x,\ y]$ such that*

$$d(x, z) = \alpha d(x, y) \text{ and } d(y, z) = (1 - \alpha)d(x, y). \qquad (2.1)$$

The notation $((1 - \alpha)x \oplus \alpha y)$ is used for the unique point z satisfying (2.1).

$CAT(0)$ space may be regarded as a metric version of Hilbert space. For example, in $CAT(0)$ space we have the following extended version of parallelogram law:

$$d(z, \alpha x \oplus (1 - \alpha)y)^2 = \alpha d(x, z)^2 + (1 - \alpha)d(z, y)^2 - \alpha(1 - \alpha)d(x, y)^2 \quad (2.2)$$

for any $\alpha \in [0, 1]$, $x, y \in X$.

If $\alpha = \frac{1}{2}$, then the inequality (2.2) becomes the (CN) inequality.

In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (cf. [5]). Complete $CAT(0)$ spaces are often called Hadmard spaces. For more on these spaces, please refer to [3, 4].

Lemma 2.2. ([14, Lemma 2.4]) *For $x, y, z \in X$ and $\alpha \in [0, 1]$, we have*

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y).$$

Let C be a nonempty closed convex subset of a $CAT(0)$ space X let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, in a complete $CAT(0)$ space, $A(C, \{x_n\})$ consists of exactly one point.

We now recall the definition of Δ-convergence in $CAT(0)$ space.

Definition 2.3. A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-lim of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

Since in a $CAT(0)$ space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence.

A $CAT(0)$ space X is said to satisfy the Opial′s property [17] if for each sequence $\{x_n\}$ in X, Δ-converges to $x \in X$, we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

for all $y \in X$ such that $y \neq x$.

Definition 2.4. A point p is called a fixed point of a mapping T if $T(p) = p$ and $F(T)$ represents the set of all fixed points of the mapping T.

Definition 2.5. Let C be a nonempty subset of a $CAT(0)$ space X.

(i) A mapping $T : C \to C$ is called a contraction if there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in C$.

(ii) A mapping $T : C \to C$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$.

(iii) A mapping is a quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have

$$d(Tx, p) \leq d(x, p).$$

In 2008, Suzuki [21] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y).$$

Suzuki [21] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness. The mapping satisfying condition (C) is called a Suzuki generalized nonexpansive mapping.

Suzuki [21] obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mapping. In 2011, Phuengrattana [18] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors, see, e.g., [22] and references therein.

The following are some basic properties of Suzuki generalized nonexpansive mappings whose proofs in the setup of $CAT(0)$ spaces follow the same lines as those of [12, Propostions 11, 14, 19] and therefore we omit them.

Proposition 2.6. *Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to C$ be any mapping.*

(i) [21, Proposition 1] *If T is nonexpansive, then T is a Suzuki generalized nonexpansive mapping.*

(ii) [21, Proposition 2] *If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii) [21, Lemma 7] *If T is a Suzuki generalized nonexpansive mapping, then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

for all $x, y \in C$.

Lemma 2.7. [21, Theorem 5] *Let C be a weakly compact convex subset of a $CAT(0)$ space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.8. [16, Lemma 2.9] *Suppose that X is a complete $CAT(0)$ space and $x \in X$. If $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that for some $r \geq 0$, we have*

$$\begin{aligned} \lim_{n\to\infty} \sup d(x_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(y_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(t_n x_n + (1 - t_n)y_n, x) &= r, \end{aligned}$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.9. [7, Proposition 2.1] *If C is a closed comvex subset of a complete $CAT(0)$ space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.10. [15] *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.11. [15, Proposition 3.7] *Let C is a closed comvex subset of a complete $CAT(0)$ space X and $T : C \to X$ be a Suzuki generalized nonexpansive mapping. Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n, x_n) \to 0$ imply $x \in C$ and $Tx = x$.*

The following is an example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 1. Define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} 1 - x \text{ if } x \in \left[0, \frac{1}{6}\right) \\ \frac{x+5}{6} \text{ if } x \in \left[\frac{1}{6}, 1\right]. \end{cases}$$

We need to prove that T is a Suzuki generalized nonexpansive but not nonexpansive.

If $x = \frac{15}{96}$ and $y = \frac{1}{6}$, then we have

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= \left|1 - \frac{15}{96} - \frac{31}{36}\right| \\ &= \frac{5}{288} \\ &> \frac{1}{96} \\ &= d(x, y). \end{aligned}$$

Hence T is not a nonexpansive mapping.

