Bishop's property, Weyl's Theorem and Riesz idempotent

Ayyoub Fellag Ariouat[1] , Aissa Nasli Bakir[2] and Abdelkader Benali[3]

Abstract. Important fundamental and spectral properties of the classes of quasi n-normal and k-quasi n-normal operators defined on a separable complex Hilbert space constitute the aim of the present paper. We prove that the considered operators satisfy Bishop's property (β) and that are polaroid, subscalar and decomposable. It's also proved that a k-quasi n-normal operator has a non trivial invariant subspace and Weyl's theorem holds for this operator. Other results related to the Riesz idempotent of elements of these classes are also established.

Mathematics Subject Classification (2010). Primary 47A30; Secondary 47B47, 47B20

Keywords. Weyl's Theorem, Isoloid operators, Bishop's property, Riesz idempotent.

1. Introduction and Background

Let H denote a separable complex Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on H. An operator $A \in B(H)$ is said to be normal if A commutes with its adjoint $A^\star$, and n-normal if $A^\star A^n = A^n A^\star$. It's obvious that if A is a n-normal operator, then A^n is normal. An operator $A \in H$ is said to be an isometry if $A^\star A = I$, where I is the identity operator on H, and a co-isomety if $A^\star$ is an isometry.

An operator A in $B(H)$ is said to have the *Single Valued Extension Property* (SVEP) at a complex number α, if for each open neighborhood U of α, the zero function is the unique analytic solution on U of the equation

$$(A-\lambda)f(\lambda)=0$$

Moreover, A is said to have SVEP if A has SVEP at each complex scalar [1] . For $A \in B(H)$, the smallest integer j for which $N\left(A^j\right) = N\left(A^{j+1}\right)$ is said to be the *ascent* of A and is denoted $p(A)$. If such integer does not exist, we shall write $p(A) = \infty$ [1].
Also, if A in $B(H)$, then the *Riesz idempotent* E with respect to an isolated point μ in the spectrum $\sigma(A)$ of A is defined by

$$E = \frac{1}{2\pi i}\int_{\partial\mathcal{D}} (z-A)^{-1}\,dz$$

where the integral is taken in the positive sense, $\mathcal{D}$ is a closed disk concentrated at λ with a small radius r satisfying $\mathcal{D} \cap \sigma(A) = \{\lambda\}$ and $\partial\mathcal{D}$ denotes its boundary, . The operator $A \in B(H)$ is said to have *Bishop's property* (β) if for each open subset G of $\mathbb{C}$, and all sequence $f_n\colon G \to H$, of analytic functions such that $(A-\lambda)f_n$ converges uniformly to 0 in norm of

compact subsets of G, $(f_n)_n$ converges uniformly to 0, in norm of compact subsets of G. See[1 ,9 ,10] for more details.
In this paper, we investigate the class of operators verifying $AA^{\star n}A^n = A^{\star n}A^{n+1}$ for an operator $A \in B(H)$, and a natural integer n, $n \geq 1$. Elements of this class are said to be quasi-normal operators of order n, [7] . For $n = 1$, the operator A is quasi normal. We show that if A is quasi-normal of order n, then A has Bishop's property (β), A is isoloid and it satisfies Weyl's Theorem. We also establish other spectral results related to the compacity and the Riesz idempotent.

Example 1. Matrices on $\mathbb{C}^2$ of the form $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ with $a \neq 0$ are quasi-normal of order 2 but not quasi normal. However, the matrix $\begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}$, $(i^2 = -1)$ is quasi-normal of order n for all integer n, $n \geq 1$.

Example 2. The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are quasi-normal of order 2. Nonetheless, the matrix $A + S$ is not quasi-normal of order 2.

Finally, $N(A)$, $R(A)$ and $|A| = (A^*A)^{\frac{1}{2}}$ denote respectively the null space, the range and the modulus of an operator A in $B(H)$.

2. Quasi-normal operators of order n

Proposition 3. [7] *The class of quasi-normal operators of order* n *contains the class of* n*-normal operators.*

Proposition 4. [7] *Let* $A \in B(H)$ *be a quasi-normal operator of order* n, *and let* $B \in B(H)$ *be unitarily equivalent to* A. *Then,* B *is a quasi-normal operator of order* n *too.*

The following example shows that if A and B are quasisimilar, then the Proposition 4 is in general not true.

Example 5. The operator $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is quasi-normal of order 2, and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible and not unitary. The matrix $B = XAX^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is not quasi-normal of order 2 since

$$\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix} = BB^{\star 2}B^2 \neq B^{\star 2}B^3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

It is also given in [7] that a quasi-normal operator of order 2 needs not to be quasi-normal operator of order 3.

Example 6. The matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 3-quasi-normal operator of order 2 but not quasi-normal of order 2.