To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in \left[0, \frac{1}{6}\right)$. Then $\frac{1}{2}d(\,x, Tx) = \frac{1-2x}{2} \in \left(\frac{1}{3}, \frac{1}{2}\right]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we have $\frac{1-2x}{2} \leq y - x$, i.e., $\frac{1}{2} \leq y$ and hence $y \in \left[\frac{1}{2}, 1\right]$. We have

$$d(Tx, Ty) = \left|\frac{y+5}{6} - (1-x)\right| = \left|\frac{y+6x-1}{6}\right| < \frac{1}{6}$$

and

$$d(x, y) = |x - y| > \left|\frac{1}{6} - \frac{1}{2}\right| = \frac{2}{6}.$$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y)$.

Case II: Let $x \in \left[\frac{1}{6}, 1\right]$. Then $\frac{1}{2}d(\,x, Tx) = \frac{1}{2}\left|\frac{x+5}{6} - x\right| = \frac{5-5x}{12} \in \left[0, \frac{25}{72}\right]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we have $\frac{5-5x}{12} \leq |y - x|$, which gives two possibilities:

(a) Let $x < y$. Then $\frac{5-5x}{12} \leq y - x \Longrightarrow y \geq \frac{5+7x}{12} \Longrightarrow y \in \left[\frac{37}{72}, 1\right] \subset \left[\frac{1}{6}, 1\right]$. So

$$d(Tx, Ty) = \left|\frac{x+5}{6} - \frac{y+5}{6}\right| = \frac{1}{6}d(x, y) \leq d(x, y).$$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y)$.

(b) Let $x > y$. Then $\frac{5-5x}{12} \leq x - y \Longrightarrow y \leq x - \frac{5-5x}{12} = \frac{17x-5}{12} \Longrightarrow y \in \left[-\frac{13}{72}, 1\right]$. Since $y \in [0, 1]$, $y \leq \frac{17x-5}{12} \Longrightarrow x \in \left[\frac{5}{12}, 1\right]$. So the case is $x \in \left[\frac{5}{12}, 1\right]$ and $y \in [0, 1]$.

Now the case that $x \in \left[\frac{5}{12}, 1\right]$ and $y \in \left[\frac{1}{6}, 1\right]$ is the same case as that of (a). So let $x \in \left[\frac{5}{12}, 1\right]$ and $y \in \left[0, \frac{1}{6}\right)$. Then

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x+5}{6} - (1-y)\right| \\ &= \left|\frac{x+6y-1}{6}\right|. \end{aligned}$$

For convenience, first we consider $x \in \left[\frac{5}{12}, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{6}\right)$. Then $d(Tx, Ty) \leq \frac{1}{12}$ and $d(x, y) \geq \frac{3}{12}$. Hence $d(Tx, Ty) \leq d(x, y)$.

TABLE 1. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.99809713998382	0.99722222222222	0.98333333333333
x_2	0.99997729192914	0.99993300629392	0.99758822658104
x_3	0.99999985210113	0.99999849779947	0.99967552468466
x_4	0.99999999971662	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

TABLE 2. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.5	0.5	0.5
x_1	0.99048569991909	0.99722222222222	0.98333333333333
x_2	0.99988645964572	0.99993300629392	0.99758822658104
x_3	0.99999926050565	0.99999926050566	0.99967552468466
x_4	0.99999999858311	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

Next consider $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{6}\right)$. Then $d(Tx, Ty) \leq \frac{1}{6}$ and $d(x, y) \geq \frac{2}{6}$. Hence $d(Tx, Ty) \leq d(x, y)$. So

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y).$$

Hence T is a Suzuki generalized nonexpansive mapping.

In order to show the efficiency of K^* iteration process, we use Example 1 with $x_0 = 0.9, x_0 = 0.5$ and get the above Tables 1 and 2. Graphic representation is given in Figure 1.

Let $n \geq 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$. Ullah and Arshad [24] introduced a new iteration process known as the K^* iteration process

$$\begin{cases} x_0 \in C \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n. \end{cases}$$

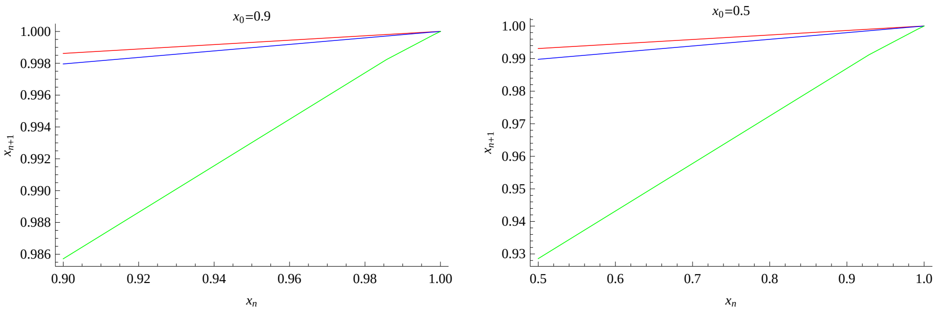

FIGURE 1. Convergence of iterative sequences generated by K^* (red line), Picard-S (blue line) and S (green line) iteration process to the fixed point 1 of the mapping T defined in Example 1.