Proposition 7. *Let $A \in B(H)$ be an invertible quasi-normal operator of order n. Then, so is its inverse A^{-1}.*

Proof. Under the hypotheses, A is n-normal. Indeed, A^n is also invertible, and

$$AA^{\star n} = AA^{\star n}A^nA^{-n} = A^{\star n}A^{n+1}A^{-n} = A^{\star n}A$$

Hence,

$$\begin{aligned} A^{-1}(A^{-1})^{\star n}A^{-n} = (A^nA^{\star n}A)^{-1} = (A^nAA^{\star n})^{-1} &= (A^{n+1}A^{\star n})^{-1} \\ &= (A^{-1})^{\star n}(A^{-1})^{n+1} \end{aligned}$$

□

Lemma 1. *If $A \in B(H)$ is a quasi-normal operator of order n, then $(A^{*n}A^n)^3 = A^{*3n}A^{3n}$.*

Proof. By the hypothesis, $(A^{*n}A^n)^2 = A^{*2n}A^{2n}$. Then,

$$(A^{*n}A^n)^3 = (A^{*n}A^n)^2A^{*n}A^n = A^{*2n}A^{2n}A^{*n}A^n.$$

Hence,

$$\begin{aligned} A^{*2n}A^{2n}A^{*n}A^n = A^{*2n}A^{2n-1}A^{*n}A^{n+1} &= A^{*2n}A^{2n-2}A^{*n}A^{n+2} \\ &= A^{*2n}A^{2n-3}A^{*n}A^{n+3} \\ &= \ldots\ldots \\ &= A^{*3n}A^{3n} \end{aligned}$$

□

Definition 8. An operator $A \in B(H)$ is said to be paranormal if for all unit vector x in H $\|Ax\|^2 \leq \|A^2x\|$.

We've then,

Proposition 9. *If $A \in B(H)$ is a quasi-normal operator of order n, then A^n is paranormal.*

Proof. According to the hypothesis and Lemma 1, $(A^{*n}A^n)^2 = A^{*2n}A^{2n}$. Then, for each unit vector x in H,

$$\|A^{*n}A^nx\|^2 = \langle (A^{*n}A^n)^2x, x\rangle = \langle A^{*2n}A^{2n}x, x\rangle = \|A^{2n}x\|^2$$

By Cauchy-Schwarz inequality, we get

$$\|A^nx\|^2 = \langle A^{*n}A^nx, x\rangle \leq \|A^{*n}A^nx\|\|x\| = \|A^{*n}A^nx\| = \|A^{2n}x\| = \|(A^n)^2x\|$$

This shows the paranormality of A.

Definition 10. [1] For $A \in H$, the smallest integer j for which $N(A^j) = N(A^{j+1})$ is said to be the *ascent* of A and is denoted $p(A)$. If such integer does not exist, we shall write $p(A) = \infty$.

In view of [1, Theorem3.8] , operators that have finite ascent have SVEP too. We've then

Theorem 11. *Let A be quasi-normal of order n. Then, $N(A^n) = N(A^{n+1})$.*

Proof. Let x be in $N(A^{n+1})$. Then, $A^{n+1}x = 0$. Since A is quasi-normal of order n,

$$AA^{\star n}A^n x = 0$$

Hence,

$$\langle AA^{\star n}A^n x, z\rangle = \langle A^{\star n}A^n x, A^{\star} z\rangle = 0$$

for all $z \in H$. Thus,

$$A^{\star n}A^n x \in R(A^{\star})^{\perp} \cap \overline{R(A^{\star n})}$$

Since $R(A^{\star})^{\perp} \subset R(A^{\star n})^{\perp}$,

$$A^{\star n}A^n x \in R(A^{\star n})^{\perp} \cap \overline{R(A^{\star n})} = \{0\}$$

Finally, $A^n x = 0$. That is $x \in N(A^n)$. This achieves the proof since the second inclusion is evident. □

Corollary 12. *If A is quasi-normal of order n, then $p(A) \leq n$.*

Corollary 13. *Quasi-normal operators of order n have SVEP at 0.*

3. k-quasi-normal operators of order n

Definition 14. An operator $A \in B(H)$ is said to be k-quasi-normal of order n. If

$$A^{\star k}(AA^{\star n}A^n - A^{\star n}A^{n+1})A^k = 0$$

1-quasi-normal operators of order n are quasi-normal of order n.

Theorem 15. *Let A be a k-quasi-normal operator of order n. Assume that $R(A^k)$ is dense in H. Then, A is quasi-normal of order n.*

Proof. Since A is k-quasi-normal of order n,

$$A^{\star k}(AA^{\star n}A^n - A^{\star n}A^{n+1})A^k = 0$$

Let x be in H. Since $\overline{R(A^k)} = H$, $x = \lim_{n\to\infty} A^k x_n$ for some sequence $(x_n)_n$ of elements of H. Since A is k-quasi-normal of order n,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle A^{\star k}(AA^{\star n}A^n - A^{\star n}A^{n+1})A^k x_n, x_n\rangle \\ &= \lim_{n\to\infty} \langle (AA^{\star n}A^n - A^{\star n}A^{n+1})A^k x_n, A^k x_n\rangle \end{aligned}$$

Then,

$$0 = \langle (AA^{\star n}A^n - A^{\star n}A^{n+1}) \lim_{n\to\infty} A^k x_n, \lim_{n\to\infty} A^k x_n\rangle$$

by the continuity of the inner product. Hence,

$$\langle AA^{\star n}A^n - A^{\star n}A^{n+1})x, x\rangle = 0$$

This shows that A is k-quasi-normal of order n. □

Corollary 16. *If A is k-quasi-normal operator of order n such that A is not quasi-normal of order n, then A is not invertible.*

Theorem 17. *The restriction of a k-quasi-normal operator $A \in B(H)$ of order n on an invariant closed subspace $M \subset H$ is also k-quasi-normal of order n.*

Proof. $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ under the orthogonal decomposition $H = M \oplus M^{\perp}$. Since A is k-quasi n-normal,

$$\begin{aligned} 0 &= A^{\star k}(AA^{\star n}A^n - A^{\star n}A^{n+1})A^k \\ &= \begin{pmatrix} A_1^{\star k}(A_1A_1^{\star n}A_1^n - A_1^{\star n}A_1^{n+1})A_1^k & R \\ S & T \end{pmatrix} \end{aligned}$$

for certain operators $R, S, T \in B(H)$. Hence,

$$A_1^{\star k}(A_1A_1^{\star n}A_1^n - A_1^{\star n}A_1^{n+1})A_1^k = 0$$

The desired result is proved.

Theorem 18. *Let A be a k-quasi-normal operator of order n, for which $\overline{R(A^k)} \neq H$. If $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ on $H = \overline{R(A^k)} \oplus N(A^{\star k})$, then*

- *A_1 is quasi-normal of order n.*
- *$A_3^k = 0$ and $\sigma(A) = \sigma(A_1) \cup \{0\}$.*

Proof. Let $x \in H$. Since A is k-quasi-nomal of order n,

$$\begin{aligned} 0 &= \langle A^{\star k}(AA^{\star n}A^n - A^{\star n}A^{n+1})A^k x, x\rangle \\ &= \langle (AA^{\star n}A^n - A^{\star n}A^{n+1})A^k x, A^k x\rangle \end{aligned}$$

Then, for all $y \in \overline{R(A^k)}$

$$\langle (AA^{\star n}A^n - A^{\star n}A^{n+1})y, y\rangle = 0$$

Hence,

$$(AA^{\star n}A^n - A^{\star n}A^{n+1})\Big|_{\overline{R(A^k)}} = A_1A_1^{\star n}A_1^n - A_1^{\star n}A_1^{n+1} = 0$$

Thus, A is quasi-normal of order n.
Let now P be the orthogonal projection on $\overline{R(A^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$,

$$\langle A_3^k x_2, y_2\rangle = \langle A^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, A^{\star k}(I-P)y\rangle = 0$$

Thus, A_3 is nilpotent of order k.

Moreover, $\sigma(A) \cup \sigma(A_3) = \sigma(A) \cup \Omega$, where Ω is the union of holes in $\sigma(A)$ which happen to be a subset of $\sigma(A) \cap \sigma(A_3)$ by [4, Corollary 7] , with the interior of $\sigma(A) \cap \sigma(A_3)$ is empty and A_3 is nilpotent. Thus, $\sigma(A) = \sigma(A_3) \cup \{0\}$.

4. Spectral study

Theorem 19. *Let $A \in B(H)$ be a quasi-normal operator of both order 2 and 3. Then, equation $Ax = \mu x$ implies $A^\star x = \overline{\mu} x$ for some $x \in H$ and a nonzero complex scalar μ.*

Proof. Since A is quasi-normal operator of order 2,

$$AA^{*2}A^2 = A^{*2}A^3$$

Then,

$$A^2A^{*2}A^3 = AA^{*2}A^4$$

and so for each $x \in H$

$$\langle A^2A^{*2}A^3x, x\rangle = \langle AA^{*2}A^4x, x\rangle$$

Since $\mu \neq 0$,

$$\|A^{*2}x\| = |\mu^2|\|x\| \qquad (19.1)$$

Thus, for all vector x in H,

$$\|(A^2 - \mu^2)^*x\|^2 = \|A^{*2}x\|^2 + |\mu|^2\|x\|^2 - 2|\mu|^2\|x\|^2 = 0$$