They also proved that the K^* iteration process is faster than the Picard-S iteration and S-iteration processes with the help of a numerical example.

3. Convergence results for Suzuki generalized nonexpansive mappings

In this section, we prove some strong and Δ-convergence theorems of a sequence generated by a K^* iteration process for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ space. The K^* iteration process in the language of $CAT(0)$ space is given by

$$\begin{array}{c} x_0 \in C \\ z_n = (1-\beta_n)x_n \oplus \beta_n T x_n \\ y_n = T((1-\alpha_n)z_n \oplus \alpha_n T z_n) \\ x_{n+1} = T y_n \end{array} . \tag{3.1}$$

Lemma 3.1. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T is a Suzuki generalized nonexpansive mapping,

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p, z) \text{ implies that } d(Tp, Tz) \leq d(p, z).$$

By Proposition 2.6 (ii), we have

$$\begin{aligned} d(z_n,p) &= d(((1-\beta_n)x_n \oplus \beta_n Tx_n),p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned} d(y_n,p) &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.3}$$

Similarly by using (3.3), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

This implies that $\{d(x_n,p)\}$ is bounded and nonincreasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} d(x_n,p)$ exists, as required. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a,b]$ for some a,b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 3.2, $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{3.4}$$

From (3.2) and (3.4), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.5}$$

By Proposition 2.6 (ii) we have

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.6}$$

On the other hand, by using (3.2), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &= d(x_n,p)-\alpha_n d(x_n,p)+\alpha_n d(z_n,p). \end{aligned}$$

This implies that

$$\frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p).$$

So

$$d(x_{n+1},p)-d(x_n,p) \leq \frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p),$$

which implies that

$$d(x_{n+1},p) \leq d(z_n,p).$$

Therefore,

$$r \leq \liminf_{n\to\infty} d(z_n,p). \tag{3.7}$$

By (3.5) and (3.7), we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n,p) \\ &= \lim_{n\to\infty} d(((1-\beta_n)x_n+\beta_n Tx_n),p) \\ &= \lim_{n\to\infty} d(\beta_n(Tx_n,p)+(1-\beta_n)(x_n,p)). \end{aligned} \tag{3.8}$$

From (3.4), (3.6), (3.8) and Lemma 2.8, we have that $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n)=0$. Let $p \in A(C,\{x_n\})$. By Proposition 2.6 (iii), we have

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,Tp) \\ &\leq \limsup_{n\to\infty}(3d(Tx_n,x_n)+d(x_n,p)) \\ &\leq \limsup_{n\to\infty} d(x_n,p) \\ &= r(p,\{x_n\}). \end{aligned}$$

This implies that $Tp \in A(C,\{x_n\})$. Since X is uniformly convex, $A(C,\{x_n\})$ is a singleton and hence we have $Tp=p$. So $F(T) \neq \emptyset$. □

Now we are in the position to prove Δ-convergence theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0, 1]$ such that $\{t_n\} \in [a, b]$ and $\{s_n\} \in [0, b]$ or $\{t_n\} \in [a, 1]$ and $\{s_n\} \in [a, b$ for some a, b with $0 < a \leq b < 1$. For an arbitrary element $x_1 \in C$, $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, by Theorem 3.3, we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $u \in w_w\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n \{v_n\} = v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, $v \in F(T)$ by Lemma 2.11. We claim that $u = v$. Suppose not. Since T is a Suzuki generalized nonexpansive mapping and $v \in F(T)$, $\lim_n d(x_n, v)$ exists by Theorem 3.2. Then by uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} sup d(v_n, v) &< \lim_{n\to\infty} sup d(v_n, u) \\
&\leq \lim_{n\to\infty} sup d(u_n, u) \\
&< \lim_{n\to\infty} sup d(u_n, v) \\
&= \lim_{n\to\infty} sup d(x_n, v)) \\
&= \lim_{n\to\infty} sup d(v_n, v),
\end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n \{v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $c \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not. Since $\{d(x_n, v)\}$ is convergent, by the uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} sup d(v_n, v) &< \lim_{n\to\infty} sup d(v_n, x) \\
&\leq \lim_{n\to\infty} sup d(x_n, x) \\
&< \lim_{n\to\infty} sup d(x_n, v) \\
&= \lim_{n\to\infty} sup d(v_n, v),
\end{aligned}$$

which is a contradiction and hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 3.4. *Let C be a nonempty compact convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.7, we have that $F(T) \neq \emptyset$ and so by Theorem 3.2 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.6 (iii), we have

$$d(x_{n_k}, Tp) \leq 3d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p), \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. By Theorem 3.2, $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and so $\{x_n\}$ converges strongly to p. ☐