By (19.1). that is,

$$A^{*2}x = \overline{\mu^2}x \qquad (19.2)$$

Analogously, since A is also quasi-normal of order 3,

$$AA^{*3}A^3 = A^{*3}A^4$$

Then,

$$\begin{aligned} A^3A^{*3}A^3 = A^2A^{*3}A^4 = AAA^{*3}A^3A = AA^{*3}A^4A &= AA^{*3}A^3A^2 \\ &= A^{*3}A^4A^2 \\ &= A^{*3}A^6 \end{aligned}$$

So for each $x \in H$,

$$\langle A^3A^{*3}A^3x, x\rangle = \langle A^{*3}A^6x, x\rangle$$

Since $\mu \neq 0$,

$$\|A^{*3}x\| = |\mu^3|\|x\| \qquad (19.3)$$

Thus, for all vector x in H,

$$\|(A^3-\mu^3)^*x\|^2 = \|A^{*3}x\|^2 + |\mu|^3\|x\|^2 - 2|\mu|^3\|x\|^2 = 0$$

by (19.3). That is,

$$A^{*3}x = \overline{\mu^3}x \tag{19.4}$$

Finally, for each $x \in H$,

$$|\overline{\mu^3}|^2\|x\|^2 = \|A^{*3}x\|^2 = \langle A^*A^{*2}x, A^*A^{*2}x\rangle = \mu^2\overline{\mu^2}\langle A^2x, A^*x\rangle = |\mu^2|^2\|A^*x\|^2$$

Thus,

$$\|A^*x\| = |\mu|\|x\|$$

$$\|(A-\mu)^*x\|^2 = \|A^*x\|^2 + |\mu|^2\|x\|^2 - 2|\mu|^2\|x\|^2 = 0$$

by (19.2) and (19.4). Then, $A^*x = \overline{\mu}x$. □

Corollary 20. *Let $A \in B(H)$ be k-quasi-normal operator of order n. Then, $N(A-\mu) = N(A-\mu)^m$, for all non-zero complex scalar μ and all integer $m \geq 1$.*

Corollary 21. *If $A \in B(H)$ is k-quasi-normal operator of order n, then A has SVEP.*

Proof. A straightforward consequence of Theorem 11 and the previous Corollary.

Definition 22. An operator $A \in B(H)$ is said to be isoloid, if every isolated point of its spectrum is an eigenvalue of A.

We've then,

Theorem 23. *If $A \in B(H)$ is a k-quasi-normal operator of both orders n and $n+1$, then A is isoloid.*

Proof. According to Theorem 18, $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$. under the decomposition $H = \overline{R(A^k)} \oplus N(A^{\star k})$. Let $\lambda \in \sigma(A)$ be an isolated point of A. Since $\sigma(A) = \sigma(A_1) \cup \{0\}$ by Theorem 18, λ is an isolated point in $\sigma(A_1)$ or $\lambda = 0$.
a. If λ is an isolated point in $\sigma(A_1)$, then λ is in the ponctual spectrum $\sigma_p(A_1)$ of A_1.
b. Assume that $\lambda = 0$ and $\lambda \notin \sigma(A_1)$. Then, for all $x \in N(A_3)$,

$$A(-A_1^{-1}A_2x \oplus x) = 0$$

That is, $u = (-A_1^{-1}A_2x \oplus x) \in N(A_3)$.

Theorem 24. *Let $A \in B(H)$ be a k-quasi-normal operator of order n, and let $M \subseteq H$ be a closed invariant subspace for A. If the restriction $A|M$ of A on M is one-to-one and normal, then M reduces A, that is, M is invariant for A^* too.*

Proof. Suppose that P is an orthogonal projection of H onto $\overline{R(A^k)}$. Since A is k-quasi-normal of order n,

$$P(A^{*2}A^2 - AA^*)P \geq 0$$

By the hypothesis, $A|_M$ is an injective and normal operator. Then, $E \leq P$ for the orthogonal projection E of H onto M, and $\overline{R(A^k|_M)} = M$ because $A|_{R(A^k)}$ and hence

$$E(A^{*2}A^2 - AA^*)E \geq 0.$$

Let

$$A = \begin{pmatrix} A|M & A_2 \\ 0 & A_3 \end{pmatrix},$$

on $M \oplus M^{\perp}$. Then,

$$AA^* = \begin{pmatrix} A|MA^*|M + A_2A_2^* & A_2A_3^* \\ A_3A_2^* & A_3A_3^* \end{pmatrix}$$

and

$$A^{*2}A^2 = \begin{pmatrix} A^{*2}|MA^2|M & S \\ T & R \end{pmatrix}$$

for some bounded linear operators S, T and R. Thus,

$$\begin{pmatrix} A|MA^*|M + A_2A_2^* & 0 \\ 0 & 0 \end{pmatrix} = E(AA^*)E = E|A^*|^2E \leq E(A^{*2}A^2)^{\frac{1}{2}}E$$

$$\leq (E(A^{*2}A^2)E)^{\frac{1}{2}}$$

$$= \begin{pmatrix} A^{*2}|MA^2|M & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}}$$

This implies that

$$A|_MA^*|_M + A_2A_2^* \leq A|_MA^*|_M.$$

Since $A|M$ is normal and $A_1A_1^*$ is positive, it follows that $A_2 = 0$. Hence M reduces A.

Remark 25. The previous result is in general false if the restriction $A|M$ is not injective. In fact, if A is a nilpotent operator of order k, such that $A^{k-1} \neq 0$, then $A\Big|\overline{R(A^{k-1})} = 0$ is a normal operator. Assume that $\overline{R(A^{k-1})}$ reduces A. Then, $A^{\star}A^{k-1}H \subset \overline{R(A^{k-1})}$. Thus, $A^{*k-1}A^{k-1}H \subset \overline{R(A^{k-1})}$and $N(A^{*k-1}) \subset N(A^{*k-1}A^{k-1}) = N(A^{k-1})$ Since $A^{*k} = A^{*k-1}A^* = 0$, $A^{k-1}A^* = 0$. Hence, $A^{k-1}A^{*k-1} = 0$. Therefore, $A^{k-1} = 0$. This contradicts the hypotheses on A.

Definition 26. For an operator A in $B(H)$, the Riesz idempotent E with respect to an isolated point μ in the spectrum $\sigma(A)$ of A is defined by

$$E = \frac{1}{2\pi i}\int_{\partial\mathcal{D}} (z - A)^{-1}\, dz$$

Here, the integral is taken in the positive sense, $\mathcal{D}$ is a closed disk concentrated at λ with a small radius r satisfying $\mathcal{D} \cap \sigma(A) = \{\lambda\}$ and $\partial\mathcal{D}$ denotes its boundary.

It's known that $E^2 = E,\ EA = AE$ and $\sigma(A|_{EH}) = \{\lambda\}$. Reader is refered to [1, 8] for more information.

Theorem 27. *Let $A \in B(H)$ be a k-quasi-normal operator of order n, and let $\mu \in \mathbb{C}$ be a nonzero isolated point of $\sigma(A)$. The Riesz idempotent E with respect to μ satisfies $EH = N(A-\mu) = N(A-\mu)^\star$ Furthermore, E is self-adjoint.*

Proof. By Theorem 23, μ is an eigenvalue of A, and $EH = N(A-\mu)$. According to Theorem 19, it sufficies to show that $N(A-\mu)^\star \subset N(A-\mu)$. The subspace $N(A-\mu)$ reduces A by Theorem 19, and the restriction of A on its reducing subspace is also k-quasi-normal of order n by Theorem 17. It follows that

$$A = \mu \oplus B \text{ on } H = N(A-\mu) \oplus \left(N(A-\mu)\right)^\perp$$

where B is k-quasi-normal of order n, and $N(B-\mu) = \{0\}$. We've

$$\mu \in \sigma(A) = \{\mu\} \cup \sigma(B)$$

and λ is isolated. Then, either $\mu \notin \sigma(B)$, or μ is an isolated point of $\sigma(B)$, which contradicts the fact that $N(B-\mu) = \{0\}$. Since B is invertible on $\left(N(A-\mu)\right)^\perp$,

$$N(A-\mu) = N(A-\mu)^\star$$

Furthermore, since $EH = N(A-\mu) = N(A-\mu)^\star$,

$$((z-A)^\star)^{-1}E = \overline{(z-\mu)^{-1}}E$$

Thus,

$$\begin{aligned} E^\star = -\frac{1}{2\pi i}\int_{\partial D} ((z-A)^\star)^{-1}\, E\, d\overline{z} &= -\frac{1}{2\pi i}\int_{\partial D} \overline{(z-\mu)^{-1}}\, E\, d\overline{z} \\ &= \overline{\frac{1}{2\pi i}\int_{\partial D} (z-\mu)^{-1}\ dz\, E} \\ &= E \end{aligned}$$

E is then self-adjoint.

Definition 28. An operator $A \in B(H)$ is said to be polaroid, if each isolated point of its spectrum is a pole of the resolvent of A.

Theorem 29. *Let $A \in B(H)$ be a k-quasi-normal operator of order n. Then, Weyl's theorem holds for A.*

Proof. According to the hypotheses and by Corollary 12, A has SVEP at zero. Suppose that A admits a representation as in Theorem 18. Either $\sigma(A) \subseteq \partial\mathcal{D}$ or $\sigma(A) = \overline{\mathcal{D}}$, where $\mathcal{D}$ is the open unit disc, and $\partial\mathcal{D}$ denotes its boundary.