Senter and Dotson [22] introduced the notion of a mappings satisfying condition (I) as follows.

A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.5. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \geq 0$. If $r = 0$, then the result follows. Suppose $r > 0$. Then from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq d(Tx_n, x_n). \tag{3.9}$$

Since $F(T) \neq \emptyset$, by Theorem 3.3, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So (3.9) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{3.10}$$

Since f is a nondecreasing function, from (3.10), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\}$, $y_k \in F(T)$, such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So using (3.4), we get

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1}, y_k) &\leq d(y_{k+1}, x_{k+1}) + d(x_{k+1}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p \in F(T)$. Hence the proof is complete. □

References

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesn. **66** (2014), 223–234.

[2] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. **8** (2007), 61–79.

[3] M. Bridson and A. Heaflinger, *Metric Space of Non-positive Curvature*, Springer-Verlag, Berlin, 1999.

[4] D. Burago, Y. Burago and S. Inavo, *A Course in Metric Geometry, in: Graduate Studies in Mathematics*, Vol. 33, Americal Mathematical Society, Providence, 2001.

[5] F. Burhat and J. Tits, *Groups reductifs sur un curps local*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251.

[6] R. Chugh, V. Kumar and S. Kumar, *Strong convergence of a new three step iterative scheme in Banach spaces*, Am. J. Comput. Math. **2** (2012), 345–357.

[7] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 35–45.

[8] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorem in $CAT(0)$ spaces*, Comput. Math. Appl. *56* (2008), 2572–2579.

[9] F. Gursoy and V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, arXiv:1403.2546v2 (2014).

[10] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147–150.

[11] I. Karahan and M. Ozdemir, *A general iterative method for approximation of fixed points and their applications*, Adv. Fixed Point Theory **3** (2013), 520–526.

[12] E. Karapinar and K. Tas, *Generalized (C)-conditions and related fixed point theorems*, Comput. Math. Appl. **61** (2011), 3370–3380.

[13] H. Khatibzadeh and S. Ranjbar, *A variational inequality in complete $CAT(0)$ spaces*, J. Fixed Point Theory Appl. **17** (2015), 557–574.

[14] S.H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl. **2013**, Article No. 69 (2013).

[15] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spacces*, Nonlinear Anal.–TMA **68** (2008), 3689–3696.

[16] W. Lawaong and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in $CAT(0)$ spaces*, Fixed Point Theory Appl. **2010**, Art. ID 367274 (2010).

[17] W.R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.

[18] W. Phuengrattana, *Approximating fixed points of Suzuki-generalized nonexpansive mappings*, Nonlinear Anal. Hybrid Syst. **5** (2011), 583–590.

[19] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math. **235** (2011), 3006–3014.

[20] D. R. Sahu and A. Petrusel, *Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces*, Nonlinear Anal.–TMA **74** (2011), 6012–6023.

[21] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340** (2008), 1088–1095.

[22] B.S Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput. **275** (2016), 147–155.

[23] G. C. Ugwunnadi, A. R. Khan and M. Abbas, *A hybrid proximal point algorithm for finding minimizers and fixed points in $CAT(0)$ spaces*, J. Fixed Point Theory Appl. **20** (2018), 20:82.

[24] K. Ullah and M. Arshad, *New three step iteration process and fixed point approximation in Banach spaces*, preprint.

[25] Z. Yang and Y.J. Pu, *Generalized Browder-type fixed point theorem with strongly geodesic convexity on Hadamard manifolds with applications*, Indian J. Pure Appl. Math. **43** (2012), 129–144.

Kifayat Ullah
Department of Mathematics, University of Science and Technology, Township Campus, 44000, Bannu, Pakistan
e-mail: kifayatmath@yahoo.com

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Korea
e-mail: dyshin@uos.ac.kr

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

Bakhat Ayaz Khan
Department of Mathematics, University of Science and Technology, Township Campus, 44000, Bannu, Pakistan
e-mail: bakhtayazkhan580@gmail.com