If $\sigma(A) \subseteq \partial\mathcal{D}$, then A has SVEP everywhere: else $\sigma(A) = \overline{\mathcal{D}}$. The operator A has SVEP on $\sigma(A)\backslash w(A)$, then $< 0\dim(A-\lambda) < \infty$. We have

$$\lambda \in \sigma_p(A) \subseteq \partial\mathcal{D} \cup \{0\}$$

An operator such that its point spectrum has empty interior has SVEP . Hence A has SVEP [1, Remark 2.4(d)] . Also, if

$$\sigma(A) = \overline{\mathcal{D}}$$

then $iso\sigma(A) = \emptyset$. If $\sigma(A) \subset \partial\mathcal{D}$, then A is polaroid. This achieves the proof by [3] .

Lemma 2. *Let A be a k-quasi-normal operator of order n but a quasi-normal of order n, then A admits at least a non-trivial closed invariant subspace.*

Proof. Suppose that A has no non-trivial closed invariant subspace. Since $A \neq 0, N(A) \neq H$ and $\overline{R(A)} \neq 0$ are closed invariant subspace for A. Thus necessarily, $N(A) = \{0\}$ and $\overline{R(A)} = H$. Thus, A is quasi normal operator of order n, which contradicts the hypothesis.

Definition 30. An operator A is said to be n-perinormal if $(A^{*n})(A^n) \geq (A^*A)^n$ for a positive integer n such that $n \geq 2$.

Lemma 3. *Let A be a quasi-normal operator of order n. Then A^n is 2-perinormal operator .*

Proof. Since A is quasi-normal operator of order n,

$$(A^{*n}A^n)^2 = A^{*2n}A^{2n}$$

Then,

$$(A^{*n})^2(A^n)^2 \geq (A^{*n}A^n)^2$$

Hence, A^n is 2-perinormal operator. □

Lemma 4. *Let A be a n-perinormal operator. Then,*

1. $\sigma_{jp}(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$.
2. $\sigma_{ja}(A) \setminus \{0\} = \sigma_a(A) \setminus \{0\}$

Theorem 31. *Let A be a quasi-normal operator of both orders n and $n+1$. Then,*

$$\sigma_{ja}(A) \setminus \{0\} = \sigma_a(A) \setminus \{0\}$$

Proof. By the hypothesis on A, both of A^n and A^{n+1} are 2-perinormal operators.

Let $\lambda \in \sigma_a(A) \setminus \{0\}$. Then, there exists a sequence of units vectors $\{x_m\}$ such that $(A-\lambda)x_m \to 0$

Hence,

$$\begin{cases}(A^n - \lambda^n)x_m \to 0 \\ (A^{n+1} - \lambda^{n+1})x_m \to 0\end{cases}$$

Since A^n and A^{n+1} are 2-perinormal,

$$\begin{cases}\left(A^{*n} - \overline{\lambda}^n\right)x_m \to 0 \\ \left(A^{*n+1} - \overline{\lambda}^{n+1}\right)x_m \to 0\end{cases}$$

Finally,

$$\left(A^* - \overline{\lambda}\right)x_m \to 0$$

□

Definition 32. [1, 9] An operator $A \in B(H)$ is said to have Bishop's property (β) if for each open subset G of $\mathbb{C}$, and all sequence $f_n: G \to H$, of analytic functions such that $(T - \lambda)f_n$ converges uniformly to 0 in norm of compact subsets of G, $(f_n)_n$ converges uniformly to 0, in norm of compact subsets of G.

Lemma 5. *Let A be a quasi-normal operator of orders n and $n+1$. Then, A has Bishop's property (β) .*

Proof. An immediate consequence of Theorem 31 and [12, Lemma 2.1] . □

As an extension of Lemma 5, we state the following result :

Theorem 33. *Let A be a k-quasi-normal operator of order n and $n+1$, then A has Bishop's property (β) .*

Proof. Let's consider two cases :

1. If $A^k(H)$ is dense, then A is quasi-normal of order n and $n+1$ and hence, A has Bishop's property (β).

2. If $A^k(H)$ is not dence, we write A on

   $$H = \overline{A^k(H)} \oplus N(A^{*k})$$

   as $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ with A_1 is a quasi-normal operator of orders n and $n+1$, and $A_3^k = 0$. Let $g_k(u)$ be analytic on $D \subseteq C$ with $(A-u)g_k(u) = 0$ uniformly on each compact K of D. Then

$$\begin{bmatrix} A_1 - u & A_2 \\ 0 & A_3 - u \end{bmatrix} \begin{bmatrix} g_{k_1}(u) \\ g_{k_2}(u) \end{bmatrix} = \begin{bmatrix} (A_1 - u)g_{k_1}(u) + A_2 g_{k_2}(u) \\ (A_3 - u)g_{k_2}(u) \end{bmatrix}$$

Since A_3 is nilpotent, A_3 satisfies Bishop's property (β).

Thus, g_{k_2} uniformly on each compact K on D. Therefore, $(A_1 - u)g_{k_1}(u) \to 0$ as $K_1 \to 0$, since A_1 satisfies Bishop's property. It follows that $g_{k_1}(u) \to 0$ and then, A has Bishop's property (β). □

Theorem 34. *Let $A \in B(H)$ be a quasi-normal operator of order n and $n+1$. If $\sigma(A) = \lambda$, then $A = \lambda$.*

Proof. Let $\sigma(A) = \lambda$. Using the spectral mapping theorem, we get

$$\begin{cases} \sigma(A^n) = \sigma(A)^n = \lambda^n \\ \sigma(A^{n+1}) = \sigma(A)^{n+1} = \lambda^{n+1} \end{cases}$$

Since A is quasi-normal of both orders n and $n+1$, and according to Proposition 9, A^n and A^{n+1} are paranormal operators. Hence

$$\begin{cases} A^n = \lambda^n \\ A^{n+1} = \lambda^{n+1} \end{cases}$$

Thus,

$$A = \lambda$$

Proposition 35. *If $A \in B(H)$ is quasi-normal of order n and $n+1$. If A^{2n} and A^{2n+2} are compacts operators then A is also compact.*

Proof. assume that $A \in B(H)$ is a quasi-normal operator of both orders n and $n+1$. By paranormality of A^n and A^{n+1} we get

$$\begin{cases} \| A^n x \|^2 \leq \| A^{2n} x \| \\ \| A^{n+1} x \|^2 \leq \| A^{2n+2} x \| \end{cases}$$

for all unit vector x. Let $\{x_m\}$ in H be weakly convergent sequence with limit 0 in H. From the compatness of A^{2n} we get that

$$\begin{cases} \| A^n x_m \|^2 \to 0 \\ \| A^{n+1} x_m \|^2 \to 0 \end{cases}$$

Thus, A^n and A^{n+1} are compacts operators. Put $y_m = A^n x_m$. Then, from the compatness of A^{n+1}, we get

$$\| A^{n+1} x_m \|^2 = \| A(A^n) x_m \|^2 = \| A y_m \|^2 \to 0$$

Therefore, A is compact operator.

Definition 36. Let $A \in B(H)$. The local resolvent set of A at a vector Let $x \in H$ denoted by $\rho_A(x)$, is defind to consist of complex element z_0 such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, for which

$$(A - z)f(z) = x$$

Theorem 37. *Let* $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ *be a k-quasi-normal operator of order n with respect to the decomposition* $H = \overline{R(A^k)} \oplus N(A^{\star k})$. *Then, for all* $x = x_1 + x_2 \in H$:

- $\sigma_{A_3}(x_2) \subset \sigma_A(x_1 + x_2)$
- $\sigma_{A_1}(x) = \sigma_{A_1}(x_1 + 0)$

Proof. a. Let $z_0 \in \rho_A(x_1 + x_2)$. Then, there exists a neighborhood U of z_0 and an analytic function $f(z)$ defined on U, with values in H, for which

$$(A - z)f(z) = x, z \in U$$

Let $f = f_1 + f_2$ where f_1, f_2 are in the spaces $O\left(U, \overline{\operatorname{ran}(A^k)}\right)$ and $O\left(U, \ker(A^{\star k})\right)$ respectively, consisting of analytic functions on U with values in H, with respect to the uniform topology [1]. Equality (2) can then be written

$$\begin{pmatrix} A_1 - z & A_2 \\ 0 & A_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then

$$(A_3 - z)f_2(z) = x_2, z \in U$$

Hence, $z_0 \in \rho_{a_3}(x_2)$. Thus, (a) holds by passing to the complement. b. If $z_1 \in \rho_A(x_1 + 0)$, then, there exists a neighborhood V_1 of z_1 and an analytic function g defined on V_1 with values in $\mathcal{H}$ verifying

$$(A - z)f(z) = x_1 + 0, z \in V_1$$

Let $g = g_1 + g_2$, where $g_1 \in O\left(V_1, \overline{\operatorname{ran}(A^k)}\right), g_2 \in O\left(V_1, \ker(A^{\star k})\right)$ are as in (a). From equation (3) we obtain and

$$(A_1 - z)g_1(z) + A_2 g_2(z) = x_1$$
$$(A_3 - z)g_2(z) = 0, z \in V_1$$

Since A_3 is nilpotent by Theorem 18, A_3 has SVEP. Thus, $g_2(z) = 0$. Consequently, $(A_1 - z)g_1(z) = x_1$. Therefore, $z_1 \in \rho_{A_1}(x_1)$, and then $\rho_A(x_1 + 0) \subset \rho_{A_1}(x_1)$. Thus, $\sigma_{A_1}(x) = \sigma_{A_1}(x_1 + 0)$.

Now, if $z_2 \in \rho_{A_1}(x_1)$, then, there exists a neighborhood V_2 of z_2 and an analytic function h from V_2 onto H, such that $(A_1 - z)h(z) = x_1$, for all $\in V_2$. Thus,

$$(A - z)(h(z) + 0) = (A_1 - z)h(z) = x_1 = x_1 + 0$$

Hence, $z_2 \in \rho_A(x_1 + 0)$. □

Proposition 38. *Let A be a regular quasi-normal operator of order n. Then, the approximate point spectrum $\sigma_a(A)$ of A lies in the set* $\{\lambda \in \mathbb{C}: \frac{1}{\|A^{-n}\|^2\|A^{2n-1}\|} \leq |\lambda| \leq \|A\|\}$

Proof. Let $x \in H$ with $\| x \|= 1$. We have

$$\| x \|^2 = \|A^{-n}A^n x\|^2 \leq \|A^{-n}\|^2\|A^n x\|^2$$

Since A^n is paranormal, $\|A^n x\|^2 \leq \|A^{2n}x\|$. Then,

$$1 = \| x \|^2 \leq \|A^{-n}\|\|A^{2n}x\| \leq \|A^{-n}\|^2\|A^{2n-1}x\| \, \| Ax \|$$

Hence,

$$\| Ax \| \geq \frac{1}{\|A^{-n}\|^2\|A^{2n-1}\|}$$

If $\lambda \in \sigma_a(A)$, then there exists a unit sequence $(x_m)_m$ in H satisfying $\|(A-\lambda)x_m\| \to 0$. Then,

$$\|Ax_m - \lambda x_m\| \geq \|Ax_m\| - |\lambda|\|x_m\| \geq \frac{1}{\|A^{-n}\|^2\|A^{2n-1}\|}$$

Letting $m \to \infty$, we obtain

$$|\lambda| \geq \frac{1}{\|A^{-n}\|^2\|A^{2n-1}\|}$$

Thus,

$$\sigma_a(A) \subseteq \left\{\lambda \in \mathbb{C}: \frac{1}{\|A^{-n}\|^2\|A^{2n-1}\|} \leq |\lambda| \leq \| A \|\right\}$$

Competing interests. The authors declare that they have no competing interests.

Availability of data and materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Author's contribution. All authors discussed the results and contribute to the final Manuscript

References

1. P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer Academic Publishers, 2004.

2. S.K. Berberian, *Introduction to Hilbert Spaces*, New York : Chelsea Publishing Company, 1976.

3. R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem* , Trans. Amer. Math. Soc., **349** (1997), 2115–2124.

4. J.K. Han, H.Y. Lee and W.Y Lee, *Invertible completions of* 2×2 *upper triangular operator matrices* , Proc. Amer. Math. Soc., **128** (1999), 119–123.

5. T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of* log-*hyponormal and several related classes* , Sci. Math., **1** (1998), 389–403.

6. W. Rudin, *Functional Analysis*, McGraw-Hill, 1973, New York.

7. L. K. Shaakir and S. S. Marai, *Quasi-nomal operators of order* n, Tikrit J. Pure Sci., **20** (2015), 167–169.

8. S. Mecheri, *Bishop's property* β *and Riesz Idempotent for* k*-quasi-paranormal operators*, Banach. J. Math. Anal., **6** (2012), 147–154.

9. S. Mecheri, Bishop's propertyβ, *hypercyclicity and hyperinvariant subspaces*, Oper. Matr., **8** (2014), 555-562.

10. S. Mecheri, *Analytic extension of n-normal operators*, Oper. Matr., **15** (2021), 615–626.

11. H. Zoo and F. Zoo, *A note on perinormal operators* , Acta. Math. Scientia, **34B** (2014), 194–198.

12. A. Uchiyama and K. Tanahashi, Bishop's property β for paranormal operators, *Oper. Matr.* **3** (2009), 517–524.

Ayyoub Fellag Ariouat[1], Aissa Nasli Bakir[2] and Abdelkader Benali[3]

[1]Department of Mathematics

Laboratory of Mathematics and Applications LMA

Faculty of Exact Sciences and Informatics

Hassiba Benbouali University of Chlef, B.P. 78C, 02180, Ouled fares.

Chlef, Algeria.

e-mail : a.fellagariouat@univ-chlef.dz

[2]Department of Mathematics

Laboratory of Mathematics and Applications LMA

Faculty of Exact Sciences and Informatics

Hassiba Benbouali University of Chlef, B.P. 78C, 02180, Ouled fares.

Chlef, Algeria.

National Higher School of Cybersecurity NSCS, Sidi Abdellah.

Algiers, Algeria.

e-mail : a.nasli@univ-chlef.dz

[2]Department of Mathematics

Laboratory of Mathematics and Applications LMA

Faculty of Exact Sciences and Informatics

Hassiba Benbouali University of Chlef, B.P. 78C, 02180, Ouled fares.

e-mail : benali4848@gmail.com